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APPENDIX

The condition of time reflection symmetry for a system in equilibrium with a constant-temperature heat bath specifies that no feature of the thermal fluctuations as a function of time allows an observer to distinguish between positive and negative sense of time. Thus if the system is observed to be in state N_0 at time t_0, the probability that it will be in state N_1 at time $t_0+\tau$ is the same as the probability that it was in state N_1 at time $t_0-\tau$. Let the former probability be $p(N_0, N_1, \tau)$ and the latter be $q(N_0, N_1, \tau)$. Furthermore let $P_0(N)$ be the unconditional probability of state N. Thus $P_0(N)$ satisfies

$$P_0(N) = \sum_{N'} P_0(N')p(N', N, \tau). \quad \text{(A1)}$$

Also, by Bayes's theorem,

$$q(N_0, N_1, \tau) = \frac{P_0(N_1)p(N_1, N_0, \tau)}{\sum_{N'} P_0(N')p(N', N_0, \tau)}. \quad \text{(A2)}$$

Using Eq. (A1),

$$q(N_0, N_1, \tau) = \frac{P_0(N_1)p(N_1, N_0, \tau)}{P_0(N_0)}. \quad \text{(A3)}$$

The required equality of $q(N_0, N_1, \tau)$ and $p(N_0, N_1, \tau)$ then immediately yields

$$P_0(N_0)p(N_0, N_1, \tau) = P_0(N_1)p(N_1, N_0, \tau). \quad \text{(A4)}$$

In terms of p^+ and p^- as defined by Eq. (1) we have, in the limit of small τ,

$$p(N, N+1, \tau) = \tau p^+(N),$$

$$p(N+1, N, \tau) = \tau p^-(N+1). \quad \text{(A5)}$$

Thus Eq. (4) follows from Eqs. (A4), (A5), and (2).

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Finite Geometries and Ideal Bose Gases*†

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It is shown that the lack of condensation in partially finite geometries is not dependent on the use of periodic boundary conditions or the symmetry-breaking technique for a class of independent-particle Bose-gas models. Condensation in an ideal Bose gas in a finite geometry is studied for one-, two-, and three-dimensional systems. Explicit expressions for the chemical potential in terms of the number of particles, the temperature, and the dimensions of the system are obtained by using a contour-integration technique to express sums as integrals plus corrections. Even though there is no true condensation, the range of the reduced density matrix for an ideal Bose gas in a thick slab ($L_1 \times L_2 \times L_3$ with L_1, $L_2 \to \infty$ and $\infty > L_3 \gg \lambda_T =$ the thermal wavelength) is found to go from order $\lambda_T T_{cB}/(T-T_{cB})$ to order $L_3 \exp\{\frac{1}{2}\sqrt{\pi}\,\zeta_{3/2}(L_3/\lambda_T)\,[(T_{cB}/T)^{3/2}-1]\}$ as the temperature is lowered through the bulk critical temperature T_{cB}. (ζ_χ is the Riemann ζ function.) The range exhibits a spatial asymmetry. The onset of this extremely long-range order is found to be reflected in PVT diagrams and the specific heat in a manner very similar to that in bulk condensation. It is conjectured that in calculations for infinite systems the existence of a long- but finite-range order may be a more relevant criterion for superfluidity and superconductivity than the usual criterion of an infinitely long-range order.

I. INTRODUCTION

The observations of persistent currents[1] and Josephson effects[2] in liquid He^4 and superconductors in thin film and pore geometries (i.e., $L \times L \times D$ or $L \times D \times D$ with $L \gg D$) show that "super" behavior exists in these geometries. Measurements of the onset temperature for persistent currents,[3] the temperature of the specific-heat maximum,[4] the critical velocity[5] for liquid He^4, and the energy gap[6] in superconductors for various D values clearly show a geometry effect.

As a first attempt to calculate the observed dependence on D it is tempting to keep D finite and let $L\to\infty$ in analogy with successful bulk calculations. However, it has been shown quite generally that, for nonzero temperatures, this leads to the vanishing of the quasi-averages[7] usually associated with superconductivity and superfluidity.[8,9] In Sec. II we investigate the necessity of two of the assumptions used in Refs. 8 and 9, namely the use of periodic boundary conditions[10] and the use of the symmetry-breaking technique.[11] Without using these assumptions, it is shown that for an independent Boson model there is no condensation[12] in partially finite geometries (i.e., one or more dimensions finite and one or more dimensions extend to infinity). This is in agreement with the results of Hohenberg and Krueger. Since there is nothing in the proofs of Hohenberg and Krueger to distinguish between noninteracting systems and those with arbitrary local interparticle interactions, the above result makes plausible the conjecture that the symmetry-breaking technique and periodic boundary conditions are convenient, but not crucial for the validity of the final general result.

If we believe (a) that nonzero quasi-averages are *not* possible in partially finite geometries and (b) that nonzero quasi-averages are *necessary* for "super" behavior, then the experimental evidence forces us to conclude that partially finite geometries are not good approximations to the experimental situation. However, the general proof only implies that there is no strong off-diagonal long-range order (ODLRO)[13] in the reduced density matrix (i.e., due to a macroscopic occupation of one level), but does not imply anything about weak ODLRO (i.e., due to many very closely spaced levels having moderately large populations as suggested by Girardeau[14]). Furthermore, the general results say nothing about the existence of an extremely long- but finite-range order.

In the absence of a necessary as well as sufficient criterion for super behavior, we are left with at least two reasonable alternatives: (a) calculate for strictly finite systems and use "condensation" into a single state as a criterion or (b) let $L\to\infty$ with D finite but use a relaxed criterion for super behavior. This relaxed criterion might take the form of an extremely long- but finite-range off-diagonal order in the reduced density matrices. In Sec. III we shall show that in some sense both alternatives are possible for the ideal Bose gas.

II. INDEPENDENT - PARTICLE BOSE GAS

The independent-particle model is studied because (a) there is no *a priori* reason for the effects of finite geometry to be absent in this model; (b) rigorous results are possible so the deviations from bulk results are due solely to the finite geometry; and (c) the general results of Refs. 8 and 9 may be verified without using the symmetry-breaking technique or periodic boundary conditions.

In the language of second quantization, the Hamiltonian of the independent-particle Bose gas is

$$H=\sum_\sigma \epsilon_\sigma a_\sigma^+ a_\sigma, \tag{1}$$

where $a_\sigma^+(a_\sigma)$ is the creation (annihilation) operator for the single-particle state which has energy ϵ_σ. The usual Bose commutation relations hold:

$$[a_\sigma, a_{\sigma'}^+]=\delta_{\sigma',\sigma}\;;\;[a_\sigma,a_{\sigma'}]=0=[a_\sigma^+,a_{\sigma'}^+]. \tag{2}$$

In the grand canonical ensemble, the expected total number of particles is given by

$$N=\sum_\sigma n_\sigma=\sum_\sigma(e^{\beta\epsilon_\sigma+\alpha}-1)^{-1}, \tag{3}$$

where $\beta^{-1}=\kappa T$, T is the temperature, κ is Boltzmann's constant, $\alpha=-\beta\mu$, and μ is the chemical potential. The dependence on geometry is implicit in the values and distribution of the ϵ_σ's. The geometry will, in general, be characterized by three lengths, $L_1\geq L_2\geq L_3$.

Solving Eq. (3) for $\alpha(N, T, L_1, L_2, L_3)$ allows us to use the density ($\equiv N/L_1L_2L_3$) as an independent variable in place of α. If we denote the lowest energy state by $\sigma=0$ and choose our zero of energy appropriately, we have $\epsilon_0=0$ and $(e^\alpha-1)^{-1}=n_0>n_\sigma$ for $\sigma\neq0$. When n_0 is of "order N" or equivalently $\alpha(N,T,L_1,L_2,L_3)$ is of "order N^{-1}" it is customary to say that the system is "condensed." In the thermodynamic limit (N, $L_1,L_2,L_3\to\infty$, with $N/L_1L_2L_3$ finite) this is made more precise by defining Bose condensation as the existence of a nonzero value of n_0/N in this limit. For strictly finite systems, many definitions are possible. For the qualitative discussion in the next paragraph, it is sufficient to use $n_0/N>\frac{1}{2}$ as the condition for condensation.[15] A less restrictive definition will be used in Sec. III.

For a strictly finite geometry, it is quite plausible that as $\alpha\to0, \sum_{\sigma\neq0}n_\sigma$ has an upper bound. This follows if there is a finite separation between adjacent energy levels and a degeneracy which is finite for finite energies and increases with energy less rapidly than exponentially. This is, of course, true for the free-particle gas. Then, for sufficiently large N, the $\sigma=0$ term in Eq. (3) will dominate and $n_0/N>\frac{1}{2}$. Thus the independent-particle Bose gas in any finite geometry at any temperature will be condensed above a finite density. Equivalently, for a finite geometry and any given density, the system will be condensed below a finite temperature. Without further assumptions about ϵ_σ, we are not able to say what the critical temperature is. In fact, it may turn out to be an abnormally low temperature for certain spectra. (See, for example, points (2) and (3) in Sec. III.) For the present investigation, the simple existence of a condensation is the important point because it shows that the general results of Hohenberg and Krueger are not applicable. An obvious possible reason is that they considered infinitely large systems, whereas we are considering strictly finite, though not necessarily small, systems. Since it is usually assumed that large finite systems are well approximated by a corresponding system in the thermodynamic limit, it is worthwhile making sure that there are not other causes for the general results; for example, they may be due to the use of periodic boundary conditions or may be spurious results introduced through the use of the symmetry-breaking technique. We now show that the latter two conjectures are not true for a class of independent-particle systems in partially finite geometries.

