Model for Pion S-Wave Phase Shifts from Current Algebra and Partial Conservation of Axial-Vector Current

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From the algebra of axial-vector charges and divergences, using extensively the principle of pion-pole dominance, we derive a set of sum rules for $(\pi\pi)$ S-wave interactions in the isospin-zero channel. In a single-particle approximation, they provide information on the conjectured σ meson. In the approximation of elastic unitarity, the sum rules take the form of an integral equation for a vertex function $\langle\pi|\sigma|\pi\rangle$, which leads to the inequality $0<\delta_0<\pi$ for the $(\pi\pi)$ S-wave phase shift δ_0 in the isospin-zero channel, whenever it is approximately valid. Exact solutions of the approximate integral equation, however, cannot be constructed, and we have to introduce an effective cutoff function as a correction to our approximation of the partial conservation of axial-vector current. The solutions for δ_0 give a scattering length slightly larger than Weinberg's result and a broad maximum of the phase shift around 700 MeV; its height is sensitive to the cutoff parameter.

1. INTRODUCTION

THE first applications of current algebras have been made to sum rules and low-energy theorems[1]; at present attempts are being made to extend the range of applications of current algebras to finite energies, the work of Schnitzer and Weinberg[2] being an example. In this line, we try to go a step further and present a model for the $(\pi\pi)$ S-wave phase shifts in the isospin-zero channel, based on the algebra of axial-vector charges and divergences and on an extensive use of pion-pole dominance. Unlike earlier authors,[3] we try to assume as little as possible beforehand about the strength and energy dependence of the $(\pi\pi)$ S-wave interaction, such as whether or not the (unknown) unitarity cut allows certain extrapolations from zero energy to threshold, or whether or not there is a σ resonance. To offset this lack of information, we have to use the principle of pion-pole dominance very extensively, far beyond what can be justified on the basis of relative distances of singularities. We maintain that even this extreme use of pion-pole dominance deserves exploration, since the limits to its applicability are hardly known at present.[4]

In Sec. 2, we derive a set of sum rules involving the off-mass-shell vertex $\langle\pi|\sigma|\pi\rangle$, to be defined below. As a preliminary test, we consider these sum rules in a model of single-particle dominance, and obtain results acceptable for the conjectured σ meson.[5] In Sec. 3, we abandon the preliminary single-particle model and, in the approximation of elastic unitarity for $(\pi\pi)$ scattering, we convert the sum rules into an integral equation for the vertex $\langle\pi|\sigma|\pi\rangle$, which leads to the inequality $0<\delta_0<\pi$ for the $(\pi\pi)$ S-wave phase shift δ_0 in the isospin-zero channel, wherever it is approximately valid. Exact solutions of the approximate integral equation, however, cannot be constructed. In Sec. 4, we apply the N/D formalism and find that our partially conserved axial-vector current (PCAC)-approximate analysis of the vertex would lead to an N function in the $(\pi\pi)$ scattering amplitude without a left-hand cut. To correct for this, we introduce an effective interaction pole, which at the same time will serve as a cutoff function in the integral equation. The solutions for δ_0 give a scattering length of $0.23/m_\pi\sim0.33/m_\pi$, which is somewhat larger than Weinberg's result,[3] and a broad maximum at 700 MeV. Its height is sensitive to the cutoff

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[1] Detailed references are given in recent monographs on current algebras; see S. L. Adler and R. F. Dashen, *Current Algebra* (W. A. Benjamin, Inc., New York, 1968); B. Renner, *Current Algebras and Their Applications* (Pergamon Press, Inc., New York, 1968).
[2] H. J. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967).
[3] T. Akiba and K. Kang, Phys. Letters **25B**, 35 (1967); Phys. Rev. **164**, 1836 (1967); T. Becherrawy, Rochester Report No. UR-875-174, 1967 (unpublished); L. S. Brown and R. L. Goble, Phys. Rev. Letters **20**, 345 (1968); A. Donnachie, Nuovo Cimento **53**, 932 (1968); N. H. Fuchs, Phys. Rev. **155**, 1785 (1967); D. F. Greenberg, Chicago Report No. EFINS 66-69 (unpublished); J. Iliopoulos, Nuovo Cimento **52A**, 192 (1967); **53**, 552 (1968); N. N. Khuri, Phys. Rev. **153**, 1477 (1967); J. Kwiecinski, Cracow Report No. PAK-NP 496/PL (unpublished); F. T. Meiere, Phys. Rev. **159**, 1467 (1967); J. Sucher and C. H. Woo, Phys. Rev. Letters **18**, 723 (1967); S. Weinberg, *ibid.* **17**, 616 (1966).
[4] S. G. Brown and G. B. West, Phys. Rev. Letters **19**, 812 (1967) and Phys. Rev. **168**, 1605 (1968); B. Renner, Cambridge Report, 1967 (unpublished).
[5] Section 2 of this paper presents the results of an unpublished report by two of the authors: A. M. S. Amatya and B. Renner.

