Reggeization of Quark Number

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In earlier work the Reggeization of approximate dynamical groups has been presented as a calculational method for classifying particles and evaluating S-matrix elements at high energies. In continuation of this work, an especially simple model is considered where just one invariant of the higher $U(6)\times U(6)$ approximate symmetry, quark-plus-antiquark number, is Reggeized. The resulting classification of particles (according to their quark content) into exploding supermultiplets of spin and unitary spin, and the formulas for computing S-matrix elements, are given for high energies, where an exchange of an N-plane trajectory in the cross channel may be expected to dominate the scattering. The hope is that this analysis may help reduce the large number of parameters now used in Regge theory by combining Regge ideas with higher symmetries. The type of Fourier expansion on a higher approximate symmetry group and the Regge technique used here for evaluating asymptotic behavior may possess wider applications than the case considered in this paper.

1. REGGE MODEL OF HIGHER SYMMETRIES

THE Regge method in strong-interaction physics originated in the study of the S matrix for complex values of angular momentum and has recently met with a certain number of successes in describing elastic and inelastic two-body reactions. Even where it has succeeded, however, it has been necessary to admit a large number of residue parameters with no guiding principle to limit their arbitrariness. A similar situation prevailed in the absorption-model description of low-energy scattering; recently, however, higher supermultiplet theories [and in particular $U(6,6)$] were used with fair success to constrain strongly the values of the coupling-constant parameters[1] that entered into the Born approximation. One may expect that a marriage of supermultiplet schemes with Regge theory would be desirable in that it may suitably reduce the number of Regge parameters. We shall describe below one attempt[2] at obtaining these correlations based upon a supermultiplet scheme that Reggeizes the quark number.

The basis of our scheme is the following. Angular momentum is but one of the conserved quantities on which the S matrix depends. In particular, if a system possesses a higher spin-containing symmetry, there may be other conserved quantities (Casimir invariants of the relevant symmetry group) which it may be more profitable to continue to complex values and Reggeize. For example, with the hydrogen atom it is well known that one obtains a deeper insight into the dynamics of the bound states if it is the principal quantum number [connected with the well-known $O(4)$ symmetry of the system] that is Reggeized rather than the angular momentum. For hadron physics the $U(6)\otimes U(6)$ group appears to be an *approximate* symmetry for classification of particles. The analogy of the principal quantum number for the hydrogen case here would seem to be with the total quark number N (half the number of quarks plus antiquarks) and an analogous Reggeization of this number appears to be indicated. One may now go further and explore the dynamical consequences for high-energy scattering of such a Reggeization procedure and it is this aspect of the scheme in terms of its practical applications which we wish to stress in this paper.[3]

The consequences of the scheme are twofold:

(i) One obtains two master trajectories [plots of ReN versus (mass)^2], one for mesons ($B=0$) and one for fermions ($B=1$). For $M^2>0$, ReN goes through 1, 2, 3, $\cdots$ for mesons and $\frac{3}{2}$, $\frac{5}{2}$, $\cdots$ for fermions. On present evidence it is not excluded that this simple picture of Regge recurrences classified according to quark content can accommodate all known semistable meson and baryon states. The idea that there should be basically only one baryon and one meson entity was proposed long ago by Weisskopf.

(ii) To evaluate the high-energy behavior of scattering amplitudes, we make the Regge assumption that the amplitude is dominated by the contributions from an exchange in the crossed channel of these master trajectories. The residue functions automatically satisfy $U_W(6)$ invariance.

It appears that this Regge model will provide a reasonably restrictive theoretical framework for analysis

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[1] R. Migneron and K. Moriarty, Phys. Rev. Letters 18, 978 (1967).

[2] Abdus Salam and J. Strathdee, Phys. Rev. Letters 19, 339 (1967) (referred to hereafter as I); R. Delbourgo, Abdus Salam, M. A. Rashid, and J. Strathdee, Phys. Rev. 170, 1477 (1968) (referred to hereafter as II).

[3] R. Delbourgo, M. A. Rashid, Abdus Salam, and J. Strathdee, in *Proceedings of the International Seminar in High-Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna, 1965), p. 455.

of experimental data. Naturally, this theory will not provide any antidote to the obvious failures of conventional Regge techniques nor will it provide a fundamental answer to the unitarity difficulties which beset supermultiplet schemes. But it does give the possibility of building unitarity into the formalism as this is always done in Regge theory, i.e., mainly through the signature factor.[4] The new formalism will, however, certainly provide relations between presently used Regge residue parameters.

2. PARTIAL-WAVE EXPANSION IN $U(6)\otimes U(6)$

The basic ideas of the approach were described in I and II. Here we shall present a simplified version of the generalized expansion technique, proceeding by direct analogy with the conventional partial-wave expansion of the S matrix. The conventional partial-wave expansion can be understood either as a consequence of rotation invariance of the S matrix—and this of course is the deeper point of view—or, alternatively, as a *mathematical expansion* in terms of an appropriately chosen complete set of functions. It is this latter point of view that we wish to stress in this paper.

