introduce an entire set of currents, and we shall stop at our simplest $\mathcal{H}^{(+)}$ structure.

Table I displays the main observational criteria. (ηN has been included for the sake of completeness, but may be irrelevant since we have as yet no experimental way of observing this reaction even indirectly for that region of s, t.) The most interesting test seems to be in πN scattering.

ACKNOWLEDGMENTS

The author would like to thank Professor F. J. Gilman for a discussion of the results of Ref. 15; Professor M. Gell-Mann for valuable comments and important criticism; and Professor L. Motchane and Professor L. Michel for the hospitality of the IHES, where part of this study was performed.

PHYSICAL REVIEW VOLUME 172, NUMBER 5 25 AUGUST 1968

Simultaneous "Partial-Wave" Expansion in the Mandelstam Variables: Crossing Symmetry for Partial Waves

A. P. BALACHANDRAN*
Physics Department, Syracuse University, Syracuse, New York 13210

AND

J. NUYTS
Laboratoire de Physique Théorique et Hautes Énergies, 91 Orsay, France

(Received 11 March 1968)

The amplitude for the elastic scattering of two spinless particles of equal mass $\frac{1}{2}$ is expanded in terms of eigenfunctions which form a complete set for a certain class of functions of the Mandelstam variables s,t,u ($s+t+u=1$) and which display the threshold behavior of the partial-wave amplitudes. The eigenfunctions are generated by a partial differential operator which commutes with the total angular momentum in any of the three channels and which is invariant under s, t, u permutations. An infinite number of *finite-dimensional* crossing relations for the partial-wave amplitudes which are necessary and sufficient for the crossing symmetry of the total amplitude are derived, as well as an explicit form for the corresponding crossing matrices. It is shown that the Fourier coefficients of the expansion satisfy a Froissart-Gribov integral representation whose kernel is determined by the imaginary parts of the partial-wave amplitudes.

I. INTRODUCTION

THIS paper attempts to generalize and extend some of the classical results from the angular-momentum theory of the scattering matrix. Because we shall consider only the elastic scattering of two spinless particles of equal mass, it is convenient to formulate our problem directly in terms of this system. The partial-wave expansion of the scattering amplitude F of such a system "displays" in a certain sense its dependence on the angle of scattering (which is related in a well-known way to the square of the momentum transfer t) since this variable appears only in the arguments of the Legendre polynomials. *We wish to find an eigenfunction expansion of F which displays its dependence on both s and t* (remember that u is not independent of s and t). Without further specification, there is no unique solution to this problem, of course. For example, expansions of this sort can be achieved quite readily by expressing each partial-wave amplitude f_l as a Fourier integral or as a power series in s. Such representations of F are, as a rule, useless and lead to no new insights into the structure of the system. In our approach, the choice of the eigenfunctions will follow from the choice of a differential operator. The properties that we shall demand of the latter will be motivated by physics and will be explained in Sec. II. The eigenfunctions will also be tabulated there in terms of known special functions and their elementary normalization and orthogonality properties stated.

Section III contains our major results. Hopefully, had we discovered the "right" set of eigenfunctions, none of the advantages of an ordinary partial-wave expansion would be lost and some new kinematical constraints on the system would also be revealed. The former is certainly true in our considerations because the eigenfunctions are diagonal in angular momentum and each s-dependent partial-wave amplitude is in effect expanded in a suitable basis. An interesting feature of this basis is that it exhibits the threshold behavior of these amplitudes explicitly. Further, the analysis shows the existence of a new "quantum number" σ which is a non-negative integer (see, however, Appendix B) and which is conserved under crossing. As a consequence, we are able to state the necessary and sufficient conditions on f_l in order that the scattering amplitude F be crossing symmetric. For each σ,

* Supported in part by the U. S. Atomic Energy Commission.

partial-wave amplitudes f_l with angular momenta l which do not exceed σ are related to each other by a $(\sigma+1)$-dimensional crossing matrix. There are also indications from the asymptotic behavior of the eigenfunctions in their indices and from the analyticity of f_l that with increasing σ, these crossing relations may decrease in importance for dynamical approximations. These indications represent the kind of insights that an effort of this sort would be expected to yield and we believe them to be significant. The reader may recall here that in the conventional formalism, it is not easy to state the implications of crossing symmetry in terms of f_l in any neat form. We shall present the solutions of the eigenvalue problem associated with the crossing matrix in a later publication.

In Sec. IV, we comment briefly on some aspects of the eigenfunction expansion which were not touched upon previously. In particular, a Froissart-Gribov representation is derived for the Fourier coefficients of the expansion. Implications of this expansion (or its analytic continuation in energy) for the analytic properties of f_l in approximate calculations are also indicated.

