Coherent Phonon States and Long-Range Order in Two-Dimensional Bose Systems

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At any finite temperature, the correlation function $\langle\psi^\dagger(r)\,\psi(0)\rangle$ of a two-dimensional Bose system is known to approach zero as the separation r approaches infinity. This paper is devoted to supporting the point of view that superfluid phenomena may still persist in such a system. To describe the low-temperature behavior of a condensed two-dimensional Bose system, we use a coherent-state representation of the low-lying phonon excitations. The expectation of the particle field operator $\langle\psi(r)\rangle$ in a coherent phonon state of the system is shown to be a well-defined function. This function has only small fluctuations in its magnitude and, therefore, it is possible to make the usual arguments for superfluid behavior based on the existence of the condensate wave function. The large phase fluctuations of the function, when the coherent phonon states are collected into the thermal ensemble, give rise to the spatial decay of the correlation function or reduced density matrix $\langle\psi^+(r)\psi(0)\rangle$. It is concluded that two-dimensional Bose systems are superfluid at sufficiently low temperatures, even in the inifinite-volume limit. An analogous argument would indicate the possibility of superconductivity in a two-dimensional Fermi system.

I. INTRODUCTION

It has been conclusively demonstrated that two-dimensional systems of interacting bosons do not possess long-range order at finite temperatures. Kane and Kadanoff[1] have shown that the correlation function $\langle\psi^\dagger(r)\psi(0)\rangle$ approaches zero in the limit $r\to\infty$. Their result was based on Hohenberg and Martin's theory[2] of liquid helium in the hydrodynamic limit. In a much more direct way Hohenberg[3] has proved that the occupation of the zero-momentum single-particle state must vanish in the limit of infinite volume. Thus the criterion of Penrose and Onsager[4] for a Bose condensation and that of Yang[5] for off-diagonal long-range order are not met.

These results leave open the question of whether or not two-dimensional Bose systems can have any of the other properties peculiar to helium II. In this paper we argue for the existence of the superfluid properties. Our considerations apply only to very low temperatures and we have no comments on the nature of the transition to the normal liquid.[6]

Our starting point is the assumption that the decay of the correlation function is solely a result of the thermal excitation of very long wavelength phonons. In the thermal ensemble and, indeed, even in a pure state of the system in which there is a definite number of phonons present, the phases of vibration of these phonons are completely indeterminate. This phase averaging inherent in the usual theory in combination with the phonon density of states characteristic of two-dimensional systems is responsible for their lack of long-range order. The formal 'cure' for this disease is apparent. One should consider states of the system in which the phases of the vibrational modes are well determined. Such states are called coherent states and have been extensively used in recent years to describe coherent optical radiation fields[7] and phonons in crystals.[8]

We shall show that the expectation of the particle field operator in one of these coherent phonon states has a magnitude which is close to that in the ground state even in the limit of infinite volume. The phase of this condensate wave function does, however, have a variation which, in two dimensions, increases without limit as the volume increases. The existence of the expectation of the field operator $\psi(r)$ can be made the basis of a discussion of the superfluid properties.[9] In the two-dimensional case, the expectation of $\psi(r)$ in a typical pure coherent phonon state can be used in a quite parallel way. In the appendix we show that the phase of $\langle\psi(r)\rangle$ is well determined by an ideal measurement of the system.

It can be shown that our use of the coherent-state representation is necessary; and that it is the use of the number representation of the usual theory and not the collection of these states into the thermal ensemble that causes the condensate wave function to vanish in the infinite-volume limit. The number of particles occupying the one-particle ground state in a typical phonon number representation state vanishes in the infinite-volume limit.

These coherent states have been recently invoked in theoretical discussions of boson systems by F. W. Cummings and J. R. Johnston,[10] and by J. S. Langer.[11] These authors treat many-boson states formed from eigenstates of the particle destruction operators. In contrast to this, we will form states of the complete system which are eigenstates of the phonon destruction operators; thus generating excited states from an assumed ground state. Our procedure is therefore more phenomenological, and depends upon the ground state satisfying the criterion of Penrose and Onsager for the existence of a Bose condensation. That the two-dimensional ground state has this property is one of the conclusions of a recent paper by Reatto and Chester.[12]

There are two quite distinct types of two-dimensional systems. One is a purely mathematical model in which the configuration space of each particle has only two components. The other is a real system of liquid helium with a small thickness and a very large extent in the other two directions. D. A. Krueger[13] has discussed the application of Hohenberg's theorem to the latter physical type of system. A clear-cut experimental answer to the question of the superfluid properties of two-dimensional systems is clearly very difficult to obtain; some preliminary work has been reported by D. F. Brewer et al.[14] We believe that the theoretical considerations of this paper apply equally to both types of systems.