The rotation symmetry of the S matrix manifests itself in the following ways:

(a) Particles at rest group themselves into $(2J+1)$-component multiplets of $SU(2)_J$. (If the masses of the particles vary with J, one has a strong suggestion towards grouping them on a Regge trajectory.)

(b) A three-point function with all particles confined to the 0-3 plane shows helicity conservation:

$$\langle\lambda|T(E)|\lambda_1\lambda_2\rangle=\delta_{\lambda,\lambda_1-\lambda_2}T_{\lambda_1\lambda_2}(E). \quad (2.1)$$

(c) A four-point function with all particles also confined collinearly (forward scattering) shows net helicity conservation:

$$\langle\lambda_3\lambda_4|T(E)|\lambda_1\lambda_2\rangle=\delta_{\lambda_3-\lambda_4,\lambda_1-\lambda_2}T_{\lambda_3\lambda_4,\lambda_1\lambda_2}(E). \quad (2.2)$$

Suppose now that we are dealing with a nonforward scattering amplitude with the final particles rotated through an angle θ out of the 0-3 plane. We can always extract the angular dependence of $T(E,\theta)$ by expanding in a complete set of orthonormal (square integrable) functions as follows:

$$T(E,\theta)=\sum_n T_n(E)f_n(\theta).$$

The completeness of f_n means that a one-one correspondence between T_n and $T(\theta)$ exists. If we know nothing about the rotational invariance of the S matrix but simply that conditions (b) and (c) hold as empirical experimental facts, it is appropriate to choose the complete set of functions f_n to be the two-labelled function $d_{\lambda\lambda'}{}^J(\theta)$, satisfying $d_{\lambda\lambda'}{}^J(0)=\delta_{\lambda\lambda'}$; as one well knows, a class of such functions is given by the rotation functions of $SU(2)_J$. Thus one writes the mathematical expansion

$$\langle\lambda_3\lambda_4|T(E,\theta)|\lambda_1\lambda_2\rangle=\sum_J(2J+1)\times T_{\lambda_3\lambda_4,\lambda_1\lambda_2}{}^J(E)d_{\lambda_3-\lambda_4,\lambda_1-\lambda_2}{}^J(-\theta). \quad (2.3)$$

Expressing the summation as a Sommerfeld-Watson integral, one may tie in (c) with (a) and (b) in the well-known manner by proposing that $T_{(\lambda i)}{}^J(E)$ exhibits poles in the expansion-parameter J according to

$$\langle\lambda_3\lambda_4|T(E,\theta)|\lambda_1\lambda_2\rangle=\sum_{J,\lambda,\lambda'}\delta_{\lambda_3-\lambda_4,\lambda'}g_{\lambda_3\lambda_4}{}^J\times\frac{d_{\lambda'\lambda}{}^J(-\theta)}{E^2-m_J{}^2}g_{\lambda_1\lambda_2}{}^J\delta_{\lambda,\lambda_1-\lambda_2}. \quad (2.4)$$

Let us generalize. If the rotational symmetry $SU(2)_J$ is combined with $SU(3)$ to give a possible *rest symmetry* $U(6)\otimes U(6)$ and if $U(6)\times U(6)$ was known to be the symmetry of at least a part of the S matrix—a very strong assumption and certainly false for the exact S matrix—the symmetry would manifest itself in the following ways:

(a′) Physical particles group themselves in $U(6)\otimes U(6)$ multiplets.[5,6] (If the first few representations are known it would be natural to attempt to trace a Regge trajectory through them.)

(b′) Three-point functions exhibit W-spin conservation[7] (generalized helicity conservation—see Appendix). Thus

$$\langle W|T(E)|W_1W_2\rangle=\sum_\zeta\langle\zeta W|W_1W_2\rangle T_{\zeta W_1W_2}(E), \quad (2.1')$$

where $\langle\zeta W|W_1W_2\rangle$ denotes the $U(6)_W$ Clebsch-Gordan coefficient which couples $D^{W_1}\otimes D^{W_2}$ to D^W. In general there is more than one independent coupling. It is therefore necessary to include a parameter ζ to distinguish among them.

(c′) Collinear scattering processes also exhibit $U(6)_W$ conservation:

$$\langle W_3W_4|T(E)|W_1W_2\rangle=\sum_{\zeta\zeta'W}\langle W_3W_4|\zeta'W\rangle\times T_{\zeta'\zeta}{}^W(E)\langle\zeta W|W_1W_2\rangle. \quad (2.2')$$

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[4] The absorption aspects of Regge theory arise, as is well known, mainly from the signature factor. This is because the Regge amplitude $(1\pm e^{i\pi\alpha})\beta(t)S^{\alpha(t)}/\sin\pi\alpha$ is real (and violates unitarity) for real α and β, if the crucial signature factor is not included.

[5] R. Delbourgo, Abdus Salam, and J. Strathdee, Proc. Roy. Soc. (London) **284A**, 146 (1965); B. Sakita and K. C. Wali, Phys. Rev. **139**, B1355 (1965).

[6] R. Dashen and M. Gell-Mann, Phys. Letters **17**, 142 (1965); H. Harari and H. Lipkin, Phys. Rev. **140**, B1617 (1965); P. G. O. Freund, Phys. Rev. Letters **14**, 803 (1965); R. Oehme, *ibid.* **14**, 664 (1965).