Appendix A contains the details of the evaluation of the crossing matrix. In the text, the eigenvalue problem generated by our differential operator is studied only in one of its compact forms. Appendix B partially extends this discussion to the corresponding noncompact forms.

The extension of this paper to systems with internal symmetry or spin is quite simple so long as all four particles have equal mass, since the techniques are effective for the individual invariant functions or reduced matrix elements. We hope to present elsewhere a more intrinsic treatment of spin. When the masses are not equal, however, a naive extrapolation of this approach fails, and at the moment we do not know what the correct generalization is.

The eigenfunction expansions which are encountered in physics are almost always associated with some Lie group which is an invariance group for the problem under a suitable definition and which is transitive on the manifold of allowed physical configurations. In a forthcoming paper,[1] the underlying group in our method will be identified with $SU(3)$. (See also Appendix B.) The eigenfunctions will be shown to correspond to the central elements in the weight diagram of (σ,σ) representations and the crossing matrices to Weyl reflections.

There is a fair amount of literature which deals with harmonic analysis on the Dalitz plot, in particular when the particles are nonrelativistic.[2] The work which most resembles ours is that of Charap,[3] who has investigated a class of Appell polynomials and the associated Fourier series for scattering amplitudes. It has been pointed out to us recently by P. Ramond that our eigenfunctions satisfy one of the defining partial differential equations of these polynomials. The solutions of this differential equation which will interest us in this paper, however, will not be those of Appell.

II. EIGENFUNCTIONS

We consider the elastic scattering of two spinless particles of equal mass characterized by a scattering amplitude $F(s,t)$. The common mass will be normalized to $\frac{1}{2}$ by a suitable choice of units so that the Mandelstam variables are subject to the constraint $s+t+u=1$. The cosines of the scattering angles in the three channels are defined by the equations

$$z_s=1+\frac{2t}{s-1},\quad z_t=1+\frac{2u}{t-1},\quad z_u=1+\frac{2s}{u-1}. \tag{2.1}$$

The partial-wave decomposition of F in the s channel can be regarded as its expansion in terms of the eigenfunctions of the differential operator

$$X^2=-\frac{\partial}{\partial z_s}(1-z_s^2)\frac{\partial}{\partial z_s} \tag{2.2}$$

since the solutions of the eigenvalue problem

$$X^2\varphi_l(z_s)=l(l+1)\varphi_l(z_s)$$

in the interval $[-1,+1]$ and with suitable boundary conditions are the Legendre polynomials. One method for generating eigenfunctions which depend on both s and t is therefore to construct a partial differential operator $\mathcal{O}$ which acts on both these variables. As indicated above, there are many such operators $\mathcal{O}$, and to restrict this class it is necessary to characterize $\mathcal{O}$ more precisely. The physical situation suggests that $\mathcal{O}$ must have the following properties:

(i) *It must commute with* X^2:

$$[\mathcal{O},X^2]=0. \tag{2.3}$$

If (2.3) is true, then it is possible to diagonalize $\mathcal{O}$ and X^2 simultaneously. As a consequence, we would forfeit none of the beautiful features of the usual partial-wave expansion.

(ii) *It must be invariant under all possible permutations of* s, t, *and* u. This requirement is also suggested by simple considerations. Equation (2.3) seems to treat the s channel preferentially while if $\mathcal{O}$ is permutation-invariant and $[\mathcal{O},X^2]=0$, then

$$[\mathcal{O},Y^2]=[\mathcal{O},Z^2]=0, \tag{2.4}$$

---

[1] A. P. Balachandran, W. Meggs, J. Nuyts, and P. Ramond (unpublished).

[2] F. T. Smith, Phys. Rev. **120**, 1058 (1960); J. Math, Phys. **3**, 735 (1962); G. C. Wick, Ann. Phys. (N. Y.) **18**, 65 (1962); A. Dragt, J. Math. Phys. **6**, 533 (1965); **6**, 1621 (1965); J. M. Lévy-Leblond and F. Lurçat, *ibid.* **6**, 1564 (1965); F. R. Halpern, Phys. Rev. **137**, B1587 (1965); J. M. Lévy-Leblond, J. Math. Phys. **7**, 2217 (1966); B. W. Lee, University of Miami Report, 1967 (unpublished).