In the limit of L going to infinity, the situation is not so simple, because the separation between energy levels is expected to vanish. Then it is possible that $\lim_{L\to\infty}(L_1L_2L_3)^{-1}\sum_{\sigma\neq 0}n_\sigma$ diverges as α vanishes. Basically the problem is how to take the limit in Eq. (3) since the summand depends on L through ϵ_σ and implicitly through $\alpha(N, T, L_1, L_2, L_3)$. The results of deGroot *et al.*[16] are useful at this point.

They considered the independent Bose gas with the following spectra for ϵ_σ:

$$\epsilon_{l_1\cdots l_w}=\frac{h^2}{8m}\left(\frac{l_1{}^\delta-1}{L_1^2}+\cdots+\frac{l_w{}^\delta-1}{L_w{}^2}\right), \qquad (4)$$

where $1\leqslant\delta\leqslant 2$, $l_i=1, 2, 3, \ldots$, and $w=1, 2, 3, \ldots$. For $\delta=2$, this is the spectrum of a free particle in a w-dimensional box with box boundary conditions; for $\delta=1$, it is the harmonic oscillator spectrum. Taking the limit $N, L_1, \ldots L_w\to\infty$ with $\eta=N/(L_1\ldots L_w)^{2/\delta}$ remaining finite, they find:[17]

(a) $w\leqslant\delta$: no condensed state exists

(b) $w>\delta$: (1) condensed state exists

(2) the critical temperature is

$$T_c=(\pi^2\hbar^2/2m\kappa)(\eta/\zeta_{w/\delta})^{\delta/w}[(\delta^{-1})!]^{-\delta}, \qquad (5)$$

where ζ_x is the Riemann ζ function $\sum_{n=1}^\infty n^{-x}$. They define condensation as the existence of a finite value of n_0/N in the infinite-volume limit.

In Appendix A it is shown that the condition for condensation (i. e., $w>\delta$) is valid even for partially finite geometries, i.e., for a spectrum

$$\tilde{\epsilon}_{l_1\cdots l_w, l_{w+1}\cdots l_{w+r}}=\epsilon_{l_1\cdots l_w}+E_{l_{w+1}\cdots l_{w+r}}, \qquad (6)$$

where E has associated lengths $D_1\cdots D_r$ and has (a) a finite separation between adjacent levels, (b) a degeneracy which is finite for finite energy and (c) a degeneracy which increases less than exponentially with energy.

If, instead of fixing η, we fix $\rho=N(L_1\cdots L_w D_1\cdots D_r)^{-1}$ in the infinite-volume limit, we see that $\eta\to 0$ unless $\delta=2$. Eq. (5) implies that a nonzero T_c occurs only when $\delta=2$; i.e., the free particle in a box. In that case, statement (a) is in complete agreement with the $(w+r)$-dimensional generalization of the results of Hohenberg and Krueger. *A priori*, the general proofs of Refs. 8 and 9 only point out some of the regions where condensation can not exist. It is interesting that, for the independent-particle model considered here, the general results delineated all of these regions. Thus it appears that neither the use of periodic boundary conditions nor the use of the symmetry-breaking technique is responsible for the vanishing of the quasi-averages in partially finite geometries.

III. FREE BOSE GAS

In the previous section it was argued that a strictly finite Bose system will condense at some finite temperature. For an ideal Bose gas, we calculate this temperature and its dependence on the size of the box. Comparison with the bulk results[18] are made. Computer calculations[19] have already shown that condensation does occur, but analytic expressions seem desirable. Previous analytic work on finite two- and three-dimensional systems has been carried out by Osborne[20] and Ziman,[21] but it is difficult to estimate the corrections to their results.

The Hamiltonian for a free Bose gas in a box of dimension $L_1\geqslant L_2\geqslant L_3$ is

$$H=\sum_{\vec{k}}(\hbar^2\vec{k}^2/2m)a_{\vec{k}}^+a_{\vec{k}} \qquad (7)$$

$$\vec{k}=2\pi(l_1/L_1, l_2/L_2, l_3/L_3);\quad l_i=0,\pm 1, \pm 2,\ldots, \qquad (8)$$

where periodic boundary conditions have been employed. Defining the thermal wavelength[22] as

$$\lambda_T\equiv[2\pi^2\hbar^2/m\kappa T]^{1/2}, \qquad (9)$$

we have $\beta\epsilon_1=\lambda_T{}^2/L_1{}^2$.

Condensation in a Finite Geometry

As noted earlier, the definition of condensation in finite systems is not unique. The property usually associated with Bose-Einstein condensation is a macroscopic occupation of a single state.[18] For an ideal Bose gas this is the zero-momentum state with occupation number $n_0=(e^\alpha-1)^{-1}$. A large value of n_0 implies $\alpha\ll 1$. Furthermore we expect that $n_0/n_1=(e^\alpha-1)^{-1}[\exp(\beta\epsilon_1+\alpha)-1]\approx 1+\alpha^{-1}\beta\epsilon_1$ is "appreciably" larger than 1, where ϵ_1 is the energy of the first excited state and $\beta\epsilon_1\ll 1$. This ratio n_0/n_1 is much greater than 1 only if $\alpha\ll\beta\epsilon_1$. At this point an arbitrariness enters. How large is "appreciably" larger than one? We arbitrarily take $n_0/n_1=2$ as the point at which condensation begins. This implies that in the condensed state $\alpha\leqslant\alpha_c\equiv\beta\epsilon_1$. From Eq. (3), we define the critical number N_c as

$$N_c(T, L_1, L_2, L_3)\equiv\sum_\sigma\{\exp[\beta(\epsilon_\sigma+\epsilon_1)]-1\}^{-1}. \qquad (10)$$

Since $(\partial N/\partial\alpha)_{T, L_1, L_2, L_3}\leqslant 0$, we see that the system will be condensed if and only if $N\geqslant N_c$.

Also since $(\partial\alpha/\partial T)_{N,L_1,L_2,L_3} \geq 0$, an equivalent definition is $T \leq T_C(N, L_1, L_2, L_3)$, where the critical temperature $T_C = (\kappa\beta_C)^{-1}$ is given by

$$N = \sum_\sigma \{\exp[\beta_c(\epsilon_\sigma + \epsilon_1)] - 1\}^{-1}. \quad (11)$$

This definition is similar to that used by London[15,18] for the bulk system ($L_1 = L_2 = L_3 \to \infty$).

Our definition requires $n_0/n_1 = 2$ at T_C but there is no reason why one could not arbitrarily choose $n_0/n_1 = 10$ or 100 or 1000, etc., as the onset of condensation. In terms of "macroscopic" occupation of a single state, this ambiguity is reflected in the question: For what n is $10^{-n}N$ macroscopic?

The important question is: How is this arbitrariness reflected in the values of the critical temperature? If there is a rapid variation in $\alpha(N, T, L_1, L_2, L_3)$ as a function of T in the region $\alpha \approx \beta\epsilon_1$, then α will take on all values from $\beta\epsilon_1$ to $1000\,\beta\epsilon_1$ in a small temperature interval. Then the critical temperature will be insensitive to the details of the definition. From the calculations of $\alpha(N, T, L_1, L_2, L_3)$ to be described later, we can see that this is the case when $L_1 \approx L_2 \approx L_3 \gg \lambda_T$. At the other extreme, $L_1 \gg \lambda_T > L_2 > L_3$, the variation of α with T is not rapid, and the critical temperature is sensitive to the details of the definition.

To calculate N_C or T_C, we must carry out a sum over k. Because we are primarily interested in $L_i < \infty$, we can *not* make the usual replacement $L^{-1}\sum_k \to \int dk/2\pi$. This difficulty is circumvented by use of a contour integration technique, which is described in Appendix B and which is applied to Eq. (3) in Appendix C. Expressions for N are given for one-, two-, and three-dimensional geometries (i.e., $L_1 \gg \lambda_T \geq L_2 \geq L_3$, $L_1 \geq L_2 \gg \lambda_T \geq L_3$, and $L_1 \geq L_2 \geq L_3 \gg \lambda_T$ respectively[23]), but we will focus our attention on the three-dimensional geometry.