[7] K. J. Barnes, Phys. Rev. Letters **14**, 789 (1965); H. Lipkin and S. Meshkov, *ibid.* **14**, 670 (1965); R. Arnold, Phys. Rev. **162**, 1334 (1967).

(d′) Noncollinear four-point functions show conservation of coplanar symmetry $U(3)\otimes U(3)$ which has no analog for the smaller rest symmetry $SU(2)_J$.

If we accept only that (a′), (b′), (c′), and (d′) hold as empirical facts (at least to a fair approximation), we may adopt the mathematical expansion theorem attitude and express nonforward scattering amplitudes in terms of the *complete* set of suitably defined functions $d_{WW'}{}^N(\theta)$ as follows:

$$\langle W_3W_4|T(E,\theta)|W_1W_2\rangle = \sum_{N\zeta W\zeta' W'} \langle W_3W_4|\zeta' W'\rangle d_{W'W}{}^N(-\theta) \times T_{\zeta' W',\zeta W}{}^N(E)\langle \zeta W|W_1W_2\rangle. \quad (2.3')$$

To satisfy the boundary conditions (b′), (c′), and (d′), the suitable definition of $d_{WW'}{}^N(\theta)$ turns out to be that these functions are $U(6)\times U(6)$ rotation functions [$d_{WW'}{}^N(0)=\delta_{WW'}$; also $d_{WW'}{}^N(\theta)$ are diagonal in $U(3)\otimes U(3)$ labels subsumed in W] but this is indicental for our present purposes.

What exactly is the nature of the relevant Casimir invariant N of $U(6)\times U(6)$? The completeness notion used here requires that we sum over a *one-parameter* family of $U(6)\times U(6)$ representations $\mathfrak{D}^N$ since we are eliminating thereby a single angle θ.[8] Moreover, if the representations $\mathfrak{D}^N$ are nondegenerate in their $U(6)_W$ content, i.e., if a complete set of basis vectors can be labelled $|NW\rangle$, then the functions

$$d_{W'W}{}^N(-\theta)=\langle NW'|e^{i\theta J_2}|NW\rangle$$

are well defined. One may show that any square-integrable function defined over the interval $0\leqslant\theta\leqslant\pi$ and satisfying the appropriate boundary conditions at $\theta=0$, π can be expanded in terms of the $d_{WW'}{}^N(\theta)$ if we characterize the representation D^N by, for example, the symmetrized $U(6)\times U(6)$ tensors $\phi_{\alpha_1\cdots\alpha_{N+\frac{3}{2}B}}{}^{\beta_1\cdots\beta_{N-\frac{3}{2}B}}$, where B denotes the baryon number and N takes the values $\frac{3}{2}B$, $\frac{3}{2}B+1$, $\frac{3}{2}B+2$, $\cdots$, i.e., *N is the quark number*.[9]

Returning to the expansion (2.3′) finally one ties in the property (a′) by assuming that $T^N(E)$ exhibits poles in the N Casimirs corresponding to $U(6)\otimes U(6)$ bound multiplets, thereby reducing expression (2.3′) to

$$\langle W_3W_4|T(E,\theta)|W_1W_2\rangle = \sum_{N\zeta W\zeta' W'} \langle W_3W_4|\zeta' W'\rangle g_{\zeta' W' W_3W_4}{}^N \times \frac{d_{W'W}{}^N(-\theta)}{E^2-m_N{}^2} g_{\zeta W W_1W_2}{}^N\langle\zeta W|W_1W_2\rangle. \quad (2.4')$$

This is the direct analog of (2.4). The rotation functions $d_{WW'}{}^N(\theta)=\langle NW|e^{-i\theta J_2}|NW'\rangle$ which make their appearance are generalized derivatives of the Gegenbauer $C_N{}^3$ (see next section) just like the $d_{\lambda\lambda'}{}^J(\theta)$, which are generalized derivatives of the Legendre. We can now pass to the Regge amplitude by making a Sommerfeld-Watson transformation:

$$\lim_{(\cos\theta\to\infty)} \langle W_3W_4|T(E,\theta)|W_1W_2\rangle \sim \sum_{\zeta W\zeta' W'} \langle W_3W_4|\zeta' W'\rangle g_{\zeta' W' W_3W_4}{}^\alpha \frac{d_{W'W}{}^\alpha(-\theta)}{\sin\pi\alpha(E)} \times g_{\zeta W W_1W_2}{}^\alpha\langle\zeta W|W_1W_2\rangle, \quad (2.5')$$

where $\alpha(m_N{}^2)=N$ is the master trajectory function. This is of course the direct analog of the normal Reggeization procedure which yields

$$\lim_{(\cos\theta\to\infty)} \langle\lambda_3\lambda_4|T(E,\theta)|\lambda_1\lambda_2\rangle \sim\sum_{\lambda\lambda'} \delta_{\lambda_3-\lambda_4,\lambda'} g_{\lambda_3\lambda_4}{}^\alpha \frac{d_{\lambda'\lambda}{}^\alpha(-\theta)}{\sin\pi\alpha(E)} g_{\lambda_1\lambda_2}{}^\alpha \delta_{\lambda,\lambda_1-\lambda_2}. \quad (2.5)$$

Note the very close correspondence between Eqs. (2.1)–(2.5) and (2.1′)–(2.5′). If we multiply expressions (2.5) or (2.5′) by the signature factors $(1\pm e^{i\pi\alpha})$, we shall be taking some account of unitarity in the sense that absorptive effects on the high-energy amplitude are incorporated through this.