[3] J. Charap (private communication).

where

$$Y^2=-\frac{\partial}{\partial z_t}(1-z_t^2)\frac{\partial}{\partial z_t}, \quad Z^2=-\frac{\partial}{\partial z_u}(1-z_u^2)\frac{\partial}{\partial z_u}. \quad (2.5)$$

Further, the eigenspace of such an $\mathcal{O}$ for each eigenvalue will be an invariant manifold under permutations. We shall see later how this property enables us to derive crossing relations for partial-wave amplitudes.

If s, t, and u are treated as independent variables, X^2 can be written in the form[4]

$$X^2=-(\partial_t-\partial_u)(tu)(\partial_t-\partial_u). \quad (2.6)$$

The expressions for Y^2 and Z^2 are obtained by permutation. We claim that an $\mathcal{O}$ which has properties (i) and (ii) is given by

$$\mathcal{O}=X^2+Y^2+Z^2. \quad (2.7)$$

(ii) is evident while (i) may be verified either by direct computation or by separating the variables in $\mathcal{O}$. For if we regard s, z_s, and $\Sigma=s+t+u$ as the independent variables, $\mathcal{O}$ becomes

$$\mathcal{O}=\frac{1}{1-s}\left[-\frac{\partial}{\partial z_s}(1-z_s^2)\frac{\partial}{\partial z_s}\right] +s(s-1)\partial_s^2+(3s-1)\partial_s. \quad (2.8)$$

The form (2.8) facilitates the study of the eigenvalue problem generated by $\mathcal{O}$. We try the ansatz

$$S_\alpha{}^l(s,t)=R_\alpha{}^l(s)P_l(z_s) \quad (2.9)$$

to solve the differential equation

$$\mathcal{O}S_\alpha{}^l(s,t)=\alpha S_\alpha{}^l(s,t) \quad (2.10)$$

and discover that $R_\alpha{}^l$ is the solution of the Sturm-Liouville problem

$$-\frac{d}{ds}\left[s(1-s)^2\frac{d}{ds}\right]R_\alpha{}^l(s) +[l(l+1)+(1-s)\alpha]R_\alpha{}^l(s)=0. \quad (2.11)$$

Since the zeros of $s(1-s)^2$ are at $s=0$ and 1, the theory of ordinary differential equations instructs us that the "natural" intervals of self-adjointness of the Sturm-Liouville operator in (2.11) with respect to the measure $ds(1-s)$ are $(-\infty,0]$, $[0,1]$, and $[1,\infty)$. We choose to solve (2.11) in the interval $[0,1]$ for two reasons. (See, however, Appendix B.) First, when $s\in[0,1]$ and $z_s\in[-1,+1]$, s, t, and u range over the interior of the Mandelstam triangle defined by the boundaries $s=0$, $t=0$, and $u=0$. This region is mapped onto itself under the interchanges of s, t, and u. Also,

$$\begin{aligned} ds(1-s)dz_s&=2\,ds\,dt\,\theta(1-s-t)\\ &=2\,ds\,dt\,du\,\delta(1-s-t-u)\\ &=dt(1-t)dz_t, \text{ etc.}, \end{aligned} \quad (2.12)$$

where it is understood that the s, t, and u variables which are explicit on the right-hand side should be integrated from 0 to 1 and z_t from -1 to $+1$. The Hilbert space induced by the measure (2.12) on the triangle is therefore permutation-invariant. It may be anticipated that there would be advantages in these symmetries for the display of the crossing properties of the scattering amplitude. Further, F is holomorphic in the interior of the triangle and questions with regard to the convergence of the expansion will trouble us least here.

After the transformation

$$R_\alpha{}^l(s)=(1-s)^l\, r_\alpha{}^l(s), \quad (2.13)$$

the differential equation (2.11) is of the hypergeometric form. Its complete, orthogonal system of solutions on the Hilbert space generated by the measure $ds(1-s)^{2l+1}$ and the interval $[0,1]$ are the Jacobi polynomials $P_n{}^{(2l+1,0)}(2s-1)$.[5] The corresponding eigenvalues are

$$\alpha\equiv\alpha_{n+l}=(n+l)(n+l+2), \quad n=0,1,2,\cdots. \quad (2.14)$$

On replacing the subscripts α in (2.9) by n, the final answer for the eigenfunction of $\mathcal{O}$ for the eigenvalue α_{n+l} can be written as

$$\begin{aligned} S_n{}^l(s,t)&=R_n{}^l(s)P_l(z_s)\\ &=(1-s)^lP_n{}^{(2l+1,0)}(2s-1)P_l(z_s). \end{aligned} \quad (2.15)$$

We need the orthogonality relations for $R_n{}^l$. These are readily inferred from those of the Jacobi polynomials[5]:

$$\int_0^1 ds(1-s)R_n{}^l(s)R_N{}^l(s)=\frac{1}{2(n+l+1)}\delta_{nN}. \quad (2.16)$$