parameter. With our phase shifts, we can saturate the Adler-Weisberger relation for $(\pi\pi)$ scattering with a reasonable cutoff value.

2. SUM RULES ON PION S-WAVE INTERACTIONS

In low-energy theorems, the equal-time commutators of axial-vector charges and axial-vector divergences,

$$\left[\int A_0{}^i(x)(d^3x),\partial^\mu A_\mu{}^j(0)\right]\equiv i\sigma^{ij}(0)\,, \qquad (1)$$

present themselves in describing the emission or absorption of two soft pions. Assuming Gell-Mann's commutators of the axial-vector charges,

$$\left[\int A_0{}^i(x)d^3x,\int A_0{}^j(y)d^3y\right]=ie^{ijk}\int V_0{}^k(x)d^3x\,, \quad (2)$$

one deduces from the conservation of the isospin current $V_\mu{}^k$ that $\sigma^{ij}(0)$ has to be symmetric in its isospin indices (ij). Following the suggestions of the quark model and of the σ model, it was conjectured[6] that only an isoscalar component is present in σ^{ij}, i.e.,

$$\sigma^{ij}(0)=\delta^{ij}\sigma(0)\,. \qquad (3)$$

Although Eq. (3) has not yet been directly confirmed, it is consistent with all applications of Eqs. (1) and (2): low-energy $(\pi\pi)$ scattering,[6,7] pion electromagnetic mass differences,[8] and nonleptonic K-meson decays.[9] From Eq. (3), the commutator

$$\left[\int A_0{}^i(x)d^3x,\sigma(0)\right]=-i\partial^\mu A_\mu{}^i(0) \qquad (4)$$

can be derived using Eq. (2) and the Jacobi identity.

In this paper, we want to extract information from the off-shell vertex $\langle\pi^i(q_1)|\sigma(0)|\pi^j(q_2)\rangle$:

$$\delta^{ij}f_\sigma(q_1{}^2,q_2{}^2;t)=-(m_\pi{}^2-q_1{}^2)(m_\pi{}^2-q_2{}^2)/F_\pi{}^2m_\pi{}^4 \times\int\int e^{iq_1x}e^{-iq_2y}\langle0|T(\partial^\mu A_\mu{}^i(x),\partial^\nu A_\nu{}^j(y),\sigma(0))|0\rangle d^4xd^4y \quad (5)$$

[$t=(q_1-q_2)^2$; F_π defined by $\langle0|A_\mu{}^i(0)|\pi^j\rangle=i\delta^{ij}F_\pi p_\mu{}^\pi$; $|F_\pi|\approx90$ MeV].

Pion-pole dominance for the integrand in Eq. (5) asserts that, for $q_1{}^2$ and $q_2{}^2$ not too far away from $m_\pi{}^2$, the variation of f_σ with $q_1{}^2$ and $q_2{}^2$ may be neglected.[7] We write the off-mass-shell vertex $f_\sigma(q_1{}^2,q_2{}^2;t)$ as a product of the on-mass-shell vertex $F_\sigma(t)=f_\sigma(m_\pi{}^2,m_\pi{}^2;t)$ and a correction factor

$$f_\sigma(q_1{}^2,q_2{}^2;t)=F_\sigma(t)X(q_1{}^2,q_2{}^2;t)\,. \qquad (6)$$

By definition, we have $X(m_\pi{}^2,m_\pi{}^2;t)=1$, and we know that $X(q_1{}^2,q_2{}^2;\ t)\approx1$, for $q_1{}^2\approx m_\pi{}^2$ and $q_2{}^2\approx m_\pi{}^2$. We do not know how far the region extends, where X may be approximated by unity. Conventional PCAC asserts that $X\approx1$ is reasonable for $q_1{}^2=0$, $q_2{}^2=0$ and $t=O(m_\pi{}^2)$. We shall try to keep $X\approx1$ in as large a domain as possible and needed. This assumption defines the model.[10]