3. ROTATION FUNCTIONS IN $U(6)\otimes U(6)$

Any further progress requires a practical knowledge of the $d_{WW'}{}^\alpha(\theta)$ functions which appear in (2.5′). This section is devoted to their computation and tabulation.[2,10] The first and most direct method would be to work directly in the basis $|NW\rangle$ and to determine the $d_{WW'}{}^N(\theta)$ by setting up differential equations for them. A second, less direct but more feasible, method which we shall use instead is to work in an auxiliary relativistic basis $|A_1\cdots A_N\rangle$ and to calculate the (M-function-like)

[8] This one-one count is easy to see for the orthogonal group. For example, in $O(4)$ the generalized helicity group is $O(3)$, while $O(2)$ plays the role of the coplanar group, which means that one encounters only the flipless amplitudes $T_{s\lambda s'}(\theta)$. The expansion technique then replaces λ and θ by the two Casimir labels j_0 and σ appropriate to $O(4)$. In detail,

$$T_{J\lambda J'}(\theta)=\sum_{j_0\sigma} T_{JJ'}{}^{j_0\sigma}d_{J\lambda J'}{}^{j_0\sigma}(\theta),$$

where $d(\theta)$ are the complete set of "rotation" functions for $O(4)$. More generally, for the case of $O(\nu)$ the expansion theorem reads

$$T_{N_{\nu-1}N_{\nu-2}N_{\nu-1}'}(\theta)=\sum_{N_\nu} T_{N_{\nu-1},N_{\nu-1}'}{}^{N_\nu}d_{N_{\nu-1}N_{\nu-2}N_{\nu-1}'}{}^{N_\nu}(\theta),$$

where N_ν stand for the Casimir operators of $O(\nu)$. Since $(N_\nu)=(N_{\nu-2})+1$, the one-one count is clearly exhibited. The same is true for our case of $U(\nu)\otimes U(\nu)$.

[9] It is of course not essential to employ the set of most degenerate representations characterized by the pair of quantum numbers N and B in defining the complete set of functions However, this choice involves the least complication since the less degenerate representations of $U(6)\otimes U(6)$ are not completely labelled by the $U(6)_W$ quantum numbers and it would be necessary to formulate more involved criteria for picking out orthonormal sets of functions $d_{WW'}{}^N(\theta)$ from among all the matrix elements of $e^{-i\theta J_2}$. This course may be forced upon us, however, if physical particles cannot all be accommodated on the trajectories obtained by Reggeizing simply the quark number N.

[10] R. Delbourgo, J. Math. Phys. (to be published).

$d_{A_1\cdots A_N,B_1\cdots B_N}(\theta)$; passing to the standard basis via the transformation wave functions $\langle A_1\cdots A_N|NW\rangle$, we recover the canonical $d_{WW'}{}^N(\theta)$. There are several advantages in following this seemingly indirect path.

(i) Crossing complications that occur in the canonical basis are avoided. All one needs is to differentiate between particle and antiparticle wave functions $u_{A_1\cdots A_N}\times(NW)=\langle A_1\cdots A_N|NW\rangle$ and $v_{A_1\cdots A_N}(NW)=\langle A_1\cdots A_N|\bar{N}\bar{W}\rangle$, respectively, in passing from one channel to another.

(ii) Tied to (i) is the problem of kinematical constraints on canonical basis amplitudes $T_{(W_i)}$ in passing from one channel to another. In the M-function approach these constraints are automatic (after contraction over the wave functions) and need not be considered separately provided that the invariant amplitudes in M are kinematic singularity free.

(iii) The use of the relativistic basis $|A_1\cdots A_N\rangle$ permits us to discuss in a simple manner the case where the total four-momentum vanishes. Moreover, off-mass-shell continuations appear to be more straightforwardly carried out for M functions than for $T_{(W_i)}$.

(iv) *The most important advantage* of using the M-function approach is that symmetry-breaking prescriptions can be readily formulated, particularly the symmetry breaking which comes about through using physical masses of particles rather than mean masses of multiplets and which affects even the Clebsch-Gordan coefficients $(W_1W_2|W)$. This is not easy to do after one has passed to $T_{(W_i)}$.

The auxiliary basis appropriate to (a′)–(d′) is of course provided by the nonunitary representations[3] of $U(6,6)$. The interpretation of the supermultiplet condition (b′) is that one is limited to couplings involving the $U(6,6)$ auxiliary fields $\psi_{A_1\cdots A_N}(p)$ and the momenta $q_A{}^B$ only, while (a′) is assured by subjecting the $U(6,6)$ fields to subsidiary Bargmann-Wigner equations. (c′) and (d′) are natural consequences of applying these rules to open diagrams.