This work was undertaken partly in the hope that it could be extended to include superconducting systems. A dynamically two-dimensional Fermi gas is present in the inverstion layer on the surface of a silicon crystal at low temperature. Fowler et al.[15] have observed magnetoresistance oscillations which directly imply that the one-electron density of states of this system is that predicted for a two-dimensional configuration space. The preparation of such a layer on the surface of a crystal with a large electron-phonon interaction should have superconducting properties. The work of T. M. Rice[16] on one- and two-dimensional Fermi systems is roughly parallel to that of Kane and Kadanoff on Bose systems.

In charged Fermi systems, the role of the phonons is played by longitudinal charge-current waves. In three-dimensional systems, these are plasmons which have high energies even for small wave vectors. In a two-dimensional geometry, however, their dispersion function $\omega(\vec{k})$ approaches zero as the wave vector $\vec{k}$ approaches the origin. This similarity makes us believe that the point of view of this paper may well be applicable to physical "two-dimensional" superconductors.

The question of persistent currents in physical "one-dimensional" superfluids and superconductors has been the subject of theoretical papers by Langer and Fisher,[17] and Langer and Ambegaokar.[18] These papers show that there exist large fluctuations of the order parameter or condensate wave function which result in the appearance of finite resistance at temperatures below the transition temperature of the bulk material. These papers leave unresolved the question of the existence of a condensate wave function in systems without long-range order as noted, for example, in Ref. 10 of the latter paper.

In a later paper, Langer[11] discusses this point from a very general point of view, which applies at all temperatures below the bulk transition temperature. Our work is a resolution of this question for the special case of two-dimensional superfluids at low temperatures. Langer's starting point and that adopted here are so different that we are unable to point out any direct relationship.

The next section demonstrates in detail how coherent phonon states may be generated by applying approximate phonon creation-destruction operators to the ground state. The form of these operators is taken from the work of Penrose,[19] and their approximate character limits the range of application of our work to small wave vectors and therefore to low temperatures. The expectation of a particle field operator is calculated in a typical coherent phonon state and shown to be a smooth well-behaved function of position and time.

The thermal ensemble of our system may be expressed as a multiple integral over the complex amplitudes of the coherent phonons. This is confirmed in the third section by calculating the leading term of the asymptotic expression for the correlation function $\langle\psi^\dagger(r)\psi(0)\rangle$ and obtaining the same form as Kane and Kadanoff.

In the fourth section a brief argument for the existence of persistent currents is given. Some comments on magnetic two-dimensional systems, one-dimensional systems, and a comparison of the over-all situation with the infrared catastrophe of quantum electrodynamics complete the body of the paper. The appendix sketches a demonstration that long-wavelength phase variation of the condensate wave function, i.e., the expectation of the field operator $\psi(x)$ in a coherent phonon state, is well determined by an ideal measurement of the system.

II. COHERENT PHONON STATES

Assuming a constant velocity of sound and a harmonic equation of motion for the phonon modes, Penrose[19] derived an expression for the phonon creation and destruction operators;

$$c_k^\dagger = (4e_k\omega_k N)^{-1/2}(\dot{\rho}_{-k} + i\omega_k\rho_{-k}),$$
$$c_k = (4e_k\omega_k N)^{-1/2}(\dot{\rho}_k - i\omega_k\rho_k), \tag{1}$$

where $\rho_k{}^\dagger = \rho_{-k}$, N is the total number of particles, $e_k = k^2/2m$ is the one-particle kinetic energy ($\hbar = 1$ throughout this paper), and $\omega_k = ck$ is the phonon frequency. The operator ρ_k is the Fourier transform of the particle number density, and $\dot{\rho}_k$ is its time derivative.

$$\rho_k = \sum_q a_q{}^\dagger a_{k+q},$$

$$\dot{\rho}_k = i\sum_q (e_q - e_{k+q}) a_q{}^\dagger a_{k+q},$$

where $a_k{}^\dagger$ and a_k are the particle creation and destruction operators. The form of $\dot{\rho}_k$ follows from the expression for the kinetic energy $T = \sum e_k a_k{}^\dagger a_k$.