The result for $L_1 \geq L_2 \geq L_3 \gg \lambda_T$ is

$$N = \pi^{3/2}\frac{L_1L_2L_3}{\lambda_T^3}F_{3/2}(\alpha) + 2\pi\frac{L_1L_2}{\lambda_T^2}F_1(2\pi L_3\sqrt{\alpha}/\lambda_T) + \frac{L_1L_2}{\lambda_T^2}G(\alpha L_2^2/\lambda_T^2) + 2\pi\frac{L_1}{\lambda_T\sqrt{\alpha}}[\exp(2\pi L_1 \times\sqrt{\alpha}/\lambda_T) - 1]^{-1} + R_3, \quad (12)$$

where $F_{3/2}$, F_1, G, and R_3 are discussed in Appendix C. This expression simplifies for α/λ_T^2 away from L_1^2, L_2^2, and L_3^2 to give approximately[24]

$$N = \pi^{3/2}\frac{L_1L_2L_3}{\lambda_T^3}F_{3/2}(\alpha); \quad \alpha > \lambda_T^2/L_3^2 \quad (13a)$$

$$N = \pi^{3/2}\frac{L_1L_2L_3}{\lambda_T^3}F_{3/2}(0) + \frac{\pi L_1L_2}{\lambda_T^2}\ln\frac{\lambda_T^2}{\alpha L_3^2}; \quad \frac{\lambda_T^2}{L_3^2} > \alpha > \frac{\lambda_T^2}{L_2^2} \quad (13b)$$

$$N = \pi^{3/2}\frac{L_1L_2L_3}{\lambda_T^3}F_{3/2}(0) + \frac{\pi L_1L_2}{\lambda_T^2}\ln\frac{L_2^2}{L_3^2} + \frac{\pi L_1}{\lambda_T\sqrt{\alpha}}; \quad \frac{\lambda_T^2}{L_2^2} > \alpha > \frac{\lambda_T^2}{L_1^2} \quad (13c)$$

$$N = \pi^{3/2}\frac{L_1L_2L_3}{\lambda_T^3}F_{3/2}(0) + \frac{\pi L_1L_2}{\lambda_T^2}\ln\frac{L_2^2}{L_3^2} + \frac{1}{\alpha}; \quad \frac{\lambda_T^2}{L_1^2} > \alpha. \quad (13d)$$

Equation (13d) implies that in the condensed state $n_0(\approx\alpha^{-1})$ increases linearly with N. Taking $\alpha = \beta\epsilon_1$ and dividing by $L_1L_2L_3$ in Eq. (12) gives the critical density

$$\rho_c \equiv \frac{N_c}{L_1L_2L_3} = \frac{\pi^{3/2}}{\lambda_T^3}F_{3/2}(\lambda_T^2/L_1^2) + \frac{2\pi}{\lambda_T^2L_3} \times F_1(2\pi L_3/L_1) + \frac{1}{\lambda_T^2L_3}G(L_2^2/L_1^2) + \frac{2\pi L_1}{\lambda_T^2L_2L_3} \times (e^{2\pi}-1)^{-1} + \frac{R_3}{L_1L_2L_3}. \quad (14)$$

If one wishes to use a definition of condensation which differs from ours, Eqs. (12), (C20), and (C21) provide a useful starting point.

From Eqs. (12)–(14) we note the following six points.

1. If $L_1 \approx L_3$, the critical temperature (or, alternatively, the critical density) is very close to the bulk critical temperature T_{cB},

$$T_{cB} = (2\pi\hbar^2/m\kappa)[N/L_1L_2L_3F_{3/2}(0)]^{2/3} \quad (15)$$

2. However, if $L_1 \gg L_3$, the F_1 and G terms in Eq. (14) give an upward shift in the critical density, $\rho_C = (L_1L_2L_3)^{-1}N_C$, for a given temperature (alternatively, a downward shift in the critical temperature at a given density). This gives ρ_C a term linear in T in addition to the $T^{3/2}$ term present in the bulk expression.[25]

3. In the extreme situation where $\ln(L_2/L_3) \to \infty$ in the infinite-volume limit, one can verify that condensation does not occur at any finite temperature and density.

4. Until now we have been discussing the critical density as a function of T, but for finite geometries ρ_c depends on L_1, L_2, and L_3 as well. For applications to physical problems, where L_3 is varied, it is interesting to note the critical L_3 as a function of ρ, T, L_1, and L_2. That is, a system can condense only if $L_3 \geq L_{3c}$. From Eq. (14) and analogous results for one- and two-dimensional systems, one finds the necessary and sufficient conditions for condensation (defined as $n_0/n_1 \geq 1+\gamma$).

Three-Dimensional System $L_1 \geq L_2 \geq L_3 \gg \lambda_T$:

$$T \leq T_{cB}$$

$$L_3 \geq L_{3c} = [1-(T/T_{cB})^{3/2}]^{-1} \times \pi^{-1}[F_{3/2}(0)]^{-2/3}\rho^{-1/3}(T/T_{cB}) \times [a(L_1/L_2) + 2\pi \ln(L_2/L_{3c}) + b] \,. \quad (16)$$

Two-Dimensional System $L_1 \geq L_2 \gg \lambda_T \gtrsim L_3$:

$$L_3 \geq L_{3c} = \pi^{-1}[F_{\frac{3}{2}}(0)]^{-2/3}\rho^{-1/3}(T/T_{cB}) \times [a(L_1/L_2) + 2\pi \ln(L_2/\lambda_T) + b]. \quad (17)$$

One-Dimensional System $L_1 \gg \lambda_T \gtrsim L_2 \geq L_3$:

$$L_3 \geq L_{3c} = \pi^{-1}[F_{\frac{3}{2}}(0)]^{-2/3}\rho^{-1/3}(T/T_{cB}) \times [a(L_1/L_2) + b], \quad (18)$$

where $a = \pi\gamma^{-1/2} + 2\pi\gamma^{-1/2}[\exp(2\pi\gamma^{1/2})-1]^{-1}$ and $|b| < 15$.[26] Our definition requires $\gamma = 1$. It is evident from these expressions that no condensation is possible in partially finite geometries (remembering our convention that $L_1 \geq L_2 \geq L_3$).

5. The results in Eq. (13) for $L_1 = L_2 = L_3$ can be compared to the usual result[18] obtained by replacing sums by integrals

$$N \approx n_0 + L^3(2\pi)^{-3} \int d^3k n_{\vec{k}} \equiv \tilde{N}, \quad (19)$$

which gives

$$\tilde{N} = (e^{\alpha}-1)^{-1} + (L^3\pi^{3/2}/\lambda_T^3)F_{3/2}(\alpha) \quad (20)$$

$$\tilde{N} = \alpha^{-1} + (L^3\pi^{3/2}/\lambda_T^3)F_{3/2}(0); \quad \alpha \ll 1. \quad (21)$$

The difference $\tilde{N}-N$ is

$$\tilde{N}-N \approx \alpha^{-1}, \quad 1 \gg \alpha > \lambda_T^2/L^2$$

$$\approx 0, \quad \alpha < \lambda_T^2/L^2,$$

and may become quite large ($\approx L^2/\lambda_T^2 \gg 1$) as the transition temperature is approached from above. However, the fractional error $(\tilde{N}-N)/N$ is less than $\lambda_T/[L\pi^{3/2}F_{3/2}(0)] \ll 1$. Thus, for a cubic geometry the usual replacement of $\sum_{\vec{k}(\neq 0)}$ by an integral is quite good except for corrections close to the transition temperature. It is evident from Eq. (14) that this replacement is less satisfactory if $L_2 \neq L_3$, and actually leads to an incorrect critical density (or temperature) due to the $(2L_1L_2/\lambda_T^2)\ln(L_2/L_3)$ term which is absent from (21). Similar conclusions hold in one- and two-dimensional finite geometries. Then the relevant approximations are

$$\sum_{k_1,k_2} n_{k_1,k_2,0} \approx n_0 + L_1L_2(2\pi)^{-2} \times \int dk_1 dk_2\, n_{k_1,k_2,0}, \quad (22)$$

$$\sum_{k_1} n_{k_1,0,0} \approx n_0 + L_1(2\pi)^{-1} \int dk_1 n_{k_1,0,0}. \quad (23)$$

The fractional errors in the one- and two-dimensional cases are, in general, appreciably larger than in the three-dimensional case. The extreme example is in one dimension where the fractional error $|N-\tilde{N}|/N$ can be as large as 30%.

6. Our results were obtained for periodic boundary conditions, and the question arises as to what differences, if any, would be obtained for box boundary conditions. Using box boundary conditions and $L_1 = L_2$, Ziman[21] has calculated N for $\alpha \ll \lambda_T^2/L_1^2$. Ziman's Eq. (9) may be compared to our Eq. (13d) if we use $\alpha = -\mu_{\text{Ziman}} + 2\alpha_{\text{Ziman}} + \beta_{\text{Ziman}}$ and $\alpha_{\text{Ziman}} = \lambda_T^2/(4L_1^2)$ and $\beta_{\text{Ziman}} = \lambda_T^2/(4L_3^2)$. The difference in densities is

$$\lambda_T^3(\rho_{\text{box}} - \rho_{\text{periodic}}) = \frac{\pi\lambda_T}{L_3}\left(1 + \frac{L_3}{L_1}\right)\ln\frac{\lambda_T}{L_3}\,.$$