The $d(\theta)$ functions may be calculated by inserting a general pole contribution specified by the quark number N into the scattering diagram. Before carrying out the contraction over external wave functions one meets $d^N(\theta)$ with a certain number of $U(6,6)$ indices (*the number depending on the external particles alone*). It is these which we list below for some simple cases rather than the contracted forms $d_{WW'}{}^N$.

Take the case of meson ($B=0$) exchange first and various simple examples.

A. $(1,1)_{\frac{1}{2}p+q}+(1,1)_{\frac{1}{2}p-q}\rightarrow(1,1)_{\frac{1}{2}p+q'}+(1,1)_{\frac{1}{2}p-q'}$

$$d^N(\theta)=C_N{}^3(\cos\theta),\quad \cos\theta=-\hat{q}\cdot\hat{q}'+\hat{q}\cdot p\hat{q}\cdot p'/m^2. \tag{3.1}$$

B. $(1,1+(6,\bar{6})_A{}^B\rightarrow(1,1)+(6,\bar{6})$

There are two separate contributions to the amplitude corresponding to the canonical functions $d_{1\,1}{}^N$ and $d_{1\,35}{}^N$. The amplitude is therefore described by the general linear combination

$$[g_1g_A{}^B+g_2\partial/\partial q_B{}^A]C_N, \tag{3.2}$$

where

$$\nu(\partial C_N/\partial q_B{}^A)=(\Gamma_{-q'})_A{}^BC_N'-(\Gamma_{-q})_A{}^BC_{N-1}', \tag{3.3}$$

$$\Gamma_{\pm k}=(1/4m^2)(p\pm m)\gamma\cdot\hat{k}(p\mp m). \tag{3.4}$$

C. $(1,1)+(6,1)_A\rightarrow(1,1)+(6,1)_B$

The linear combination here is modified to

$$[g_1\delta_A{}^B+g_2\partial/\partial q_B{}^A]C_N. \tag{3.5}$$

D. $(1,1)+(6,\bar{6})_A{}^B\rightarrow(1,1)+(6,\bar{6})_{A'}{}^{B'}$

This is a generalization of process B, the amplitude now containing a double derivative

$$[g_1q_A{}^B+g_2\partial/\partial q_B{}^A][g_1q'_{B'}{}^{A'}+g_2\partial/\partial q'_{A'}{}^{B'}]C_N. \tag{3.6}$$

The single differentiation formula has been written above; the double differentiation gives

$$\begin{aligned}\nu(\nu+2)(\partial^2C_N/\partial q'_{A'}{}^{B'}\partial q_B{}^A)=&(\Gamma_+)_{B'}{}^B(\Gamma_-)_A{}^{A'}C_{N+1}''\\&+[(\Gamma_{+q})_{B'}{}^{A'}(\Gamma_{-q'})_A{}^B-(\Gamma_{+q'q})_{B'}{}^B(\Gamma_-)_A{}^{A'}\\&-(\Gamma_+)_{B'}{}^B(\Gamma_{-qq'})_A{}^{A'}]C_N''+[(\Gamma_{+qq'})_{B'}{}^B(\Gamma_{-qq'})_A{}^{A'}\\&-(\Gamma_{+q})_{B'}{}^{A'}(\Gamma_{-q})_A{}^B-(\Gamma_{+q'})_{B'}{}^{A'}(\Gamma_{-q'})_A{}^B]C_{N-1}''\\&+(\Gamma_{+q'})_{B'}{}^{A'}(\Gamma_{-q})_A{}^BC_{N-2}'',\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}\Gamma_\pm&=(1/2m)(p\pm m),\\ \Gamma_{\pm kk'}&=(1/8m^3)(p\pm m)\gamma\cdot\hat{k}(p\mp m)\gamma\cdot\hat{k}'(p\pm m).\end{aligned} \tag{3.8}$$

Contraction of $\partial^2C/\partial q'\partial q$ over the external wave functions provides $d_{35\,35}{}^N(\theta)$. The calculations for more complicated $d(\theta)$ involving further derivatives have not as yet been carried out.

We now turn to the simple cases involving baryonic exchanges. For the single-quark family exchange $B=\frac{1}{3}$ there is the basic process

C′. $(1,1)+(6,1)\rightarrow(1,1)+(6,1)$

In this case we arrive at

$$(d^N(\theta))_A{}^B=(\Gamma_+)_A{}^BC_{N+1/2}'-(\Gamma_{+q'q})_A{}^BC_{N-1/2}',\quad N=\tfrac{1}{2},\tfrac{3}{2},\cdots \tag{3.9}$$

relating to $d_{6,6}{}^N(\theta)$. On the other hand, for the more practical case of $B=1$ exchanges we must consider the basic process

E. $(1,1)+56,1)(1,1)+(56,1)$

Suppressing the six obvious multispinor indices

$$\begin{aligned}d^N(\theta)=&\Gamma_+\Gamma_+\Gamma_+C_{N+3}'''-3\Gamma_{+q'q}\Gamma_+\Gamma_+C_{N+2}'''\\&+3\Gamma_{+q'q}\Gamma_{+q'q}\Gamma_+C_{N+1}'''-\Gamma_{+q'q}\Gamma_{+q'q}\Gamma_{+q'q}C_N'''.\end{aligned} \tag{3.10}$$

The $d_{56,56}{}^N$ functions[10] which may be deduced from this have been given in detail elsewhere.