These phonon operators only approximately satisfy boson commutation rules. For example,

$$[c_k, c_k{}^\dagger] = 1 + [kP/m\omega_k N], \tag{2}$$

where $P = \sum k a_k{}^\dagger a_k$ is the total-momentum operator. The term containing P has an expectation value of unity in a state in which every particle has a velocity equal to the velocity of sound c. The expectation of the square of this operator is therefore very small in a thermal ensemble at low temperatures. The other commutators of the c_k's and $c_k{}^\dagger$'s are also small at low temperatures.

Because all our results depend upon these approximate forms for the phonon operators, they will be exact only in the limit of long wavelength and low temperatures. In forming the thermal ensemble, for example, we will neglect the effect of the small deviations of the phonon operator commutators, Eq. (2), from the exact Bose commutators.

We now wish to consider states of the system in which the assumed harmonic and independent phonon modes are excited into coherent states. To remind the reader of the properties of these states, we quote the following expressions which are derived in Glauber's paper.[7] A coherent state of a single degree of freedom is specified by a complex number α which is the right eigenvalue of the step-down or destruction operator $c|\alpha\rangle = \alpha|\alpha\rangle$. It may be expanded in terms of energy eigenstates by

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_n \alpha^n (n!)^{-\frac{1}{2}}|n\rangle.$$

The coherent states form an overcomplete set with overlap $\langle\alpha|\beta\rangle = \exp(\alpha^*\beta - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2)$. Their completeness is expressed by expanding the unit operator $1 = \pi^{-1}\int d^2\alpha\,|\alpha\rangle\langle\alpha|$. The density operator which represents the thermal equilibrium ensemble at temperature T is written

$$\rho_T = (\pi n_T)^{-1}\int d^2\alpha\,|\alpha\rangle e^{-|\alpha|^2/n_T}\langle\alpha|, \tag{3}$$

where $n_T = (e^{\omega/KT} - 1)^{-1}$ is the average excitation number at temperature T.

The extension of these relations to the product space of the phonon coordinates of the low-temperature Bose system is straightforward.

A coherent phonon state of our system is specified by assigning values of the complex phonon amplitudes α_k. It may be generated from the ground state $|G\rangle$ of the system by a unitary transformation,[20,21]

$$\begin{aligned}|\{\alpha_k\}\rangle &= \exp[\sum_k(\alpha_k c_k{}^\dagger - \alpha_k{}^* c_k)]|G\rangle \\ &= U\{\alpha_k\}|G\rangle.\end{aligned} \tag{4}$$

In our approximation, the time evolution of the operator $U\{\alpha_k\}$ is given by $\alpha_k(t) = \exp(-i\omega_k t)|\alpha\ (0)|$. The exact time dependence is much more complicated, not only because of our use of approximate operators, but also because of phonon-phonon interactions.

By using the commutation relations for the α_k's we obtain the commutator

$$\begin{aligned}&[\sum_k(\alpha_k c_k{}^\dagger - \alpha_k{}^* c_k), \psi(r)] \\ &\quad = -[iS(r) + \tfrac{1}{2}\nabla^2 T(r) + (\nabla T)\cdot\nabla]\psi(r),\end{aligned} \tag{5}$$

where

$$\begin{aligned}S(r) &= \sum_k (N\epsilon_k\omega_k)^{-1/2}\omega_k|\alpha_k|\cos(kr+\theta_k), \\ T(r) &= \sum_k (N\epsilon_k\omega_k m^2)^{-1/2}|\alpha_k|\sin(kr+\theta_k),\end{aligned} \tag{6}$$

where

$$\alpha_k = |\alpha_k|\exp(i\theta_k), \quad \psi(r) = \Omega^{-\frac{1}{2}}\sum e^{ikr}\alpha_k$$

is the particle field operator, and Ω the total volume.

It now follows that the unitary transformation of Eq. (4) can be exactly represented by a transformation of the particle field operators $\psi(r)$ and $\psi^\dagger(r)$.

$$\begin{aligned}&U^{-1}\{\alpha_k\}\psi(r)U\{\alpha_k\} \\ &\quad = \exp[iS(r) + \tfrac{1}{2}\nabla^2 T(r) + (\nabla T)\cdot\nabla]\psi(r).\end{aligned} \tag{7a}$$

The transformation of $\psi^\dagger(r)$ is the adjoint equation. This is derived from the identity

$$e^Q P e^{-Q} = \sum_{n=0}^{\infty}(R_n/n!), \tag{7b}$$

where R_n is the nth commutator of Q upon P; i.e., $R_0 = P$ and $R_{n+1} = [Q, R_n]$. In our case R_n is the nth power of the derivative operator which is the argument of the exponential of Eq. (7a).