On defining the inner product

$$(f,g)=\int_0^1 ds\int_0^1 dt\,\theta(1-s-t)f^*(s,t)g(s,t) \quad (2.17)$$

for functions of s and t, we find

$$(S_n{}^l,S_N{}^L)=[2(n+l+1)(2l+1)]^{-1}\delta_{lL}\delta_{nN}. \quad (2.18)$$

The scattering amplitude F can be expanded in the form

$$F(s,t)=\sum_{n,l=0}^{\infty} 2(n+l+1)(2l+1)a_n{}^lS_n{}^l(s,t). \quad (2.19)$$

[4] Since $(\partial_t-\partial_u)(s+t+u)=0$, X^2F is the same regardless of whether we eliminate u through the constraint $u=1-s-t$ before or after the differentiations. It may be helpful to recall here the example of the angular momentum operator $\mathbf{J}=\mathbf{r}\times\mathbf{p}$ which is actually independent of the magnitude of $\mathbf{r}$ and depends only on the polar angles. We treat s, t, and u as independent variables for reasons of symmetry.

[5] See, e.g., *Bateman Manuscript Project*, edited by A. Erdelyi (McGraw-Hill Book Co., Inc., New York, 1953) Vol. II, p. 169.

The inversion formulas for the Fourier coefficients are

$$a_n{}^l=(S_n{}^l,F) \tag{2.20}$$

$$=\int_0^1 ds(1-s)R_n{}^l(s)f_l(s)\,, \tag{2.21}$$

where f_l is the lth partial-wave amplitude:

$$f_l(s)=\frac{1}{2}\int_{-1}^{+1} dz_s P_l(z_s)F(s,t)\,, \tag{2.22}$$

It is a most interesting feature of this expansion that it explicitly exhibits the threshold zeros $(1-s)^l$ of the partial-wave amplitudes.

Some brief remarks regarding the convergence of the series (2.19) are in order. It is not difficult to see that any polynomial in s and t can be expressed as a finite linear combination of $S_n{}^l$. (Change variables to s and z_s and observe that the coefficient of $z_s{}^l$ is a polynomial in s with a zero of order l at $s=1$. The rest is easy.) Since these polynomials form a dense subset of the L^2 space with the inner product (2.17), the expansion (2.19) will converge in the corresponding norm to F provided that F is an element of this Hilbert space. Similar results are true for the series

$$f_l(s)=\sum_{n=0}^{\infty} 2(n+l+1)a_n{}^lR_n{}^l(s) \tag{2.23}$$

which may be inferred from (2.19). These are weak requirements and, for instance, allow for square integrable singularities of f_l at $s=0$ and 1. We restrict ourselves hereafter to those F and f_l which are square integrable over the appropriate sets and measures. Since F is holomorphic within the triangle and f_l in the open interval $0<s<1$, the expansions should also have vastly improved convergence properties there.

III. CROSSING RELATIONS FOR PARTIAL WAVES

To save on notation, we specialize to scattering amplitudes which are invariant under s, t, and u permutations. The eigenfunctions which correspond to $S_n{}^l$ in the t channel are

$$T_n{}^l(s,t)=R_n{}^l(t)P_l(z_t)\,. \tag{3.1}$$

Since F remains the same when $s\to t$, $t\to u$, $u\to s$, (2.19) implies the identity

$$\sum_{n,l} 2(n+l+1)(2l+1)a_n{}^lS_n{}^l(s,t) =\sum_{N,L} 2(N+L+1)(2L+1)a_N{}^LT_N{}^L(s,t)\,. \tag{3.2}$$

But $S_n{}^l$ and $T_N{}^L$ are eigenfunctions of $\mathcal{O}$ with eigenvalues $(n+l)(n+l+2)$ and $(N+L)(N+L+2)$, respectively, and $\mathcal{O}$ is self-adjoint in the scalar product (2.17). Therefore,

$$(S_n{}^l,T_N{}^L)=0 \quad\text{if}\quad n+l\neq N+L\,. \tag{3.3}$$

We define

$$\sigma=n+l \tag{3.4}$$

and learn from (3.2) that

$$a_{\sigma-l}{}^l=2(\sigma+1)\sum_{L=0}^{\sigma}(S_{\sigma-l}{}^l,T_{\sigma-L}{}^L)(2L+1)a_{\sigma-L}{}^L\,,\quad l=0,1,2,\cdots,\sigma\,,\quad \sigma=0,1,2,\cdots. \tag{3.5}$$

The interchange of the integral and the sum is justified by the L^2 convergence of the series. Because the $a_n{}^l$ can be eliminated in favor of f_l through (2.21), these are our crossing relations for partial-wave amplitudes. It is evident that (3.5) guarantees the crossing symmetry of F when s and t are within the triangle, and hence, by analytic continuation, for all those s and t for which F is defined. If $t\leftrightarrow u$ invariance is imposed, there is the restriction of the sums in (3.2) to even values of l and L. Identities like $F(s,t)=F(u,t)$ then lead to nothing new.