All these functions need to be multiplied by the threshold factor $(|q||q'|)^N$ which appears naturally in M-function calculations. This was shown explicitly in I. The Regge formulas therefore appear in the form (omitting signature factors) $\approx g^\alpha(t)g^\alpha(t)(|q||q'|)^\alpha d^\alpha/\sin\pi\alpha$, where the β's are reduced residues.

4. FEYNMAN TRAJECTORIES AND THEIR DECOMPOSITION INTO REGGE TRAJECTORIES

The master N trajectories [we shall sometimes refer to them as leaning Feynman towers since the particles on them correspond to the most degenerate tower

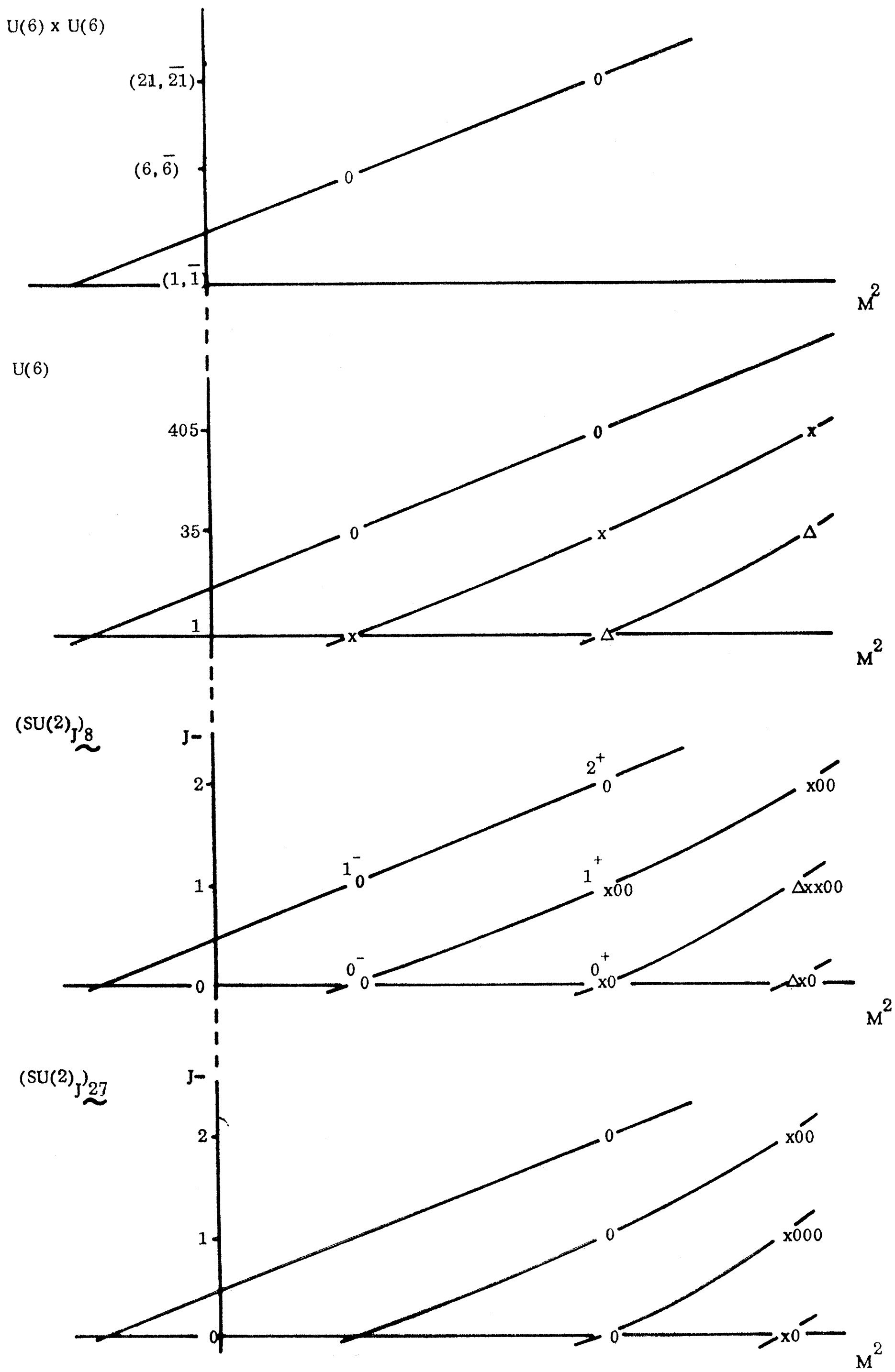

FIG. 1. The master boson trajectory decomposed into $SU(6)$ satellites (identified by 0, x, Δ, ..., etc.) which are further decomposed into $SU(3)$ pieces. Notice that there is more than one satellite trajectory of a given $SU(3)$ type. The symmetry breaking is expected to shift the trajectories from the positions shown. The known octets of $J^P=0^-$, 1^-, and 2^+ (and possibly 1^+) are shown in the third pattern.