Since $\lambda_T \ll L_3$, $L_3 \leq L_1$, and $\rho\lambda_T^3 > 1$, the difference will be small and the fractional difference is small. We know our expression for $\lambda_T^3\rho_{\text{periodic}}$ has errors no greater than order $\lambda_T/L_3 \ll 1$ and $\lambda_T^3\rho_{\text{box}}$ has errors at least of that order so the above difference is meaningless except when $\ln(L_3/\lambda_T) \gg 1$. Unfortunately, because of the truncation of the terms in the Jacobi transformation, it is very difficult to determine the errors in Ziman's calculation; so it is not possible to say whether the above difference is real or due to approximations in the calculation of $\lambda_T^3\rho_{\text{box}}$. Using the contour-integration technique, a precise comparison for a one-dimensional system is possible, and we find[27] for $\alpha \ll \lambda_T^2/L^2 \ll 1$:

$$N_{\text{periodic}} \equiv \sum_{l=-\infty}^{\infty} [\exp(l^2\lambda_T^2/L_p^2+\alpha)-1]^{-1}$$

$$=\frac{1}{\alpha}+\frac{\pi^2 L_p^2}{3\lambda_T^2}-1.460\frac{\sqrt{\pi}L_p}{\lambda_T}+O\left(\frac{\alpha L_p}{\lambda_T}\right)+O\left(\frac{\alpha L_p^4}{\lambda_T^4}\right) \quad (24)$$

$$N_{\text{box}} \equiv \sum_{l=1}^{\infty} \{\exp[(l^2-1)\lambda_T^2/(4L_B^2)+\alpha]-1\}^{-1}$$

$$=\frac{1}{\alpha}+\frac{3L_B^2}{\lambda_T^2}-1.460\frac{\sqrt{\pi}L_B}{\lambda_T}+O(1)+O\left(\frac{\alpha L_B^4}{\lambda_T^4}\right). \quad (25)$$

For equality of the number of particles outside the condensate to leading order in L/λ_T, we must have $3L_{\text{box}}=\pi L_{\text{periodic}}$. Thus changing the boundary conditions does have a quantitative effect, but the qualitative effect is unaltered, i.e., condensation does occur for both periodic and box boundary conditions.

Long-Range Order in a Partially Finite Geometry

Even though there is no condensation in partially finite geometries, we now show that there is still a strong similarity between slab and bulk geometries. In a three-dimensional slab (i.e., L_1, $L_2 \to \infty$ with $\infty > L_3 \gg \lambda_T$), there is a fairly sharp transition in the range of the off-diagonal order in the reduced density matrix as the temperature is lowered through the bulk critical temperature.[28]

From Eq. (12) we know that $\lim_{L_1,L_2\to\infty}\alpha(\rho, L_1, L_2, L_3, T)$ is finite, so there is no difficulty in taking the $L_1, L_2 \to \infty$ limit in the reduced density matrix, $\langle\psi^\dagger(\vec{x})\psi(0)\rangle$, for a slab

$$\langle\psi^\dagger(\vec{x})\psi(0)\rangle=\frac{1}{L_3}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_2}{2\pi}\sum_{k_3}\frac{e^{-i\vec{k}\cdot\vec{x}}}{e^{\beta\vec{k}^2/2m+\alpha}-1}. \quad (26)$$

Since Eq. (26) is a Fourier transform we can obtain the range of $\langle\psi^\dagger(\vec{x})\psi(0)\rangle$ from the uncertainty relation $\Delta k\,\Delta x\approx\hbar$.[29] Taking Δk as the half width of n_k at half maximum we find $\beta(\Delta k)^2/2m=\alpha=\beta|\mu|$. Thus the range r is given by

$$r=\Delta x\approx\hbar/\Delta k=\hbar(\beta/2m\alpha)^{1/2}=\lambda_T/(2\pi\sqrt{\alpha}). \quad (27)$$

Note that r decreases monotonically as T increases at fixed (ρ, L_3).

Expressions for $\alpha(\rho, T, L_3)$ may be obtained from Eq. (12) where the third and fourth terms vanish in the limit $L_1, L_2\to\infty$ with ρ finite. The result is

$$\lambda_T^3\rho=\pi^{3/2}F_{3/2}(\alpha)+(2\pi\lambda_T/L_3)F_1(2\pi L_3\sqrt{\alpha}/\lambda_T)+O[(\lambda_T/L_3)\exp(-\pi^{3/2}L_3/2\lambda_T)]. \quad (28)$$

Since $\lambda_T \ll L_3$, the correction term will be dropped. For $\alpha\ll\lambda_T^2/L_3^2$, this reduces to

$$\lambda_T^3\rho=\pi^{3/2}F_{3/2}(0)-(\pi\lambda_T/L_3)\ln(\alpha L_3^2/\lambda_T^2)$$

which shows that α may become exponentially small but remains finite at finite (ρ, T, L_3).

First consider temperatures greater than the bulk critical temperature given by Eq. (15). If $0.5\geq(T-T_{cB})/T_{cB}\geq\lambda_T/L_3$ we obtain α from Eq. (28) by dropping the F_1 term and keeping just the first two terms in the expansion (C17) of $F_{3/2}(\alpha)$. With an error of less than 5% we have

$$r=\tfrac{2}{3}(\lambda_T/\sqrt{\pi})T_{cB}/(T-T_{cB}). \quad (29a)$$

Next consider temperatures less than the bulk critical temperature. If $1\geq(T_{cB}-T)/T_{cB}\geq\lambda_T/L_3$, we obtain α from Eq. (28) by taking $F_{3/2}(\alpha)=F_{3/2}(0)=\zeta_{3/2}$ and keeping just the first term in the expansion (C15) of F_1. With an error of less than 1% we have

$$r=L_3\exp\{\zeta_{3/2}(\sqrt{\pi}/2)(L_3/\lambda_T)\times[(T_{cB}/T)^{3/2}-1]\}. \quad (29b)$$

Thus, for $T<0.9T_{cB}$, the range is exponentially larger than L_3. Since lengths in the L_3 direction are modulo L_3, we see that there is a spatial asymmetry in the range of the reduced density matrix below the bulk critical temperature.

In Appendix D we find explicit expressions if $T<T_{cB}$, $L_3\gg\lambda_T[(T_{cB}/T)^{3/2}-1]^{-1}$, and $\vec{x}=(x_1, x_2, 0)$,

$$\lambda_T^3\langle\psi^\dagger(\vec{x})\psi(0)\rangle\approx\lambda_T^3\rho>1;\quad x<\lambda_T \quad (30)$$

$$=\lambda_T^3\rho\left[1-\left(\frac{T}{T_{cB}}\right)^{3/2}\right]+2\pi\frac{\lambda_T}{L_3}\ln\frac{L_3}{x};\quad \lambda_T\left[\left(\frac{T_{cB}}{T}\right)^{3/2}-1\right]^{-1}\ll x\ll r \quad (31)$$

$$=(\lambda_T/L_3)\{r/2\pi x\}^{1/2}e^{-x/r};\quad x>r, \quad (32)$$

where $x=|\vec{x}|$. It is interesting to note that the first term in the expression for $\lambda[(T_{cB}/T)^{3/2}-1]^{-1}\ll x\ll r$ is the same as that due to the $k=0$ term in the bulk result.

It should be emphasized that this extremely long-but finite-range order occurs even though there is neither the usual Bose condensation nor a generalized Bose condensation.[14]

From calculations of the specific heat and PvT diagrams for a three-dimensional slab, we find that the onset of this extremely long-range order is reflected in a manner very similar to the onset of true ODLRO in the bulk system. Our results for C_v are qualitatively similar[30] to those of Goble and Trainor who used box boundary conditions. Relative to the bulk results, the two basic effects of reducing L_3/λ_T are to shift the peak in C_v to higher temperatures and to broaden the peak. The PvT diagrams show little deviation from the bulk result. For $1/\rho \equiv v < v_{cB} \equiv 1/\rho_{cB}$, the bulk pressure is independent of v. For the slab geometry, there is a very slight dependence of P on v for $v < v_{cB}$, which may be characterized by the derivative $(\partial P/\partial v)_T$,

$$\left(\frac{\partial P}{\partial v}\right)_{T, v=v_-} \Big/ \left(\frac{\partial P}{\partial v}\right)_{T, v=v_+} = \frac{L_3}{\lambda_T} \frac{v_c v_+^2}{v_-^3} \times \frac{2\pi^{3/2}}{\zeta_{3/2}} \exp\left[\frac{-L_3}{\lambda_T} \frac{\sqrt{\pi}\zeta_{3/2}}{2}\left(\frac{v_{cB}}{v_-} - 1\right)\right], \quad (33)$$

with $v_+ > v_{cB}$, $v_- < v_{cB}$, and v_{cB} the bulk critical volume per particle. Since $L_3 \gg \lambda_T$, this ratio is very small. The pressure for $v > v_{cB}$ is also very close to the bulk value. Thus the onset of the long- but finite-range order in a three-dimensional slab is reflected in C_v and PvT diagrams in a way very similar to that for the onset of true ODLRO. Further, it is conceivable that the theoretical predictions for the slab would be the same as those for the bulk or the strictly finite geometry calculation within some small error; and for comparison with experiment the three theoretical models would be equivalent. It is in this spirit that one would say that the onset of extremely long- but finite-range order in a partially finite geometry is equivalent to condensation.

If the thickness of the slab is decreased sufficiently ($L_3 \lesssim \lambda_T$), there are qualitative differences between the slab and bulk results. The effects are: an increase in the range of the reduced density matrix compared to the three-dimensional slab,[31] an increase in the range in temperatures over which the transition from short- to long-range order occurs, the complete disappearance of the specific-heat peak, and a large increase of the slab pressure over the bulk pressure for a given T and $v < v_{cB}$. Since $L_3 \lesssim \lambda_T$ implies that motion in one direction is essentially frozen out, it is not surprising to find deviations from the bulk results.