If we consider the amplitudes α_k to be random variables whose distribution produces the thermal ensemble according to Eq. (3), then $S(r)$ and $T(r)$ are independent Gaussian random functions of the two space components.[22]

The exponential factor of Eq. (7a) is a rather general type of derivative operator because of the presence of the gradient in its argument. This gradient, of course, operates on the field operator as well as the other terms of the argument of the exponential. Equation (7a) allows us to express any matrix element of products of the field operators in the coherent phonon representation in terms of expectations of the transformed operators in the ground state.

The simplest of such matrix elements is the expectation of a single field destruction operator.

$$\Phi(r) = \langle\{\alpha_k\}|\psi(r)|\{\alpha_k\}\rangle. \tag{8a}$$

We will refer to this function as the condensate wave function, although that term is traditionally used for the expectation of the field operator in a statistical ensemble. The state in this definition of the condensate wave function must, of course, be derived from a ground state containing an indefinite number of particles as discussed, for example, by Hohenberg and Martin.[2]

To find an explicit expression for the condensate wave function it is necessary to express the exponential factor of Eq. (7a) as a product of a function of r times an exponential of the derivative only:

$$\begin{aligned}&\exp[iS(r) + \tfrac{1}{2}\nabla^2 T(r) + (\nabla T)\cdot\nabla] \\ &\quad = \exp[iS'(r) + \tfrac{1}{2}\nabla^2 T'(r)]\exp\}[\nabla T(r)]\cdot\nabla\{,\end{aligned} \tag{8b}$$

where

$$S'(r) = \sum_{n=0}^{\infty}[\nabla T(r)\cdot\nabla]^n S(r)/(n+1)!$$

$$T'(r) = \sum_{n=0}^{\infty}[\nabla T(r)\cdot\nabla]^n T(r)/(n+1)!$$

This is a generalized Baker-Hausdorff theorem[23] and is confirmed by taking the derivative with respect to λ of $\exp\lambda(Q+P) = \exp\lambda Q'\exp\lambda P$ and using the identity of Eq. (7b).

When Eq. (8b) is applied to find the condensate wave function (8a), the derivative vanishes because it acts on the condensate wave function of the ground state which is independent of r.

$$\Phi(r) = n_0{}^{1/2}\exp[iS'(r) + \tfrac{1}{2}[\nabla^2 T'(r)]], \tag{9}$$

where $n_0 = |\langle G|\psi(r)|G\rangle|^2$ is the occupation number of the zero-momentum single-particle state. The fluctuations in the magnitude of the above function are not correctly given by Eq. (9).[24]

To assure ourselves that the infinite series which gives $S'(r)$ in terms of $S(r)$ converges to define a well-behaved function, consider the special case expressed by the formula

$$\exp[tf(x)+t(d/dx)] = \exp[\int_x^{x+t} f(x')dx'] \exp[t(d/dx)],$$

where $f(x)$ is an ordinary function of one variable. This formula is easily verified by taking its derivative with respect to t. To generalize this to cover Eq. (9), one must replace x by r which has two components and multiply the derivative by another function of r. The result of this generalization is a much more complicated expression than the above one, in which the argument of the first exponential on the right-hand side is a line integral over a streamline of the vector field $\nabla T(r)$ with a weight function which depends on $T(r)$. The form of this expression implies that if $S(r)$ and $T(r)$ are smooth, well-behaved functions, then $S'(r)$ and $T'(r)$ are also.

We have, therefore, derived an explicit expression for the condensate wave function of a two-dimensional system with an arbitrary coherent phonon excitation. In the sequel we shall use the existence of this function to argue in the conventional way for the existence of persistent currents in two-dimensional Bose systems.

III. THE ASYMPTOTIC CORRELATION FUNCTION

From the thermal ensemble in the coherent phonon representation, we will calculate the leading term of the correlation function $\langle\psi^\dagger(r_1)\psi(r_2)\rangle$ in the limit as $|r_1-r_2|\to\infty$, and thus confirm that it predicts the same long-range properties as the conventional theory.