studied by Feynman[11] for the noncompact $U(6,6)$ for mesons and baryons] contain all the relations between the J-Regge parameters. To see exactly what these relations turn out to be, we have to carry out the reduction chain $SU(6)\otimes SU(6)\rightarrow SU(6)\rightarrow SU(3)_F\otimes SU(2)_J$ of the master trajectories into the $SU(2)_J$ satellites for specific $SU(3)$ representations. Mathematically this reduction corresponds to the decomposition of particular $SU(3)$ components of the $SU(6)\otimes SU(6)$ rotation functions $C_N{}^3$ into the $SU(2)_J$ rotation functions $P_J=C_J{}^{1/2}$. The relevant formula is obtained from

$$C_N{}^\lambda(\cos\theta)=\sum_{\kappa\geq 0} a_{N\kappa}C_{N-2\kappa}{}^{\lambda'}(\cos\theta),$$

where the summation terminates at the background[12] and $a_{N\kappa}$ is a ${}_4F_3$ function (a sum of Γ-function ratios). For the simple case of the reduction

$$C_N{}^3=\sum_\kappa a_{N\kappa}P_{N-2\kappa} \tag{4.1}$$

the explicit formula was given in Eq. (15) of II. In the next section when we consider symmetry breaking, we shall need this reduction. In the M-function approach of Sec. 3, where all d^N's are expressed in terms of $C_N{}^3$ and its derivatives, it is just the formula (4.1) which is repeatedly needed.

To illustrate the consequences of this type of reduction graphically, let us plot a few satellite trajectories for the meson case. The master trajectory is shown in Fig. 1. It gives rise to the satellites shown in the lower diagrams. The rotation function $d_{WW'}{}^N$ pertaining to the master trajectories is a sum of rotation functions for all satellites. The general properties of these satellites have been noted in II. Here let us reemphasize the main physical points.

A. Parallel Satellites

If the symmetry were exact, all satellites would be parallel to the parent. Since empirically different $SU(2)$ *and indeed* $SU(3)$ trajectories are found to be roughly parallel (with the exception of the Pomeranchukon which may be a fixed pole) higher symmetry may provide the simplest explanation of this fact.

B. Residue Relations

In the asymptotic limit $P_J(\cos\theta)\approx(\cos\theta)^J$; it is clear from this that the leading satellite trajectory contained in the expansion (4.1) will dominate the scattering amplitude, e.g., the $SU(3)$ octet piece will show a dominance of the ρ trajectory over the π-trajectory contribution. On the other hand, the octet part of function $d_{35\,35}{}^N$ automatically includes the contribution of both trajectories and the decomposition $C_N=a_{N0}P_N+a_{N1}P_{N-2}+\cdots$ shows how, for example, the ρ- and π-trajectory contributions emerge. This reduction provides group-theoretic relations between the Regge residues a_{N1}/a_{N0} automatically.

C. Symmetry Breaking

Since the high-energy behavior in Regge theory depends so critically on $\alpha(t)$, and in particular on the intercept $\alpha(0)$, it is evident that any mass shifts[13] produced by the symmetry breaking will shift the resultant satellites, and their asymptotic contributions will differ markedly from the exact symmetry predictions. This is in contrast to the effect of symmetry breaking for vertices where, barring certain exceptional cases, one hopes that symmetry breaking may be wholly accounted for just by change of kinematical factors, e.g., by using physical masses in the invariant couplings (and Bargmann-Wigner equations) rather than mean supermultiplet masses. To show how critical a role this trajectory shifting can play, take the example of pure **27** of $SU(3)$ exchange that occurs in a process like $K^-p\rightarrow\pi^++Y^-$, which shows no forward peak, and a high-energy behavior $E^{-3.5\pm0.7}$ corresponding to $\alpha_{27}(0)\approx-0.7$. Assigning the **27** as well as the 2^+ octet $(f,A_2,K^{**},\cdots)$ to the same **405** of $SU(6)$, it is clear that an $SU(3)$-dependent mass shift between the **27** and the **8** of the order of no more than 300 MeV (without change of slope) can shift[14,15] $\alpha_8(0)$ from its value of about 0.4 down to $\alpha_{27}(0)\approx-0.7$.

[11] A. O. Barut, P. Budini, and C. Fonsdal, Phys. Rev. Letters **14**, 968 (1965); Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Letters **17**, 148 (1965) (the Feynman towers with the same content as the master trajectories were first presented in this paper); C. Fronsdal, in *Proceedings of the International Seminar in High-Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna, 1965), p. 665; R. Delbourgo, Abdus Salam, and J. Strathdee, Proc. Roy. Soc. (London) **289A**, 177 (1966); Abdus Salam and J. Strathdee, Phys. Rev. **148B**, 1352 (1966).

[12] The exact form of the general formulas, which take into account also the background terms when N is complex, will be the subject of a further publication. The complicating point about such formulas is that the backgrounds of C^λ and $C^{\lambda'}$ occur at different places, namely, at $N=-\lambda$ and $N=-\lambda'$, respectively.