IV. DISCUSSION AND CONCLUSIONS

In an attempt to understand recent general proofs of the vanishing of quasi-averages usually associated with superconductivity and superfluidity in partially finite geometries, we have done three things. First, we have tested two of the assumptions of the general proofs, namely the use of periodic boundary conditions and the symmetry-breaking technique, and found that they were not crucial in an independent Bose-gas model. Second, we found the necessary and sufficient conditions for condensation in a strictly finite ideal Bose gas. These turn out to be the same conditions as for the bulk system, if the dimensions of the finite system satisfy $L_1 \approx L_2 \approx L_3 \gg \lambda_T$. And third, we have shown that, even though a partially finite ideal Bose system will not condense, it does undergo a transition from a short-range order ($\approx \lambda_T$) to an extremely long- but finite-range order ($\approx L_3 \exp\{(\zeta_{3/2}\sqrt{\pi}\, L_3/2\lambda_T)[(T_{cB}/T)^{3/2}-1]\}$ with $L_3 \gg \lambda_T$) which might be "experimentally indistinguishable" from a condensation in the bulk system. This last possibility suggests that it may be useful to relax the criterion for superfluidity or superconductivity.

Even though most of the results have been obtained only for the ideal Bose gas, one might conjecture that the qualitative features have their foundation in the existence of a long-range order and are not simply reflections of the idiosyncrasies of the ideal gas. At present, the validity of this conjecture remains an open question.

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APPENDIX A

In this appendix we show that the results of deGroot *et al.*[16] may be extended to partially finite geometries (i.e., $L_1 \times \cdots \times L_w \times D_1 \times \cdots \times D_\gamma$ with $L_i \to \infty$ and $D_j < \infty$). Consider the independent Bose particle energies to be a sum $\epsilon_\sigma + E_{\sigma'}$, where ϵ_σ is the energy given in Eq. (4) (with associated lengths $L_1 \cdots L_w$) and $E_{\sigma'}$ is the energy associated with excitations in the finite directions (with associated lengths $D_1 \cdots D_2$). Actually all we need assume is that the levels $E_{\sigma'}$ have a finite separation and a degeneracy which is finite at finite energies and increases less rapidly than exponentially with energy. The number of particles is then

$$N = \sum_{\sigma,\sigma'} \{\exp[\beta(\epsilon_\sigma + E_{\sigma'}) + \alpha] - 1\}^{-1}. \quad \text{(A1)}$$

Now $\epsilon_0 = 0$, and, by a suitable shift in α, we may take $E_0 = 0$ also. Defining the quantities $\eta_A(z)$ and $\eta_B(\alpha)$

$$\eta_A(z) \equiv (L_1 \cdots L_w)^{-2/\delta} \sum_\sigma (e^{\beta\epsilon_\sigma + z} - 1)^{-1} \quad \text{(A2)}$$

$$\eta_B(\alpha) \equiv \textstyle\sum_{\sigma>0} \eta_A(\alpha+\beta E_\sigma) \tag{A3}$$

we have $\eta \equiv N(L_1\cdots L_w)^{-2/\delta} = \eta_A(\alpha)+\eta_B(\alpha).$ (A4)

We will consider the limit $N, L_1\cdots L_w \to \infty$ with η finite. Our $\eta_A(\alpha)$ was called η by deGroot *ét al.* It is convenient to study the condensation by increasing the density at fixed temperature. There are two cases to consider.

For $w \leq \delta$, it was shown in Ref. 16 that $\eta_A(\alpha)$ may become arbitrarily large without condensation occurring (i.e., $N\to\infty$ but $n_0/N \to 0$). Since $\eta_B(\alpha) \geq 0$, we see that η may become arbitrarily large without condensation taking place.

For $w > \delta$, it was shown in Ref. 16 that condensation is necessary for $\eta_A(\alpha)$ greater than some finite value. From the assumed properties of E_σ we know $\eta_B(\alpha)$ has a finite upper bound for $0 \leq \alpha \leq \infty$. By increasing η, we may force $\eta_A(\alpha)$ to take on arbitrarily large values from which condensation follows. The critical "density" will be increased and is given by

$$\eta_c = [(\delta^{-1})!]^w (2m\kappa T/\pi^2\hbar^2)^{w/\delta}\zeta_{w/\delta} + \eta_B(0). \tag{A5}$$

Since $\eta_B(0)$ depends on temperature, it is not possible, in general, to invert this equation to find T_c in terms of η; but it is clear that the addition of $\eta_B(0)$ will decrease the critical temperature. As in the system studied in Ref. 16, if $N(L_1\cdots L_w D_1\cdots D_r)^{-1}$ is held finite as $L_i\to\infty$, then $T_c\to 0$ unless $\delta=2$.

APPENDIX B

A contour-integration technique for obtaining sums in terms of integrals plus corrections is demonstrated in this appendix. As an example, we will consider

$$S = \sum_{l=-\infty}^{\infty} (e^{\beta' l^2+\alpha}-1)^{-1} = \sum_{l=-\infty}^{\infty} f(l), \tag{B1}$$

where $\beta' \ll 1$ and $\alpha > 0$. Consider the integral

$$I = \int_c dz f(z) = -\int_c dz f(z)/(e^{-2\pi i z}-1) - \int_c dz f(z)/(e^{2\pi i z}-1), \tag{B2}$$

where the contour c extends from $-\infty+i\epsilon$ to $+\infty+i\epsilon$ and where ϵ is arbitrarily small but greater than zero. The integrands have poles at all integers ($z=0, \pm1, \pm2, \ldots$) due to the $(e^{\pm 2\pi i z}-1)^{-1}$ factors. These factors also give an exponential damping away from the real z axis and allow us to close the contour in the upper (lower) half plane for the first (second) term in equation (B2). Using Cauchy's theorem and denoting the poles of $f(z)$ in the upper (lower) half plane by $z_u(z_L)$ we have

$$\sum_{l=-\infty}^{\infty} f(l) = \int_{-\infty}^{\infty} dl\, f(1) + 2\pi i \sum_{z_u} (e^{-2\pi i z_u}-1)^{-1}[\text{Residue } f(z_u)] - 2\pi i \sum_{z_L}[e^{2\pi i z_L}-1]^{-1}[\text{Residue } f(z_L)]. \tag{B3}$$

For our example, this is explicitly

$$\sum_{l=-\infty}^{\infty}[e^{\beta' l^2+\alpha}-1]^{-1} = \int_{-\infty}^{\infty} dl\,[e^{\beta' l^2+\alpha}-1]^{-1} + \frac{\pi i}{\beta'}\sum_{\substack{m=-\infty\\(y_m>0)}}^{\infty}\frac{1}{z_m}\frac{1}{e^{-2\pi i z_m}-1} - \frac{\pi i}{\beta'}\sum_{\substack{m=-\infty\\(y_m<0)}}^{\infty}\frac{1}{z_m}\frac{1}{e^{2\pi i z_m}-1}, \tag{B4}$$

where $z_m = x_m + i y_m$ and

$$x_m = \pi m\sqrt{2/(\alpha\beta')}\,[1+(1+4\pi^2 m^2/\alpha^2)^{1/2}]^{-1/2} \tag{B5}$$

$$y_m = \pm\sqrt{\alpha/(2\beta')}\,[1+(1+4\pi^2m^2/\alpha^2)^{1/2}]^{1/2}. \tag{B6}$$

From (B6) it is evident that $|y_m| > \sqrt{\pi |m|/\beta'}$. Since $\beta' \ll 1$ we have $|y_m| \gg 1$ for $m \neq 0$. This implies that the $m \neq 0$ terms in (B4) are of order $\exp(-2\pi^{3/2}\sqrt{|m|/\beta'}) \ll 1$. Since this decreases exponentially with $\sqrt{|m|}$, it follows that $\sum_{m=1}^{\infty}$ is of the same order. Thus, in practice, keeping only the $m=0$ pole in (B4) introduces only exponentially small corrections. This is the essential simplification for double and triple sums. For other sums, slightly different contours must be used. It should be pointed out also that for other $\epsilon_{\vec{k}}$ the method may be more complicated and may be of no value. For example, if $\epsilon_{\vec{k}} \sim \sqrt{k^2}$, branch points in $f(z)$ appear instead of simple poles as before.