In Eq. (5), we make the usual partial integration with respect to y, use Eqs. (3) and (4), and take $(q_2)_\mu=0$:

$$\begin{aligned}\delta^{ij}f_\sigma(t,0;t)&=(-i)\frac{(m_\pi{}^2-t)}{m_\pi{}^2F_\pi{}^2}\\&\quad\times\int e^{iq_1x}[\langle0|T(\partial^\mu A_\mu{}^i(x),\partial^\nu A_\nu{}^i(0))|0\rangle \qquad (7)\\&\quad-\delta^{ij}\langle0|T(\sigma(x),\sigma(0))|0\rangle]d^4x\\&=\delta^{ij}F_\sigma(t)X(t,0;t)\equiv\delta^{ij}F_\sigma(t)x(t)\,.\end{aligned}$$

We introduce intermediate states into the propagators and, in line with the pion-pole-dominance principle, we keep only the one-pion state in the pseudoscalar propagator[11] and a yet unspecified continuum in the scalar propagator:

$$F_\sigma(t)x(t)=-m_\pi{}^2+\frac{(m_\pi{}^2-t)}{m_\pi{}^2F_\pi{}^2}\int_{4m_\pi{}^2}^\infty\frac{ds\,\rho_\sigma(s)}{s-t}\,, \qquad (8)$$

where

$$\rho_\sigma(s)=\textstyle\sum_n\delta^3(p_n-q_1)\delta(m_n{}^2-s)|\langle0|\sigma|n\rangle|^2;$$

the summation over states does not include the vacuum.

Obviously, Eq. (8) can only become useful when $x(t)$ has been specified, if possible, by setting $x(t)=1$. This may raise the objection that to this level of accuracy one should consistently also abandon the integral term in Eq. (8), because it carries a factor $m_\pi{}^2-t$ as $x(t)-1$ does. As a counterargument, we observe that through Eq. (8) a unitarity cut in the $(\pi\pi)$ channel is introduced, which, starting at $4m_\pi{}^2$, is presumably the next important singularity for the off-shell vertex after the pion poles,[12] whereas the nature of the PCAC corrections in $x(t)$ is unclear at present. In terms of diagrams, we would say that by setting $x(t)=1$ we keep the pion-pole terms and any t-dependent structure in the σ channel, as displayed in Fig. 1. Figure 2 shows a dia-

[6] S. Weinberg, Ref. 3.

[7] N. N. Khuri, Ref. 3.

[8] T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, Phys. Rev. Letters 18, 759 (1967); C. L. Cook, L. E. Evans, M. Y. Han, N. R. Lipshutz, and N. Straumann, Phys. Rev. Letters 20, 295 (1968).

[9] L. S. Kisslinger, Phys. Rev. Letters 18, 861 (1967); D. A. Nutbrown, Cambridge Report, 1967 (unpublished).

[10] This assumption is related in spirit to the technique of Schnitzer and Weinberg (Ref. 2), factoring out propagators and assuming constant or slowly varying proper vertices.

[11] Strictly speaking, we should have another PCAC correction here from the three-pion cut; we neglect it.

[12] This is why we preferred the off-shell extrapolation defined by Eq. (5) to a successive reduction of the two pions, the latter one fails to produce the scalar propagator.

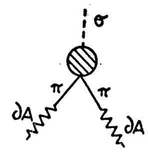

FIG. 1. For this class of graphs, $X(t)=1$ is satisfied; the criterion is the requirement that the π lines do not interact before meeting.

gram which we would neglect in this approximation; we shall correct for its effects in Sec. 4.

To see if such an approximation is at all reasonable, we test Eq. (8) in a single-particle model, using the conjectured σ resonance, which will be introduced as a pole into $F_s(t)$ and as the dominant single-particle state into $\rho_\sigma(s)$:

$$\rho_\sigma(s)\approx\delta(s-m_\sigma^2)|\langle 0|\sigma(0)|\sigma\rangle|^2=\delta(s-m_\sigma^2)g_\sigma^2, \quad (9)$$

$$F_\sigma(t)\approx g_\sigma G_{\sigma\pi\pi}/(t-m_\sigma^2). \quad (10)$$

We introduce Eqs. (9) and (10) into Eq. (8) with $x(t)=1$, compare coefficients in t, and obtain

$$g_\sigma^2\approx m_\pi^4F_\pi^2, \quad (11)$$

$$F_\pi G_{\sigma\pi\pi}\approx(m_\sigma^2-m_\pi^2). \quad (12)$$

Equation (11) is equivalent to interpreting $[\int A_0^i(x)\times d^3x,\varphi_\pi(y)]=\sigma(y)/m_\pi^2F_\pi$ as the interpolating σ field[13]; Eq. (12) can also be derived in the σ model[14] in lowest-order perturbation theory. A combination of the two equations can also be derived from somewhat weaker assumptions (Appendix A). The width of σ is predicted by Eq. (12) as a function of its mass.