We defer the evaluation of the crossing matrix to Appendix A and record the answer here:

$$\begin{aligned}(S_{\sigma-l}{}^l,T_{\sigma-L}{}^L)&=\frac{(-1)^{\sigma+L}(-\sigma)_L}{2(\sigma+1)(\sigma+1)_{L+1}}\times\sum_{\rho=0}^{m}\frac{(-L)_\rho(L+1)_\rho(l-\sigma)_\rho(-\sigma-l-1)_\rho}{[(-\sigma)_\rho]^2[\rho\,!]^2}\\&=(-1)^\sigma\frac{[\sigma\,!]^2}{(2\sigma+2)}\frac{1}{(\sigma-L)\,!(\sigma+L+1)\,!}\times{}_4F_3(-L,\,L+1,\,l-\sigma,\,-\sigma-l-1;\\&\qquad -\sigma,\,-\sigma,\,1;1)\,.\end{aligned} \tag{3.6}$$

In the above, ${}_4F_3$ denotes the standard generalized hypergeometric function[6]:

$${}_4F_3(-L,L+1,l-\sigma,-\sigma-l-1;-\sigma,-\sigma,1;1) =\sum_{\rho=0}\frac{(-L)_\rho(L+1)_\rho(l-\sigma)_\rho(-\sigma-l-1)_\rho}{(-\sigma)_\rho(-\sigma)_\rho(1)_\rho\,\rho\,!}\,, \tag{3.7}$$

where $m=\min(\sigma-l,\,L)$ and

$$(a)_\rho=a(a+1)\cdots(a+\rho-1)\,.$$

We emphasize that for each given σ, the crossing matrix in (3.5) is *finite-dimensional*. It is in fact a

[6] See, e.g., *Bateman Manuscript Project*, edited by A. Erdelyi (McGraw-Hill Book Co., Inc., New York, 1953) Vol. I, p. 183.

$(\sigma+1)\times(\sigma+1)$ matrix which relates all partial-wave amplitudes with angular momenta which do not exceed σ. It is also easy to see from the L^2 convergence of the series (2.23) that $a_{\sigma-l}{}^l$ converges to zero as σ becomes large. It is not impossible, therefore, that these equations will be useful for the implementation of crossing symmetry in dynamical schemes.

IV. MISCELLANY

(1) The series (2.19) is valid in the first instance on the Mandelstam triangle while the series (2.23) is valid for $s\in[0,1]$. On the boundaries of these regions, the singularities of F due to the onset of physical thresholds are located at $s=1$, $t=1$, and $u=1$ while those of f_l from unitarity and crossing are at $s=1$ and 0. To escape these restrictions on the series, it is necessary to continue them analytically somehow. Experience with complex angular-momentum theory suggests that as a preparation for this task, we must first continue $a_n{}^l$ to complex values of the index n with the aid of a Froissart-Gribov representation. We now proceed to derive such a representation.

If we ignore subtractions for convenience, the amplitude f_l satisfies the dispersion relation

$$f_l(s)=\frac{(s-1)^l}{\pi}\int_{-\infty}^{\infty}ds'\frac{\operatorname{Im}f_l(s')}{(s'-1)^l(s'-s)}, \tag{4.1}$$

where the factor $(s-1)^l$ accounts for its threshold behavior and $\operatorname{Im}f_l=0$ if $s'\in(0,1)$. We know also that the Jacobi functions of the second kind can be defined by the integral relation[7]

$$Q_n{}^{(\alpha,\beta)}(x)=\frac{1}{2}\frac{1}{(x-1)^\alpha(x+1)^\beta}\times\int_{-1}^{+1}d\xi\frac{(1-\xi)^\alpha(1+\xi)^\beta P_n{}^{(\alpha,\beta)}(\xi)}{x-\xi} \tag{4.2}$$

which, together with the reflection property[5]

$$P_n{}^{(\alpha,\beta)}(-\xi)=(-1)^nP_n{}^{(\beta,\alpha)}(\xi), \tag{4.3}$$

shows that

$$Q_n{}^{(\alpha,\beta)}(-x)=(-1)^{n+\alpha+\beta+1}Q_n{}^{(\beta,\alpha)}(x) \tag{4.4}$$