APPENDIX C

This appendix is devoted to the calculation of sums using the contour-integral technique discussed in Appendix B. The number of particles in a box $(L_1 \geq L_2 \geq L_3)$ is given by

$$N=\sum_{\vec{k}}(e^{\beta\epsilon_{\vec{k}}+\alpha}-1)^{-1}, \tag{C1}$$

with $\alpha > 0$ and $\vec{k}=2\pi(l_1/L_1, l_2/L_2, l_3/L_3)$ so $\beta\epsilon_{\vec{k}}=(\lambda_T/L_1)^2 l_1{}^2+(\lambda_T/L_2)^2 l_2{}^2+(\lambda_T/L_3)^2 l_3{}^2$. Using periodic boundary conditions, we have $l_i=0, \pm 1, \pm 2 \cdots$. Equation (B4) may be written schematically as

$$\sum_{l_1}=\int_{l_1}+(\text{first pole})_{l_1}+(\text{higher poles})_{l_1}, \tag{C2}$$

where (first pole) is the $m=0$ contribution and (higher poles) is the $m \neq 0$ contribution. To separate out the dominant terms, it is convenient to do the sums in order of decreasing L_i. Schematically we have

$$\sum_{\vec{k}}=\sum_{l_2,l_3}\Big[\int_{l_1}+(\text{first pole})_{l_1}+(\text{higher poles})_{l_1}\Big] \tag{C3}$$

$$\sum_{\vec{k}}=\sum_{l_3}\Big[\int_{l_1 l_2}+\int_{l_1}(\text{first pole})_{l_2}+\int_{l_1}(\text{higher poles})_{l_2}\Big]+\sum_{l_2,l_3}\Big[(\text{first pole})_{l_1}+(\text{higher poles})_{l_1}\Big] \tag{C4}$$

$$\sum_{\vec{k}}=\int_{l_1 l_2 l_3}+\int_{l_1 l_2}(\text{first pole})_{l_3}+\int_{l_1 l_2}(\text{higher poles})_{l_3}+\sum_{l_3}\int_{l_1}(\text{first poles})_{l_2}+\sum_{l_3}\int_{l_1}(\text{higher poles})_{l_2}$$
$$+\sum_{l_2,l_3}\Big[(\text{first pole})_{l_1}+(\text{higher poles})_{l_1}\Big] \tag{C5}$$

In all cases the (higher poles) terms contribute very little. Also $\sum_{l_2,l_3}$ (first pole)$_{l_1} \approx$ (first pole)$_{l_1}$, $l_2=0=l_3$ with small error if $L_1 \gg \lambda_T$. If $L_i \leq \lambda_T$, then $\sum_{l_i \neq 0}$ will be small also. Finally, $\sum_{l_3 \neq 0}\int_{l_1}$(first pole)$_{l_2}$ is small if $L_3 \gg \lambda_T$. Of course all comparisons are made with respect to the other terms in the expression.

Equations (C3), (C4), and (C5) are used for the one- , two- , and three-dimensional geometries, respectively. The small terms pointed out above are included in a remainder R_i and rigorous upper bounds are given. The results are given below, where λ_T is the thermal wavelength $[\lambda_T{}^2=h^2/(2m\kappa T)]$ and the terms retained are in the same order as in equations (C3)–(C5).

$L_1 \gg \lambda_T \gtrsim L_2 \gtrsim L_3$ (One-Dimensional Geometry):

$$N=L_1(2\pi)^{-1}\int_{-\infty}^{\infty}dk_1[\exp(\beta k_1{}^2/2m+\alpha)-1]^{-1}+(L_1/\lambda_T)(2\pi/\sqrt{\alpha})[\exp(2\pi L_1\sqrt{\alpha}/\lambda_T)-1]^{-1}+R_1. \tag{C6}$$

$$|R_1|<(L_1/\lambda_T)\,10^2[\exp(-\lambda_T{}^2/L_2{}^2)+40\exp(-L_1\sqrt{\pi}/2\lambda_T)]. \tag{C7}$$

$L_1 \geq L_2 \gg \lambda_T \gtrsim L_3$ (Two-Dimensional Geometry):

$$N=L_1L_2(2\pi)^{-2}\int_{-\infty}^{\infty}dk_1\int_{-\infty}^{\infty}dk_2\{\exp[\beta(k_1{}^2+k_2{}^2)/2m+\alpha]-1\}^{-1}+(L_1L_2/\lambda_T)\int_{-\infty}^{\infty}dk_1(\alpha+\beta k_1{}^2/2m)^{-1/2}$$
$$\times\{\exp[2\pi L_2(\alpha+\beta k_1{}^2/2m)^{1/2}/\lambda_T]-1\}^{-1}+(L_1/\lambda_T)(2\pi/\sqrt{\alpha})[\exp(2\pi L_1\sqrt{\alpha}/\lambda_T)-1]^{-1}+R_2. \tag{C8}$$

$$|R_2|<(L_1L_2/\lambda_T{}^2)\{8\pi\exp(-\lambda_T{}^2/L_3{}^2)+48\exp(-\pi L_2/L_3)+3000(\lambda_T/L_2)\exp(-\pi^{3/2}L_2/\lambda_T)+13{,}000\,(\lambda_T/L_2)$$
$$\times\exp(-\sqrt{\pi}L_1/2\lambda_T)+48\pi\exp[-\pi(L_1/L_2+L_1\sqrt{2\alpha}/\lambda_T)]\}. \tag{C9}$$

$L_1 \geq L_2 \geq L_3 \gg \lambda_T$ (Three-Dimensional Geometry):

$$N = L_1 L_2 L_3 (2\pi)^{-3} \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} dk_3 \{\exp[\beta(k_1^2 + k_2^2 + k_3^2)/2m + \alpha] - 1\}^{-1} + L_1 L_2 L_3/\lambda_T)(2\pi)^{-1}$$

$$\times \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 [\alpha + \beta(k_1^2 + k_2^2)/2m]^{-1/2} (\exp\{2\pi L_3[\alpha + \beta(k_1^2 + k_2^2)/2m]^{1/2}/\lambda_T\} - 1)^{-1} + (L_1 L_2/\lambda_T)$$

$$\times \int_{-\infty}^{\infty} dk_1 [\alpha + \beta k_1^2/2m]^{-1/2} \{\exp[2\pi L_2(\alpha + \beta k_1^2/2m)^{1/2}/\lambda_T] - 1\}^{-1} + (L_1/\lambda_T)(2\pi/\sqrt{\alpha})$$

$$\times \{\exp[2\pi L_1 \sqrt{\alpha}/\lambda_T] - 1\}^{-1} + R_3. \qquad \text{(C10)}$$

$$|R_3| < (32\pi L_1 L_2/\lambda_T^2)\{\exp[-\pi(L_2^2/L_3^2 + 2\alpha L_2^2/\lambda_T^2)^{1/2}] + \exp[-\pi(L_1^2/L_2^2 + 2\alpha L_1^2/\lambda_T^2)^{1/2}] + (200\lambda_T/L_3)$$

$$\times \exp(-\pi^{3/2} L_3/2\lambda_T) + \tfrac{1}{2}(L_3/L_2)\exp[-\pi(L_1^2/L_3^2 + \alpha L_1^2/\lambda_T^2)^{1/2}]\}. \qquad \text{(C11)}$$

These expressions are useful for $0 < \alpha \leq 1$. In obtaining the bounds on R_i the following elementary inequalities were useful:

(a) $(u+v)^{1/2} \geq \frac{1}{2}(\sqrt{u} + \sqrt{v})$ for $u \geq 0$ and $v \geq 0$,

(b) $(e^x - 1)^{-1} \leq 2e^{-x}$ for $x \geq 1$,

(c) $|\operatorname{Im}[z^{-1}(e^{-2\pi i z} - 1)^{-1}]| \leq 6y^{-1}e^{-2\pi y}$ for $z = x + iy$ and $y \geq 1$, and

(d) $\sum_{l=1}^{\infty} e^{-\sigma l^m} \leq e^{-\sigma} + \int_1^{\infty} dl e^{-\sigma l^m} \leq 5e^{-\sigma}$ for $\sigma \geq 1$ and $m \geq \frac{1}{2}$.

The inequalities in (a)–(b) could be strengthened and the bounds on R_i could be reduced, but the present bounds are sufficient to show that R_i is negligible compared to the other terms in almost all applications.[32]
The first integrals appearing in Eqs. (C6), (C8), and (C10) are Bose integrals:[33]

$$(2\pi)^{-1} \int_{-\infty}^{\infty} dk_1 [\exp(\beta k_1^2/2m + \alpha) - 1]^{-1} = (\sqrt{\pi}/\lambda_T) F_{1/2}(\alpha). \qquad \text{(C12)}$$

$$F_{1/2}(\alpha) = (\pi/\alpha)^{\frac{1}{2}} - 1.460 + 0.208\alpha + 0(\alpha^2); \quad \alpha < 1. \qquad \text{(C13)}$$

$$(2\pi)^{-2} \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \{\exp[\beta(k_1^2 + k_2^2)/2m + \alpha] - 1\}^{-1} = (\pi/\lambda_T^2) F_1(\alpha). \qquad \text{(C14)}$$

$$F_1(\alpha) = -\ln(1 - e^{-\alpha}). \qquad \text{(C15)}$$

$$(2\pi)^{-3} \int d^3k [\exp(\beta \vec{k}^2/2m + \alpha) - 1]^{-1} = (\pi^{3/2}/\lambda_T^3) F_{3/2}(\alpha). \qquad \text{(C16)}$$

$$F_{3/2}(\alpha) = 2.612 - 3.545\sqrt{\alpha} + 1.460\alpha + 0(\alpha^2); \quad \alpha < 1. \qquad \text{(C17)}$$

The remaining integral in Eq. (C8) is given by

$$\lambda_T^{-1} G(\alpha L_2^2/\lambda_T^2) \equiv \int_{-\infty}^{\infty} dk [\alpha + \beta k^2/2m]^{-1/2} \{\exp[2\pi L_2(\alpha + \beta k^2/2m)^{1/2}/\lambda_T] - 1\}^{-1} \qquad \text{(C18a)}$$

$$= 4\pi\lambda_T^{-1} \sum_{l=1}^{\infty} K_0(l 2\pi L_2 \sqrt{\alpha}/\lambda_T) \qquad \text{(C18b)}$$

$$= \pi/L_2\sqrt{\alpha} + (\pi/\lambda_T)\ln(\alpha L_2^2/\lambda_T^2) - 0.7286/\lambda_T + 0(\sqrt{\alpha}\, L_2/\lambda_T^2). \qquad \text{(C18c)}$$