$$\Gamma_\sigma=\tfrac{3}{2}(1/8\pi)(G_{\sigma\pi\pi}/m_\sigma)^2p_\pi, \quad (13)$$

as shown in Table I. These values are consistent with the results of Brown and Singer,[15] recently confirmed through current-algebra sum rules,[16] but they would also allow a substantially broader state at a higher mass.[17]

If the Adler-Weisberger relation for $(\pi\pi)$ scattering[18] is saturated with σ, ρ, and f mesons,[16,18] the σ state is required to contribute 60%, with

$$F_\pi^2[G_{\sigma\pi\pi}/(m_\sigma^2-m_\pi^2)]^2\approx 0.6. \quad (14)$$

Equation (12) provides 100% saturation through the σ state; this is not very far from (14).

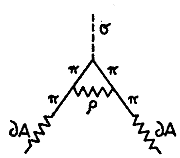

FIG. 2. Example of a graph which does not satisfy $X(t)=1$.

[13] We wish to thank Dr. R. J. Oakes for this remark.
[14] M. Gell-Mann and J. Lévy, Nuovo Cimento 16, 705 (1960).
[15] L. M. Brown and P. Singer, Phys. Rev. 133, B812 (1964).
[16] K. Kawarabayashi, W. D. McGlinn, and W. W. Wada, Phys. Rev. Letters 22, 332 (1967); G. Furlan and C. Rossetti, Phys. Letters 23, 499 (1967).
[17] C. Lovelace, R. M. Heinz, and A. Donnachie, Phys. Letters 22, 332 (1967).
[18] S. L. Adler, Phys. Rev. 140, B736 (1965).

TABLE I. Γ_σ as a function of m_σ, according to Eq. (13).

m_σ (MeV)	Γ_σ (MeV)
350	70
400	130
450	220
500	330
550	470

3. INTEGRAL EQUATION FOR PION S-WAVE SCATTERING

Strictly speaking, the single-particle model of Sec. 2 is not a solution of Eq. (8), because it ignores unitarity corrections. The only purpose of this model was to demonstrate that the approach is not misleading. Now we abandon the single-particle model and introduce a continuum of intermediate states into $\rho_\sigma(s)$. The two-pion contribution[19] comes out proportional to $|F_\sigma(s)|^2$; explicitly,

$$\rho_\sigma^{(2\pi)}(s)=\frac{3}{32\pi^2}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}|F_\sigma(s)|^2. \quad (15)$$

For $s>(4m_\pi)^2$, there are also inelastic contributions. We account for these by introducing a factor $R(s)$:

$$\rho_\sigma(s)=R(s)\rho_\sigma^{(2\pi)}(s), \quad (16)$$

with $R(s)=1$, for $s<(4m_\pi)^2$, and $R(s)>1$, for $s>(4m_\pi)^2$. We obtain from Eq. (8)

$$F_\sigma(t)x(t)=-m_\pi^2+\frac{3(m_\pi^2-t)}{32\pi^2m_\pi^2F_\pi^2}\int_{4m_\pi^2}^{\infty}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}\times R(s)|F_\sigma(s)|^2\frac{ds}{s-t}. \quad (17)$$

For convergence of the integral, we require $\lim_{s\to\infty}F_\sigma(s)=0$.[20] Further information on $F_\sigma(t)$ can be obtained from the linearized unitarity relations for vertex functions, which allow us to represent $F_\sigma(t)$ as an Omnès function:

$$F_\sigma(t\pm i\epsilon)=-m_\pi^2\times\exp\left(\frac{t-m_\pi^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds\,\delta_\sigma(s)}{(s-t\mp i\epsilon)(s-m_\pi^2)}\right)=-|F_\sigma(t)|e^{\pm i\delta_\sigma(t)}. \quad (18)$$