if α and β are integers. Insert (4.1) into (2.21) and use (4.2) and (4.4) to obtain the Froissart-Gribov representation

$$\begin{aligned}a_{\sigma-l}{}^l(\pm)=\frac{2}{\pi}(-1)^l\int_1^\infty ds'[(s'-1)^{l+1}&\\ \times\operatorname{Im}f_l(s')Q_{\sigma-l}{}^{(2l+1,0)}(2s'-1)\mp s'^{l+1}&\\ \times\operatorname{Im}f_l(1-s')Q_{\sigma-l}{}^{(0,2l+1)}(2s'-1)],&\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}a_{\sigma-l}{}^l&=a_{\sigma-l}{}^l(+)\quad\text{if}\quad\sigma\text{ is even}\\&=a_{\sigma-l}{}^l(-)\quad\text{if}\quad\sigma\text{ is odd}.\end{aligned} \tag{4.6}$$

[7] Reference 5, p. 171.

Equation (4.5) appears to be in the right form for the continuation of $a_{\sigma-l}{}^l$ into the complex σ plane. This point is at present under investigation with particular emphasis on the possibility of simultaneously continuing the crossing relations (3.5) to such values of σ.

(2) It seems likely that after even and odd l are separated, (4.5) also defines $a_n{}^l$ as an analytic function of l such that it fulfills a generalized crossing relation which is to be inferred from (3.5). We are trying to develop this line of thought and would not be surprised if it yields some new insights into the structure of complex angular-momentum theory. For example, it might indicate how the Gribov-Pomeranchuk singularity at $l=-1$ propagates through crossing in the l plane.

(3) We finally make some remarks which may be significant for dynamical approximations designed to exploit the preceding expansions. First, we note that since each term in the series (2.23) has the factor $(1-s)^l$, its truncation would still preserve the threshold behavior of the partial-wave amplitudes. Second, if this series is Watson-transformed on the index n, perhaps with the intention of reaching the physical region, $R_n{}^l$ develops a cut in the interval $(-\infty,0]$. Any distortion of this representation to obtain manageable forms for f_l is thus guaranteed for exhibit at least the correct location of its left-hand cut. (Note, however, that the discontinuities across the cuts will not in general be correctly described by such approximations.) The best method of implementing unitarity in this scheme is not entirely clear. An easy solution, of course, is to work with f_l all the way through and supplement the N/D equations, say, with the crossing relations (3.5).

ACKNOWLEDGMENTS

We are most grateful to Dr. J. J. Loeffel for his advice and counsel during the course of this work. Conversations with Professor H. Bacry, Professor V. Bargmann, Professor F. Lurçat, Professor E. Nelson, Professor J. Schechter, and Dr. N. J. Papastamatiou are also acknowledged. Finally, we wish to thank the Institute for Advanced Study, Princeton, New Jersey for its hospitality during the year 1966–67. Many of the ideas expressed in this paper were conceived during our stay there.

APPENDIX A: EVALUATION OF CROSSING MATRIX

We wish to evaluate the inner product

$$I=(S_n{}^l,T_N{}^L)=\frac{1}{2}\int_0^1 ds(1-s)\times\int_{-1}^{+1}dz_sR_n{}^l(s)P_l(z_s)T_N{}^L(s,t), \tag{A1}$$

where

$$T_N{}^L(s,t)=(1-t)^L\binom{N+2L+1}{N}\times{}_2F_1(-N,N+2L+2;2L+2;1-t)\times{}_2F_1(-L,L+1;1;\tfrac{1}{2}(1-z_t)).\quad\text{(A2)}$$

The Jacobi and Legendre polynomials have been expressed in terms of hypergeometric functions.[8] On using $P_L(z_t)=(-1)^LP_L(-z_t)$ and (2.1), (A2) becomes

$$T_N{}^L(s,t)=(-1)^L\binom{N+2L+1}{N}\times\sum_{\nu=0}^{N}\sum_{\rho=0}^{L}\frac{(-N)_\nu(N+2L+2)_\nu(-L)_\rho(L+1)_\rho}{(2L+2)_\nu\,\nu!\,(\rho!)^2}\times s^\rho(1-t)^{L+\nu-\rho}.\quad\text{(A3)}$$

We write

$$(1-t)^{L+\nu-\rho}=[1+\tfrac{1}{2}(s-1)(1-z_s)]^{L+\nu-\rho}=\sum_{\tau=0}^{L+\nu-\rho}\binom{L+\nu-\rho}{\tau}\left(\frac{s-1}{2}\right)^\tau(1-z_s)^\tau.\quad\text{(A4)}$$