At $\alpha = \lambda_T^2/4\pi^2 L_2^2$, expansion (C18c) is in error by less than 2%. For α greater than this, the sum in (C18b) is useful. The remaining integral in Eq. (C10) may be done exactly:

$$(2\pi)^{-2} \int_{-\infty}^{\infty} dk_1 \int dk_2 z^{-1} [\exp(2\pi L_3 z/\lambda_T) - 1]^{-1} = -(\lambda_T L_3)^{-1} \ln[1 - \exp(-2\pi L_3 \sqrt{\alpha}/\lambda_T)]$$

$$= (\lambda_T L_3)^{-1} F_1(2\pi L_3 \sqrt{\alpha}/\lambda_T), \qquad \text{(C19)}$$

where $z=[\alpha+\beta(k_1^2+k_2^2)/2m]^{1/2}$. Combining the above for $L_1 \geq L_2 \geq L_3 \gg \lambda_T$, one obtains Eq. (12) for N. The resulting expressions for two- and one-dimensional systems are

$L_1 \geq L_2 \gg \lambda_T > L_3$ (Two-Dimensional):

$$N=(\pi L_1 L_2/\lambda_T^2)F_1(\alpha)+(L_1 L_2/\lambda_T^2)G(\alpha L_2^2/\lambda_T^2)+(2\pi L_1/\lambda_T\sqrt{\alpha})[\exp(2\pi L_1\sqrt{\alpha}/\lambda_T)-1]^{-1}+R_2. \quad \text{(C20)}$$

$L_1 \gg \lambda_T > L_2 \geq L_3$ (One-Dimensional):

$$N=\sqrt{\pi}\,(L_1/\lambda_T)F_{1/2}(\alpha)+(2\pi L_1/\lambda_T\sqrt{\alpha})[\exp(2\pi L_1\sqrt{\alpha}/\lambda_T)-1]^{-1}+R_1. \quad \text{(C21)}$$

The contour-integral technique has also been used to calculate the thermodynamic potential and number density for a one-dimensional system governed by the Hamiltonian

$$H=\sum_k (k-eA/c)^2 a_k^+ a_k. \quad \text{(C22)}$$

This Hamiltonian arises when considering an ideal gas confined to a ring geometry ($L\times L\times D$ with $L\gg\lambda_T\gg D$) in the presence of an external magnetic field. The thermodynamic potential is given by

$$\beta\Omega=-\ln(\mathrm{Tr}\,e^{-\beta H-\alpha N})=\sum_{l=-\infty}^{\infty}\ln\{1-\exp[-\lambda_T^2(l-\phi)^2/L^2-\alpha]\}$$

$$\beta\Omega=\ln|4\sin\pi(\phi-\xi)\sin\pi(\phi+\xi)|+P\int_{-\infty}^{\infty}dz\,\ln|1-\exp(-\lambda_T^2 z^2/L^2-\alpha)|+O(e^{-L/\lambda_T});$$

$$\alpha<0 \text{ and } -\tfrac{1}{2}\leq\phi\leq\tfrac{1}{2}, \quad \text{(C23)}$$

where $\xi\equiv L\sqrt{|\alpha|}/\lambda_T$, $\phi=eAL/2\pi c$, and P denotes the Cauchy principal-value integral. The number of particles is given by

$$N=\sum_{l=-\infty}^{\infty}\{\exp[\lambda_T^2(l-\phi)^2/L^2+\alpha]-1\}^{-1}$$

$$N=(\pi L^2/2\lambda_T^2)\xi^{-1}[\cot\pi(\phi-\xi)-\cot\pi(\phi+\xi)]+P\int_{-\infty}^{\infty}dz[\exp(\lambda_T^2 z^2/L^2+\alpha)-1]^{-1}+O(e^{-L/\lambda_T});$$

$$\alpha<0 \text{ and } -\tfrac{1}{2}\leq\phi\leq\tfrac{1}{2}. \quad \text{(C24)}$$

For $\alpha\geq 0$ and $-\frac{1}{2}\leq\phi\leq\frac{1}{2}$, we find

$$\beta\Omega=\int_{-\infty}^{\infty}dz\,\ln[1-\exp(-\lambda_T^2 z^2/L^2-\alpha)]+\ln[1+\exp(-4\pi\xi)-2\exp(-2\pi\xi)\cos 2\pi\phi]+O(e^{-L/\lambda_T}) \quad \text{(C25)}$$

$$N=\int_{-\infty}^{\infty}dz[\exp(\lambda_T^2 z^2/L^2+\alpha)-1]^{-1}-(2\pi L/\lambda_T\sqrt{\alpha})[\exp(2\pi\xi)\cos\pi\phi-1]/[\exp(4\pi\xi)-2\exp(2\pi\xi)\cos\pi\phi+1]$$

$$+O[\exp(-L/\lambda_T)]. \quad \text{(C26)}$$

Expansions for integrals in (C23)-(C26) have been given by Dingle.[33] Due to the branch points in the summand in Eq. (C23) and the pole on the real z axis in the summand in Eq. (C24), the contour has been chosen differently from that used in the previous calculation of N. However the technique is basically the same as before.

APPENDIX D

The reduced density matrix $\langle\psi^\dagger(\vec{x})\psi(0)\rangle$ for an ideal Bose gas in a slab (with dimensions $\infty\times\infty\times L_3$ with $L_3\gg\lambda_T$) is evaluated in this appendix. The starting point is Eq. (26).

$$\langle\psi^\dagger(\vec{x})\psi(0)\rangle=(1/4\pi^2 L_3)\int_{-\infty}^{\infty}dk_1\int dk_2\, e^{-i\vec{k}\cdot\vec{x}}\sum_{k_3}[\exp(\beta\vec{k}^2/2m+\alpha)-1]^{-1}, \quad \text{(26)}$$

where $\vec{x}=(x_1,x_2,0)$. The sum on k_3 is done using Eq. (B4). We drop the $m\neq 0$ pole terms which introduces an error of order $\exp(-L_3/\lambda_T)\ll 1$ to obtain

$$\langle\psi^\dagger(\vec{x})\psi(0)\rangle = A + B \quad \text{(D1)}$$

$$A = (2\pi)^{-3}\int d^3k e^{-i\vec{k}\cdot\vec{x}}[\exp(\beta\vec{k}^2/2m+\alpha)-1]^{-1} \quad \text{(D2)}$$

$$B = (2\pi\lambda_T)^{-1}\int d^2k e^{-i\vec{k}\cdot\vec{x}}(\beta\vec{k}^2/2m+\alpha)^{-1/2}\{\exp[2\pi L_3(\beta\vec{k}^2/2m+\alpha)^{1/2}/\lambda_T]-1\}^{-1}. \quad \text{(D3)}$$

Doing the angular integrations and using Cauchy's theorem for the radial integral provides a straightforward evaluation of A

$$A = (\pi/\lambda_T^2 x)\{\exp(-2\pi x\alpha^{1/2}/\lambda_T) + O[\exp(-2\pi x/\lambda_T)]\}. \quad \text{(D4)}$$

Since we are primarily interested in $x \equiv |\vec{x}| > \lambda_T$ and $\alpha \ll 1$, the last term in (D4) is dropped.

In B the angular integration introduces the cylindrical Bessel function $J_0(kx)$. Using $(e^x-1)^{-1} = \sum_{l=1}^\infty e^{-lx}$ and[34]

$$\int_0^\infty z\,dz(z^2+b)^{-1/2}J_0(zy)\exp[-2\pi l(z^2+b)^{1/2}] = (y^2+4\pi^2l^2)^{-1/2}\exp[-(by^2+4\pi^2l^2b)^{1/2}],$$

we obtain $B = (2\pi/\lambda_T^2)\sum_{l=1}^\infty(x^2+L_3^2l^2)^{-1/2}\exp[-2\pi\alpha^{1/2}(x^2+l^2L_3^2)^{1/2}/\lambda_T] \geq 0.$ (D5)

If $f(l) \geq 0$ and $df/dl \leq 0$ for $0 \leq l \leq \infty$, we have

$$\int_0^\infty f(l)dl \geq \sum_{l=1}^\infty f(l) \geq \int_0^\infty f(l)dl - f(0). \quad \text{(D6)}$$

Evaluation of the integral in (D6) as applied to (D5) is done by using[35]

$$\int_1^\infty dz(z(z-1))^{-1/2}e^{-yz^{1/2}} = 2K_0(y). \quad \text{(D7)}$$

$K_0(y)$ is the modified Bessel function of the second kind and has the following asymptotic behavior

$$K_0(y) \sim -\ln y, \quad |y| \ll 1; \quad K_0(y) \sim (\pi/2y)^{1/2}e^{-y}, \quad |y| \gg 1.$$

Combining (D5)–(D7), we have

$$-(2\pi/\lambda_T^2 x)e^{-x/r} + (2\pi/\lambda_T^2L_3)K_0(x/r) \leq B \leq (2\pi/\lambda_T^2L_3)K_0(x/r), \quad \text{(D8)}$$

where $r = \lambda_T/(2\pi\alpha^{1/2})$. Using (D1)–(D4) and (D8), we have

$$(2\pi/\lambda_T^2L_3)K_0(x/r) - A \leq \langle\psi^\dagger(\vec{x})\psi(0)\rangle \leq (2\pi/\lambda_T^2L_3)K_0(x/r) + A. \quad \text{(D9)}$$

For $x \gg r$, this implies an exponential decrease in $\langle\psi^\dagger(\vec{x})\psi(0)\rangle$ with a characteristic falloff distance r in agreement with the uncertainty relation.