For $t<(4m_\pi)^2$, the phase $\delta_\sigma(t)$ is equal to the $(\pi\pi)$ S-wave scattering phase $\delta_0(t)$ in the isospin-zero channel, and this relation remains approximately true as long as inelasticity can be neglected. We compare the dis-

[19] In Ref. 5, a factor of $\frac{1}{2}$ was omitted in the corresponding Eq. (9). This had no consequences there.
[20] At $t\to\infty$, if $x(t)F_\sigma(t)\to\infty$, we reproduce a sum rule of C. H. Woo [Phys. Rev. Letters 19, 537 (1967)], which the author has proposed by assuming asymptotic chiral invariance. His failure to saturate his sum rule may be connected to our difficulties in solving Eq. (17).

continuities of $F_\sigma(t)$ according to Eqs. (17) and (18) [assuming that any discontinuity of $x(t)$ may be neglected]:

$$2\pi i\frac{3}{32\pi^2}\frac{(m_\pi^2-t)}{m_\pi^2F_\pi^2}\left(\frac{t-4m_\pi^2}{t}\right)^{1/2}\frac{R(t)}{x(t)}|F_\sigma(t)|^2 = -2i\sin\delta_\sigma(t)|F_\sigma(t)|\,, \quad (19)$$

and find that, as long as $x(t)>0$,

$$\sin\delta_0(t)\approx\sin\delta_\sigma(t)>0,\quad 0\lesssim\delta_0(t)\lesssim\pi. \quad (20)$$

We emphasize that this result requires only local validity of our assumptions in the range of t under consideration; it is not affected by the difficulties and ambiguities which we shall encounter in trying to solve Eq. (17).

To solve Eq. (17), we make the familiar ansatz

$$F_\sigma(t)=-m_\pi^2/D_\sigma(t) \quad (21)$$

and find the discontinuity of $D_\sigma(t)$:

$$\mathrm{disc}D_\sigma(t)=2i\,\mathrm{Im}D_\sigma(t) = (2\pi i)\frac{3}{32\pi^2}\frac{(m_\pi^2-t)}{F_\pi^2}\left(\frac{t-4m_\pi^2}{t}\right)^{1/2}\frac{R(t)}{x(t)}, \quad (22)$$

for $t>4m_\pi^2$.

In Appendix B, we show that with $x(t)=1$, $F_\sigma(t)$ has no zeros, so that there should not be any Castillejo-Dalitz-Dyson poles in $D_\sigma(t)$. Using its discontinuity in Eq. (22), we can represent $D_\sigma(t)$ as a dispersion integral, subtracted at $t=m_\pi^2$ with $D_\sigma(m_\pi^2)=1$. The addition of arbitrary subtraction polynomials is limited, because they would lead to superconvergence of $F_\sigma(t)$: $\lim_{t\to\infty} tF_\sigma(t)=0$; this is not compatible with a negative definite $\mathrm{Im}F_\sigma(t)$ [unless $x(t)$ changes sign].

Integrating $D_\sigma^{(1)}(t)$ with $R(t)=x(t)=1$ (neglecting inelasticity and PCAC corrections) leads into difficulties: We would obtain[21]

$$\begin{aligned}
D_\sigma^{(1)}(t)&=1+a(m_\pi^2-t)+(m_\pi^2-t)^2\frac{3}{32\pi^2}\frac{1}{F_\pi^2}\int_{4m_\pi^2}^{\infty}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}\frac{ds}{(m_\pi^2-s)(s-t)}\\
&=1+(m_\pi^2-t)\left\{a'+\frac{3}{32\pi^2F_\pi^2}\left(\frac{t-4m_\pi^2}{t}\right)^{1/2}\ln\left(\frac{(4m_\pi^2-t)^{1/2}-(-t)^{1/2}}{(4m_\pi^2-t)^{1/2}+(-t)^{1/2}}\right)\right\}\quad\text{for } t<0\\
&=1+(m_\pi^2-t)\left\{a'-\frac{3}{16\pi^2F_\pi^2}\left(\frac{4m_\pi^2-t}{t}\right)^{1/2}\arctan\left[\left(\frac{t}{4m_\pi^2-t}\right)^{1/2}\right]\right\}\quad\text{for } 0<t<4m_\pi^2\\
&=1+(m_\pi^2-t)\left\{a'+\frac{3}{32\pi^2F_\pi^2}\left(\frac{t-4m_\pi^2}{t}\right)^{1/2}\left[\ln\left(\frac{t^{1/2}-(t-4m_\pi^2)^{1/2}}{t^{1/2}+(t-4m_\pi^2)^{1/2}}\right)+i\pi\right]\right\}\\
&\qquad\text{for } t=t_R\pm i\epsilon,\ t_R>4m_\pi^2. \quad (23)
\end{aligned}$$