The limit of this expectation in the two-dimensional ground state as shown by Reatto and Chester[12] is just the constant n_0. The transformation of the field operators, Eq. (7a), then gives us the result,

$$\lim_{|r_1-r_2|\to\infty} \langle\psi^\dagger(r_1)\psi(r_2)\rangle = \langle\exp[-iS'(r_i)+\tfrac{1}{2}\nabla^2T'(r_1)+iS'(r_2) + \tfrac{1}{2}\nabla^2T'(r_2)]\rangle. \tag{10}$$

The averaging on the right-hand side is to be done by a multiple integral over the coherent phonon amplitudes α_k. This can be thought of as the stochastic averaging over the random functions specified by the Gaussian random variables, the α_k. The contribution of the real part of the exponent is negligible at large separations compared to the imaginary part, and so we will drop the $T'(r)$ in the remaining discussion. This well-known property of the condensate wave function could be confirmed here by including the ∇^2T' term in the remaining discussion of this section.

The limit of the correlation function is easily evaluated by a cumulant expansion,[25]

$$\langle\exp X\rangle = \exp(\sum_{n=1}^{\infty} \langle X^n\rangle_c/n!),$$

where the second and fourth cumulants are given in terms of the ordinary moments by: $\langle X^2\rangle_c=\langle X^2\rangle$, $\langle X^4\rangle_c=\langle X^4\rangle-3\langle X^2\rangle^2$; assuming $\langle X\rangle=\langle X^3\rangle=0$. The cumulant series terminates at the second moment for a Gaussian random function. It is, therefore, easier to evaluate the first few terms of the cumulant expansion if we base it on the derivative form of Eq. (7a) rather than the explicit function $S'(r)$. Kubo has discussed the extension of the cumulant expansion to operators.[25] Odd-order averages vanish and the second moment involves only $S(r)$.

According to Eq. (3), the amplitudes α_k have their phases uniformly distributed and their magnitudes Gaussian-distributed,

$$\langle\alpha_k\alpha_q\rangle=0;$$
$$\langle\alpha_k\alpha_q^*\rangle\approx\delta_{k,q}\times\begin{cases}k_0/k, & k<k_0\\ 0, & \text{otherwise}\end{cases}$$

where we have for convenience approximated the Planck distribution by the classical distribution cutoff at $k=k_0=KT/c$. We then pass to the infinite-volume limit and obtain,

$$\begin{aligned}\langle[S(r_1)-S(r_2)]^2\rangle &\approx (ck_0/N)\sum\epsilon_k^{-1}(1-\cos kr)\\ &\approx(mr_0^2ck_0/2\pi^2)\int_{k<k_0}(d^2k/k^2)\\ &\quad\times(1-\cos kr),\end{aligned} \tag{11}$$

where $r_0^2=\Omega/N$ and $r=r_1-r_2$. A typical term contributing to the fourth cumulant is:

$$3\langle\{[\nabla_1T(r_1)\cdot\nabla_1S(r_1)]-[\nabla_2T(r_2)\cdot\nabla_2S(r_2)]\}^2\rangle.$$

The integrals which evaluate this expression and the other contributing terms decrease at large separations as $|r_1-r_2|^{-2}$. The result for the asymptotic value of the correlation function, therefore, arises from the logarithmic term of the integral of Eq. (11), and has the same form as that obtained by Kane and Kadanoff.

$$\lim_{|r_1-r_2|\to\infty}\langle\psi(r_1)\psi(r_2)\rangle = (\tfrac{1}{2}k_0|r_1-r_2|)^{-(mr_0^2ck_0/\pi)}. \tag{12}$$

IV. THE ARGUMENT FOR PERSISTENT CURRENTS

Our starting point is to assume that at sufficiently low temperatures the thermal ensemble of a two-dimensional Bose system can be exactly expressed as a multiple integral over the amplitudes of coherent phonon states. The formal developments of this paper show that this is possible in the particularly simple approximation that has been made in assuming Penrose's form for the phonon operators, and in assuming they exactly obey Bose commutation rules. It seems unlikely to us that corrections to these approximations would change the form of the theory so profoundly as to falsify our conclusions. The approximate phonon operators are apparently exact in the limit of long wavelength and zero temperature. The effect of phonon-phonon interactions, on the other hand, can presumably be represented by a suitable nonlinear differential equation for the time evolution of the coherent amplitudes.