If $T < T_{cB}$ and $L_3 \gg \lambda_T[(T_{cB}/T)^{3/2}-1]^{-1}$ and $x \gg \lambda_T[(T_{cB}/T)^{3/2}-1]^{-1}$, then A is negligible compared to the K_0 term, and Eq. (D9) implies $\langle\psi^\dagger(\vec{x})\psi(0)\rangle \approx (2\pi/\lambda_T^2L_3)K_0(x/r)$. Using the asymptotic expansions for K_0, we obtain Eqs. (31) and (32).

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[2]P. W. Anderson and A. H. Dayem, Phys. Rev. Letters 13, 195 (1964); B. M. Khorana and B. S. Chandrasekhar, Phys. Rev. Letters 18, 230 (1967); and S. Shapiro, Phys. Rev. Letters 11, 80 (1963).

[3]D. F. Brewer, in *Superfluid Helium*, edited by J. F. Allen (Academic Press, Inc., New York, 1966), p. 159.

[4]H. P. R. Frederikse, Physica 15, 860(1949); See also Ref. 3.

[5]W. M. van Alphen, G. J. van Haasteren, R. de Bruyn Ouboter, and K. W. Taconis, Phys. Letters 20, 474 (1966).

[6]D. H. Douglass, Phys. Rev. Letters 7, 14 (1961).

[7]The theoretical characterization of "super systems" in the past has been primarily concerned with the bulk system (i.e., $L\times L\times L$ with $L\to\infty$). The existence of a persistent current has generally been assumed to

imply the existence of an order parameter or quasi-average. See D. Pines, in *Quantum Fluids*, edited by D. F. Brewer (North-Holland Publishing Co., Amsterdam, 1966), p. 338.

[8]P. C. Hohenberg, Phys. Rev. 158, 383 (1967).

[9]D. A. Krueger, Phys. Rev. Letters 19, 563 (1967).

[10]Periodic boundary conditions require that $\phi(x+L) = \phi(x)$, where $\phi(x)$ is the wave function and L is the size of the box. Box boundary conditions require that $\phi(x)=0$ for x on the surface of the box. Periodic boundary conditions were used in obtaining the f sum rule which was used in obtaining the form of the inequality on $\langle a_{\vec{k}}^{+} a_{\vec{k}} \rangle$ used in Refs. (8) and (9).

[11]H. Wagner, Z. Physik 195, 273 (1966). For the Bose system, the symmetry-breaking technique consists of (a) adding to the Hamiltonian a term $\frac{1}{2}(-\nu)\sqrt{V} \times (a_0 + a_0^{+})$, where V is the volume, $a_0^{+}(a_0)$ is the creation (annihilation) operator for the zero-momentum state, and ν is a positive parameter; (b) calculating all quantities for ν finite; (c) taking the infinite-volume limit; and (d) taking the $\nu \to 0$ limit.

[12]For these systems, a nonzero quasi-average is equivalent to condensation.

[13]C. N. Yang, Rev. Modern Phys. 34, 694 (1962).

[14]M. D. Girardeau, J. Math. Phys. 6, 1083 (1965).

[15]Any definition of condensation for a finite system is somewhat arbitrary. The Ehrenfest definitions of phase transitions are not applicable for strictly finite systems, because it is only in the thermodynamic limit that *mathematical* singularities and discontinuities of the derivatives of the thermodynamic potentials may occur. [See, for example, K. Huang, *Statistical Mechanics* (John Wiley & Sons, Inc., New York, 1963), Chap. 15.] It still may be that the transitions will appear sharp to the experimentalist because of the finite resolution of his instruments. Theoretically the "transitions" in a finite geometry will appear as large changes of derivatives over a small temperature region. Due to this fuzzy nature of the transition, even the definition of a "critical temperature" becomes uncertain, and it is necessary to talk of transition regions. D. F. Goble and L. E. H. Trainor [Can. J. Phys. 44, 27 (1966)] have suggested six possible definitions of the critical temperature for an ideal Bose gas in a finite geometry, but no one of them is clearly superior to the others. Qualitatively one finds that if $L_1 \simeq L_2 \simeq L_3 \gg \lambda_T$, then the transition is quite sharp. For $L_1 \simeq L_2 \gg \lambda_T > L_3$, it is rounded substantially, and is fairly diffuse when $L_1 \gg \lambda_T > L_2 > L_3$.

[16]S. R. deGroot, G. J. Hooyman, and C. A. ten Seldam, Proc. Roy. Soc. (London) A203, 266 (1950).

[17]The results for $\delta = 2$ have also been obtained by R. M. May, Phys. Rev. 135, A1515 (1964). Note that η is the density only if $\delta = 2$.

[18]The bulk ideal Bose gas is discussed at length by Fritz London, *Superfluids* (Dover Publications, Inc., New York, 1961), 2nd Revised Ed., Vol. II.

[19]D. L. Mills, Phys. Rev. 134, A306 (1964); See also Goble and Trainor (Ref. 15) and Phys. Rev. 157, 167 (1967).

[20]M. F. M. Osborne, Phys. Rev. 76, 396 (1949).

[21]J. M. Ziman, Phil. Mag. 44, 548 (1953).

[22]Using the mass of He^4 for m, we have $\lambda_T = 15.5\,T^{-\frac{1}{2}}$Å, where T is in °K. Our thermal wavelength is $\sqrt{\pi}$ times that used by some authors.

[23]The motivation for our nomenclature is that for $L_i \gg \lambda_T$ many states ($\approx L_i/\lambda_T$) contribute significantly to the thermodynamic quantities, whereas for $\lambda_T > L_i$ only the lowest mode in the ith direction contributes appreciably. In the latter case, we say that motion in the ith direction is "frozen out" and the system has effectively lost one degree of freedom.

[24]We have dropped the small remainder R_3. A bound on this term is given in Appendix C. In addition, we have dropped corrections which are appreciable only when α is close to the stated bounds. The ranges in α over which these are important correspond, in general, to a relatively small density interval.

[25]From Eq. (14) we can also see that the thermodynamic limit must be specified by not only $L_1, L_2, L_3, N \to \infty$ with $N/(L_1, L_2, L_3)$ finite but also how L_1, L_2 and $L_3 \to \infty$. For example, if L_2 goes to infinity exponentially faster than L_3, then there will be a shift in the critical temperature relative to the $L_1 = L_2 = L_3 \to \infty$ case. This is not in contradiction with the result $n_0/N = 1-(T/T_0)^{w/\delta}$ of de Groot *et al.*, because their proof that their sum $A_1 + A_2 + \cdots + A_{w-1}$ vanishes in the infinite-volume limit implicitly assumes that $L_j^{-1} \ln L_i \to 0$ for all i and j. [See their Eq. (E.20)].

[26]The precise value of b is unknown because it includes the effects of the remainders R_i for which only upper bounds have been found.

[27]The coefficients of the L^2/λ_T^2 terms are simply $\sum_{l=-\infty(\neq 0)}^{\infty} l^{-2}$ and $4\sum_{l=2}^{\infty} (l^2-1)^{-1}$ for periodic and box boundary conditions, respectively.

[28]A similar situation obtains for the two-dimensional slab, but the temperature region over which the transition occurs is broadened.

[29]F. Bloch, Phys. Rev. 137, A787 (1965) and Phys. Today 19, 27 (1966), has discussed the implications of ODLRO on the shape of the distribution function for both bosons and fermions.

[30]Our results for C_v using periodic boundary conditions show less deviation from the bulk result and consequently a sharper peak than found by Goble and Trainor (Refs. 15 and 19). Denote $C_{v\,\mathrm{max}}$ as the maximum value of C_v as a function of T for fixed ρ and L_3. We find that $C_{v\,\mathrm{max}}$ is a monotonically increasing function of L_3, contrary to the behavior found by Goble and Trainor (Ref. 19). These differences may be due to the use of box boundary conditions by Goble and Trainor and to the use of periodic boundary conditions in our calculation.

[31]The existence of a very long- but finite-range order in a two-dimensional charged Bose system is implicit in the work of R. M. May, Phys. Rev. 115, 254 (1959), who found an "essentially perfect" Meissner effect.

[32]Probably the largest error due to dropping R_i occurs in the two-dimensional system where it is still less than 1% if $L_1 \gtrsim 100\lambda_T$.

[33]R. B. Dingle, Appl. Sci. Res. B6, 240 (1957); See also the Appendix in Ref. 18.

[34]*Tables of Integral Transforms*, edited by A. Erdelyi *et al.* (McGraw-Hill Publishing Co., New York, 1953) Vol II, Sec. 8.2, Eq. (24).

[35]See Ref. 34, Sec. 13.2, Eq. (17).