The unknown subtraction constant a' has been redefined from a in the course of the calculation. It is easy to see that none of these solutions is acceptable. At $t=m_\pi^2$, we have $D_\sigma(m_\pi^2)=1$, at $t\to-\infty$ the last term dominates and we obtain $D_\sigma^{(1)}(t)\to-\infty$. So $D_\sigma^{(1)}(t)$ must have a zero, which would lead to a pole of $F_\sigma(t)$ not contained in Eq. (17). By keeping $R(s)\geq1$, the negative term is only enhanced. So we conclude that Eq. (17) has no solutions with $x(t)=1$.

4. CORRECTIONS TO PION-POLE DOMINANCE

Being forced to introduce corrections to pion-pole dominance, it will be our aim for this paper to keep the model simple and to avoid having too many undetermined and unmotivated parameters. In comparing Eq. (22) with the usual N/D equations of $(\pi\pi)$ S-wave scattering, we have

$$T_{l=0,I=0}(t)=32\pi\left(\frac{t}{t-4m_\pi^2}\right)^{1/2}e^{i\delta_0(t)}\sin\delta_0(t)=\frac{N(t)}{D(t)}, \quad (24)$$

$$\mathrm{disc}D(t)=2i\frac{(-1)}{32\pi}\left(\frac{t-4m_\pi^2}{t}\right)^{1/2}N(t). \quad (25)$$

With $D(t)$ given by $D_\sigma(t)$, we would have a linear N function

$$N_\sigma(t)=(-3)\frac{(m_\pi^2-t)}{F_\pi^2}\left|\frac{R(t)}{x(t)}\right| \quad (26)$$

up to inelasticity and corrections to PCAC. This may well be a good approximation for t in the neighborhood of m_π^2, but it fails at negative t, where $N(t)$ should have its left-hand cut. A linear rise of $N(t)$ at large t appears also unlikely. On these grounds it seems reasonable to correct $N(t)$ by introducing a factor $R(t)/x(t) =(m^2+m_\pi^2)/(m^2+t)$ to simulate some effects of the omitted left-hand cut for large t in the integration region $t>4m_\pi^2$. Not to distort the current-algebra predictions for small t, we should choose $m^2\gg m_\pi^2$. As to the size of m^2, we have taken different values in the ρ-exchange region and above $m=600$, 760, 1000, 1500, and 2000 MeV. We do not want to commit ourselves to a final statement on this point and, for the moment, prefer to regard m^2 as some cutoff parameter. The precise nature

[21] This integral is related to the finite part of the $(\pi\pi)$ self-energy graph.

of the required PCAC correction will be investigated further; we shall try to exploit the role of Eq. (17) as the unitarity equation for T, approximated in terms of F_σ at low energies.[6] The emergence of this unitarity-like equation from current algebra and PCAC is worth noting.