Let us consider a large two-dimensional system which is periodic in the x direction. That is, we envisage a physically doubly connected two-dimensional space, such as the surface of a cylinder. For the moment we will ignore the necessity of including vortices in our discussion. From the ground state one may generate a series of states of nonzero circulation by multiplying the ground state configuration wave function by $\exp[2\pi\nu i \times \sum_j (x_j/L)]$ where x_n is the coordinate of the nth particle and L is the spatial period. Our condensate wave function derived from such a state of circulation number ν is:

$$\Phi_\nu(r) = n_0^{1/2} \exp[iS(r) + \tfrac{1}{2}\nabla^2 T(r) + \nabla T(r) \circ \nabla] \times \exp(2\pi i \nu x/L). \qquad (13)$$

This is a well-defined function whose magnitude suffers only relatively small fluctuations, and whose value is relatively well-determined by an ideal measurement as is shown in the appendix. If our conjecture is correct that phonon-phonon interactions are represented by a nonlinear differential equation, as has been derived by Carruthers and Dy[8] for phonons in crystals, then the circulation number will be exactly conserved. In a two-dimensional system, of course, fluctuations that completely disrupt the continuity of the condensate wave function for all possible closed paths in the x direction will have a vanishing probability in the limit of large two-dimensional volume.

To sum up our point of view, we believe that all states of the systems with finite probability in the infinite-volume limit are characterized by a classically well-determined condensate wave function of definite circulation number whose value is exactly conserved in the time evolution.

When the occurrence of vortices is taken into account, this simple picture is considerably altered in both two- and three-dimensional systems. Persistent currents of helium II are only observed in containers of finely divided powder which serves to prevent the motion of vortices[26] and inhibit their nucleation.[27] We see no reason to believe that the effects of vortices on persistent currents will be of a different nature in two- and three-dimensional systems.

V. CONCLUDING REMARKS

In the case of two-dimensional magnetic systems, the physical manifestations of the lack of long-range order are more dramatic than in superfluid or superconductive systems. Mermin and Wagner[28] have shown that at finite temperature the spontaneous moment of a two-dimensional Heisenberg model approaches zero as the external field approaches zero. In this system the analog of the existence of a condensate wave function is the existence of a local classical moment. The thermal excitation of spin waves then causes the direction of this moment to wander in space and time. In a special Heisenberg model in which the atomic spins and moments have only two components, the local moment may be described by a complex number, and there is an obvious analog to persistent current states. Some theoretical results have been obtained which favor the existence of phase transitions in two-dimensional magnetic systems.[29]

Our discussion does not apply to one-dimensional systems. The ground state of a one-dimensional interacting Bose system does not meet the Penrose-Onsager criterion for the presence of a condensate.[30] Physical systems having a large extent in only one direction will have a finite linear density of regions in which the condensate wave function will vanish.[31]

The infrared catastrophe of quantum electrodynamics provides a rough parallel with these questions of long-range order in two-dimensional systems. In both cases an elegant and powerful theory has to be extended to deal with a certain set of problems because of the effects of classically excited modes on long wavelength. V Chung[32] has treated the infrared divergence problem by a coherent state representation of the modes of the radiation field.

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APPENDIX

The precision with which our condensate wave function, Eq. (8a), is determined by an ideal measurement may be calculated as follows. Arthurs and Kelly[33] have shown that the amplitude α of a single degree of freedom may be determined by a simultaneous measurement of its real and imaginary parts with an uncertainty $\langle[\mathrm{Re}(\alpha_m-\alpha)]^2\rangle = \langle[I_m(\alpha_m-\alpha)]^2\rangle = 1$, where α_m is the measured value of the amplitude and the system is in the state $|\alpha\rangle$ before the measurement. This result has been discussed in terms of coherent states by Gordon and Louisell.[34] Using this result, one may calculate $[\langle|\Phi_1(r)-\Phi_2(r)|^2\rangle/n_0]$, where $\Phi_{1,2}$ are the values of the condensate wave function, Eq. (9), as determined by two independent measurements on systems prepared in the same coherent phonon state (4). The angular brackets indicate averaging over the result of the two measurements and the thermal ensemble for the measured state. Assuming that only the amplitudes in the thermal range, i.e., α_k for $k < k_0$ need be determined, we find in the approximations of Sec. III that this relative uncertainty in the measured value of the condensate wave function is $\pi^{-1} c m k_0 r_0^2$, which is much less than unity for $T < T_\lambda$.

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