Since $P_l(z_s)$ is orthogonal to all polynomials in z_s of degree less than l and since because of the orthogonality properties of the Jacobi polynomials, we have

$$\int_0^1 ds(1-s)^{l+1}s^pR_n{}^l(s)=0,\quad p=0,1,2,\cdots,n-1,\quad\text{(A5)}$$

we can restrict the set of indices in (A3), (A4) by the following inequalities for the evaluation of I:

$$0\leqslant\tau\leqslant L+\nu-\rho,\quad\text{(A6a)}$$

$$\tau\geqslant l,\quad\text{(A6b)}$$

$$0\leqslant\rho\leqslant L,\quad\text{(A6c)}$$

$$\rho+\tau-l\geqslant n,\quad\text{(A6d)}$$

$$0\leqslant\nu\leqslant N.\quad\text{(A6e)}$$

From (A6d), (A6a), and (A6e), we find

$$n+l\leqslant\rho+\tau\leqslant L+\nu\leqslant L+N.\quad\text{(A7)}$$

But a simple change of variables gives the symmetry relation

$$(S_n{}^l,T_N{}^L)=(-1)^{l+L}(S_N{}^L,T_n{}^l).\quad\text{(A8)}$$

It follows that there is also the inequality $N+L\leqslant n+l$ and therefore

$$n+l=N+L,\quad\text{(A9a)}$$

$$\tau=n+l-\rho,\quad\text{(A9b)}$$

$$\nu=N,\quad\text{(A9c)}$$

$$0\leqslant\rho\leqslant\min(n,L)=m,\quad\text{say},\quad\text{(A9d)}$$

where (A9d) is a consequence of (A6b), (A6c), and (A9b). Thus, in so far as I is concerned, $T_N{}^L$ can be modified as follows:

$$T_N{}^L(s,t)\to\delta_{n+l,N+L}(-1)^{l+L}\binom{N+2L+1}{N}(1-s)^ls^n\times\sum_{\rho=0}^{m}\frac{(-N)_N(N+2L+2)_N(-L)_\rho(L+1)_\rho}{(2L+2)_N\,N!\,(\rho!)^2}\times\left(\frac{1-z_s}{2}\right)^{N+L-\rho}.\quad\text{(A10)}$$

We have replaced the factor $s^\rho(s-1)^{n-\rho}$ by s^n, which is permissible owing to (A5).

We need two more formulas. First, from

$$P_l(z_s)={}_2F_1(-l,l+1;1;\tfrac{1}{2}(1-z_s))$$

we derive the integral relation

$$\int_{-1}^{+1}dz_s(1-z_s)^\nu P_l(z_s)=2^{\nu+1}\sum_{\tau=0}^{l}\frac{(-l)_\tau(l+1)_\tau}{(\tau!)^2}\frac{1}{\tau+\nu+1}=\frac{2^{\nu+1}}{\nu+1}{}_3F_2(-l,l+1,\nu+1;1,\nu+2;1),$$

since

$$(\tau+\nu+1)^{-1}=(\nu+1)^{-1}(\nu+1)_\tau/(\nu+2)_\tau.$$

The ${}_3F_2$ is summable with the aid of Saalschütz's theorem[9]:

$${}_3F_2(-l,l+1,\nu+1;1,\nu+2;1)=(-1)^l(-\nu)_l/(-l-\nu-1)_l.$$

Therefore,

$$\int_{-1}^{+1}dz_s(1-z_s)^{n+l-\rho}P_l(z_s)=(-1)^l\frac{2^{n+l-\rho+1}}{(n+l-\rho+1)}\frac{(-n-l+\rho)_l}{(-n-2l-1+\rho)_l}.\quad\text{(A11)}$$

The second formula needed is

$$\int_0^1 ds(1-s)^{l+1}s^nR_n{}^l(s)=\frac{1}{2}\frac{(n+2l+1)!\,n!}{(n+l+1)(2n+2l+1)!}.\quad\text{(A12)}$$

This can be derived from[5]

$$\int_{-1}^{+1}dx(1-x)^{2l+1}[P_n{}^{(2l+1,0)}(x)]^2=\frac{2^{2l+1}}{n+l+1}$$

[8] Reference 5, pp. 170, 180.