Introducing this cutoff function, we can integrate a corrected function

$$\begin{aligned} D_\sigma^{(m)}(t) &= 1+(m_\pi^2-t)\frac{3}{32\pi^2}\frac{1}{F_\pi^2}\int_{4m_\pi^2}^{\infty}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}\frac{m^2+m_\pi^2}{m^2+s}\frac{ds}{s-t} \\ &= 1+\frac{3(m^2+m_\pi^2)(m_\pi^2-t)}{32\pi^2F_\pi^2(m^2+t)}\left[\left(\frac{4m_\pi^2+m^2}{m^2}\right)^{1/2}\ln\left(\frac{(4m_\pi^2+m^2)^{1/2}+m}{(4m_\pi^2+m^2)^{1/2}-m}\right)-\left(\frac{t-4m_\pi^2}{t}\right)^{1/2}\ln\left(\frac{(4m_\pi^2-t)^{1/2}+(-t)^{1/2}}{(4m_\pi^2-t)^{1/2}-(-t)^{1/2}}\right)\right] \\ &\qquad\qquad \text{for } t<0 \\ &= 1+\frac{3(m^2+m_\pi^2)(m_\pi^2-t)}{32\pi^2F_\pi^2(m^2+t)}\left\{\left(\frac{4m_\pi^2+m^2}{m^2}\right)^{1/2}\ln\left(\frac{(4m_\pi^2+m^2)^{1/2}+m}{(4m_\pi^2+m^2)^{1/2}-m}\right)-2\left(\frac{4m_\pi^2-t}{t}\right)^{1/2}\arctan\left[\left(\frac{t}{4m_\pi^2-t}\right)^{1/2}\right]\right\} \\ &= 1+\frac{3(m^2+m_\pi^2)(m_\pi^2-t)}{32\pi^2F_\pi^2(m^2+t)}\left\{\left(\frac{4m_\pi^2+m^2}{m^2}\right)^{1/2}\ln\left(\frac{(4m_\pi^2+m^2)^{1/2}+m}{(4m_\pi^2+m^2)^{1/2}-m}\right)\right. \\ &\qquad \left.-\left(\frac{t-4m_\pi^2}{t}\right)^{1/2}\left[\ln\left(\frac{t^{1/2}+(t-4m_\pi^2)^{1/2}}{t^{1/2}-(t-4m_\pi^2)^{1/2}}\right)\mp i\pi\right]\right\} \\ &\qquad\qquad \text{for } t=t_R\pm i\epsilon,\ t_R>4m_\pi^2. \end{aligned} \tag{27}$$

For $t<4m_\pi^2$, $D_\sigma^{(m)}(t)$ is real and has no zeros (in the range of m considered). For $t>4m_\pi^2$, neglecting inelasticity, its phase is the negative of the s-wave $(\pi\pi)$ phase shift δ_0 in the isospin-zero channel.

$$\delta_0^{(m)}(t)\approx -\arg D_\sigma^{(m)}(t) \qquad \text{for } t>4m_\pi^2. \tag{28}$$

In Fig. 3, we have plotted $\delta_0^{(m)}(t)$ for different values of m. The resulting scattering lengths

$$\begin{aligned} a(m) &= \lim_{t\to 4m_\pi^2}\frac{(-2)\,\mathrm{Im}D_\sigma^{(m)}(t)}{(t-4m_\pi^2)^{1/2}\,\mathrm{Re}D_\sigma^{(m)}(t)} \\ &\approx \frac{\pi}{32\pi^2F_\pi^2/9m_\pi^2-2\ln(m/m_\pi)}\frac{1}{m_\pi} \qquad \text{for } m/m_\pi\gg 1 \end{aligned} \tag{29}$$

come out somewhat larger than Weinberg's value: $0.23/m_\pi \sim 0.33/m_\pi$. They are given in Table II. The phase shift exhibits a broad maximum at about 700 MeV falling off very slowly at larger energies. This shape resembles qualitatively the results of Lovelace, Heinz, and Donnachie,[17] but we prefer to reserve our opinion at present, because the height of the maximum is sensitive to the cutoff parameter M, and there is every reason to regard the tail at $t \gtrsim m^2$ as cutoff-dependent.

TABLE II. S-wave scattering length a_0 as a function of the cutoff mass m.

m (MeV)	a_0 (m_π^{-1})
600	0.231
760	0.252
1000	0.276
1500	0.307
2000	0.330

Using the calculated values of $\delta_0^{(m)}(t)$ in dependence of m, we examine the saturation of the Adler-Weisberger relation for $(\pi\pi)$ scattering[18] with the resonances ρ and f (contributing about 40%) and the isospin-zero s-wave continuum. We require,[22] therefore,

$$0.60\approx |F_\pi|^2\int_{4m_\pi^2}^{\infty}\frac{32}{3}\frac{ds}{(s-m_\pi^2)^2}\left(\frac{s}{s-4m_\pi^2}\right)^{1/2}\times\sin^2\delta_0^{(m)}(s)ds\,, \tag{30}$$

and find approximate saturation for $m\approx 1300$ MeV (Fig. 4). This value may appear reasonable on independent dynamical grounds. The sum rule of Woo[20] is not satisfied by our solution.

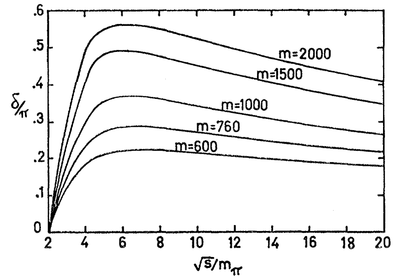

FIG. 3. Pion S-wave phase shift as a function of the cutoff mass in MeV.