[9] Reference 6, p. 188.

if it is also remembered that because of (A5) only the term involving x^n in one of the Jacobi polynomials, which is

$$\binom{n+2l+1}{n}\frac{(-n)_n(n+2l+2)_n}{(2l+2)_n\,n!}(-1)^n\frac{x^n}{2^n},$$

contributes to the integral. In obtaining (A12), we used the identities

$$\begin{aligned}(-a)_n&=(-1)^n a!/(a-n)!\,,\quad a\geqslant n\\(a+1)_n&=(a+n)!/a!\end{aligned}\tag{A13}$$

where a and n are non-negative integers.

Insert (A10) into (A1) and use (A11), (A12), (A13), and

$$(b+\rho)_l=(b)_l(b+l)_\rho/(b)_\rho$$

[valid if $(b)_\rho\neq 0$] to derive the formula in the text:

$$\begin{aligned}(S_n{}^l,T_N{}^L)=\delta_{n+l,N+L}&\frac{(-1)^{n+l}[(n+l)!]^2}{(2n+2l+2)N!(N+2L+1)!}\\&\times{}_4F_3(-L,\,L+1,\,-n,\,-n-2l-1;\\&\qquad -n-l,\,-n-l,\,1;\,1)\,.\end{aligned}\tag{A14}$$

APPENDIX B: EIGENFUNCTIONS FOR PHYSICAL REGION

In the text, we focused our attention on the inside of the Mandelstam triangle $(s\geqslant 0, t\geqslant 0, u\geqslant 0, s+t+u=1)$. The differential equation we have written, however, makes sense outside this triangle, and in particular in the physical region $s\geqslant 1$, $1-s\leqslant t\leqslant 0$. In this Appendix we present a few remarks regarding its solutions in this region.

Since we are still in the compact interval $[-1,+1]$ for z_s, X^2 continues to be the Casimir operator of $SO(3)$ [locally $SU(2)$] when restricted to functions of s and t while Y^2 and Z^2 are the Casimir operators of $SO(2,1)$ [locally $SU(1,1)$] under a corresponding restriction. [If $\mathbf{J}_t$ and $\mathbf{J}_u$ are the Hermitian generators of these groups, then $\mathbf{J}_t{}^2F(s,t)=-Y^2F(s,t)$ and $\mathbf{J}_u{}^2F(s,t)=-Z^2F(s,t)$.] The variable z_s is the cosine of a real angle while z_t and z_u are hyperbolic cosines of real angles. Their ranges of variation are $+1\leqslant z_t<\infty$ and $-\infty<z_u\leqslant -1$ [cf. (2.1)]. The diagonalization of X^2 leads to

$$X^2P_l(z_s)=l(l+1)P_l(z_s)\,,\quad l=0,1,2,\cdots\tag{B1}$$

while the diagonalization of Y^2 leads to

$$\begin{aligned}Y^2P_\lambda(z_t)&=\lambda(\lambda+1)P_\lambda(z_t)\,,\\\lambda&=-\tfrac{1}{2}+i\rho\,,\quad -\infty<\rho<\infty\end{aligned}\tag{B2}$$

where $P_\lambda(z_t)$ are the familiar conical functions. There is a similar equation for Z^2.

Since Θ commutes with X^2, Y^2, and Z^2, it can be diagonalized together with any one of these operators. We first discuss the possibility (B1). The equation to solve is still (2.11), but we are interested in the region where s is larger than unity. The solutions can be expressed in terms of hypergeometric functions:

$$\begin{aligned}R_{-l-1+i\rho}{}^l(s)=(s-1)^l{}_2F_1(&l+1+i\rho,\,l+1-i\rho;\\&2l+2;\,1-s)\,,\ -\infty<\rho<\infty\,.\end{aligned}\tag{B3}$$

The notation is that of (2.15). The associated eigenvalues of Θ are $(-1+i\rho)(1+i\rho)$. These Jacobi functions ${}_2F_1$ of complex indices form a complete orthogonal basis for functions which are square integrable on the interval $[1,\infty)$ with measure $ds(1-s)^{2l+1}$.

The connection between these functions and the base vectors of the principal series of $SU(2,1)$ and their detailed properties will be discussed elsewhere.[1] Analogous statements can be made if we start with (B2). The answer can be obtained by replacing the integer l in (B3) by the complex λ of (B2) and of s by t.

The series which correspond to these eigenfunctions for the noncompact intervals bear the same relationship to (2.19) that the background integral in the Regge theory bears to the partial-wave expansion for real scattering angles.[10]

[10] For a detailed exposition of eigenfunction expansions on various little groups, see, e.g., J. Strathdee, J. F. Boyce, R. Delbourgo, and Abdus Salam, Trieste Report 1967 (unpublished). References to previous work may also be found here.