[22] In this application, Adler's soft-pion corrections cancel each other and we may use the uncorrected sum rule.

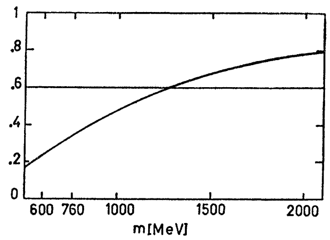

FIG. 4. Saturation of the Adler-Weisberger sum rule. The graph represents the right-hand side of Eq. (30) versus the cutoff mass m.

Finally, to test the dependence of our result on the shape of the cutoff function, we have repeated our calculation with a few different cutoff functions $[R(t)/x(t)]$:

(a) two poles $\left(\frac{2M^2}{M^2+t}-\frac{m^2}{m^2+t}\right)$;

(b) pole and dipole, $\frac{1}{1-\beta}\left(\frac{M^2}{M^2+t}-\frac{\beta M^4}{(M^2+t)^2}\right)$,

$$0<\beta<1.$$

Qualitatively, the shapes of the phase shifts are the same as before. Again we find a maximum near 700 MeV, its height being dependent on the parameters.

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APPENDIX A

We consider the vertex $\langle\pi^i(q)|\sigma(0)|\pi^j(p)\rangle$ with one pion off its mass shell:

$$\delta^{ij}f(q^2,m_\pi^2,t)=i\int e^{iqx}\frac{(m_\pi^2+q^2)}{F_\pi m_\pi^2}\langle 0|T(\partial^\mu A_\mu^i(x),\sigma(0))|\pi^j(p)\rangle d^4x, \quad \text{(A1)}$$

$$\delta^{ij}f(0,m_\pi^2,m_\pi^2)=-i\frac{1}{F_\pi}\times\left\langle 0\left|\left[\int d^3x\, A_0^i(x),\sigma(0)\right]\right|\pi^j(p)\right\rangle=-\delta^{ij}m_\pi^2. \quad \text{(A2)}$$

Dispersing (A1) in the variable $\nu=(p\cdot q)/m_\pi$ at fixed $q^2=0$, we find

$$f(0,m_\pi^2;m_\pi^2)=\frac{1}{2\pi F_\pi}\int\frac{d\nu'}{\nu'}\times(2\pi)^4\sum_n\left\{(-1)\left\langle 0\left|\sigma(0)\frac{|n\rangle\langle n|}{(2\pi)^3 2E_n}\partial^\mu A_\mu^i\right|\pi^j(p)\right\rangle\delta^4(p_n+q'-p)\right\}. \quad \text{(A3)}$$

Keeping only the σ state, we get

$$-F_\pi m_\pi^2=\frac{1}{m_\sigma^2-m_\pi^2}\frac{g_\sigma G_{\sigma\pi\pi}F_\pi m_\pi^2}{-m_\pi^2}, \quad \text{(A4)}$$

$$g_\sigma G_{\sigma\pi\pi}=(m_\sigma^2-m_\pi^2)m_\pi^2. \quad \text{(A5)}$$

APPENDIX B

To show that $F_\sigma(t)\neq 0$ for $X(t)=1$, subtract

$$F_\sigma(\infty)=0=-m_\pi^2+\frac{3}{32\pi^2m_\pi^2F_\pi^2}\int_{4m_\pi}^{\infty}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}\times R(s)|F_\sigma(s)|^2 \quad \text{(B1)}$$

from Eq. (17),

$$F_\sigma(t)=\frac{3}{32\pi^2m_\pi^2F_\pi^2}\int_{4m_\pi^2}^{\infty}\left(\frac{s-4m_\pi^2}{s-t}\right)^{1/2}\times R(s)\left(\frac{m_\pi^2-s}{s-t}\right)|F_\sigma(s)|^2ds. \quad \text{(B2)}$$

For real $t<4m_\pi^2$, $F_\sigma(t)$ is always negative, because the integrand is negative definite. For complex t, one uses

$$\operatorname{Im}F_\sigma(t)=\frac{3}{32\pi^2m_\pi^2F_\pi^2}\int_{4m_\pi^2}^{\infty}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}\times R(s)\frac{(m_\pi^2-s)|F_\sigma(s)|^2}{(s-\operatorname{Re}t)^2+(\operatorname{Im}t)^2}(\operatorname{Im}t)ds,$$

$$\neq 0 \text{ if } \operatorname{Im}t\neq 0. \quad \text{(B3)}$$