[13] F. Gürsey, A. Pais, and L. Radicati, Phys. Rev. Letters **13**, 299 (1964); M. Bég and V. Sigh, *ibid.* **13**, 418 (1964); H. Harari and M. A. Rashid, Phys. Rev. **143**, 1354 (1966).

[14] If $\alpha_{27}(0)$ is so critically shifted it is obvious that within the present Reggeization scheme, J. D. Jackson's analysis [Phys. Rev. Letters **15**, 990 (1965)] of Johnson-Treiman-like relations [J. D. Carter, J. J. Coyne, D. Horn, M. Kugler, H. Lipkin, and S. Meshkov, *ibid.* **15**, 373 (1965)] and his negative conclusion about $SU(6)_W$ predictions in the forward direction for **405** exchanges no longer applies. On the positive side, R. Arnold in Ref. 7 has considered processes not involving **27** exchange where the ρ_1A_2 trajectory dominates in a $U(6)\times U(6)\times O(3)$ scheme. He finds reasonable disagreement with experimental results if he assumes an $SU_W(6)$ symmetry between residues. Also, as Jackson himself noted, no account was taken in his analysis of mass differences due to symmetry breaking. As pointed out in (iv) of Sec. 3, the use of M-function formalism is superior for this reason to the direct W-spin formalism since it allows mass differences to be taken account of in the residues.

[15] Barut has given a one-parameter mass formula for mesons in the traceless $SU(3)$ form, $m^2=m_0{}^2+\lambda^2\{j(j+1)-\frac{1}{2}[I(I+1)-\frac{1}{4}Y^2-C]\}$, with $m_0{}^2=16\times10^4$ (MeV)^2 and $\lambda^2=28\times10^4$ (MeV)^2. The constant C which guarantees tracelessness depends on $SU(3)$ Casimirs and equals $+1$ for octets and 8/3 for 27-folds. The formula fits the known octets with accuracy and would predict a mass more than 300 MeV higher for the spin two doubly charged **27**'s. It is also worth remarking that the leading trajectory never contains **10**, $\overline{\mathbf{10}}$, **35**, $\overline{\mathbf{35}}$, or other non-self-conjugate multiplets of $SU(3)$. This means that the exchanges of such multiplets are definitely suppressed at high energies.

D. Mass Formulas and Trajectory Shifts

To take account of trajectory shifts on account of symmetry breaking, we need mass formulas, which in general may have the form[13]

$$\begin{aligned} M^2 &= M^2(N,J,F) \\ &= M_0{}^2(N)+M_1{}^2(F)+M_2{}^2(F,J)\,, \end{aligned} \tag{4.2}$$

where F denotes the $SU(3)$ labels (including I and Y). To incorporate the trajectory shifts, go back to the expression of Sec. 2:

$$T=\int \frac{dN}{\sin\pi N}\frac{b^N C_N(\cos\theta)}{t-M^2(N)}\,.$$

One may replace C_N exactly by[16] $\sum a_{J\kappa}P_J$; if we further decide to incorporate symmetry breaking by replacing $M^2(N)$ by $M^2(N,J,F)$, we obtain

$$T\sim\sum_\kappa \int \frac{dJ}{\sin\pi(J+\kappa)}\frac{b^{J+\kappa}a_{J\kappa}P_J(\cos\theta)}{t-M^2(\kappa,J,F)}\,. \tag{4.3}$$

The trajectory function $J=\alpha(t,\kappa,F)$ [obtained from solving for J the equation $t-M^2(\kappa,J,F)=0$] now allows (1) for possible $SU(3)$ shifts given by $M_1{}^2(F)$ in (4.1) and (2) for departures from parallelism among the satellite trajectories arising from the $M_2{}^2(F,J)$ term. In keeping with our program, we shall not interfere with the residues $\beta_{J+\kappa}(t)$. Let us examine the simplified form of a mass formula (4.2), where

$$M_0{}^2(N)=NM_0{}^2,\quad M_2{}^2(F,J)=J(J+1)M_2{}^2(F)\,; \tag{4.4}$$

i.e., the master trajectory in the N plane rises linearly. It is a simple matter to solve out for the trajectory function from $t-M^2=0$; we get

$$\alpha(t,\kappa,F)=[t-M_1{}^2(F)]\left(\frac{1}{M_0{}^2}+\frac{(2\kappa-1)M_2{}^2(F)}{M_0{}^4}\right) -\kappa(\kappa-1)\frac{M_2{}^2(F)}{M_0{}^2} \tag{4.5}$$

to lowest order in $M_2{}^2/M_0{}^2$. To this order the trajectories remain linear but with modified slopes, exhibiting an $SU(3)$ mass shift which depends on which satellite we are considering.

Since at present we have no reliable theoretical means for computing mass formulas—except perhaps as tadpole effects or as estimates from second-order self-energy graphs written in the language of current algebra—we have to take the trajectory parameters from experiment. This is a weakness of the present scheme.

5. RELATIVISTIC ASPECTS OF $U(6,6)$

Just as for forward scattering of equal-mass particles the little group enlarges from $O(3)$ to $O(3,1)$, likewise here $U(6)\otimes U(6)$ enlarges to $U(6,6)$ itself. The $O(3,1)$ partial-wave analysis at $P_\mu=0$ which was originally carried out by Toller[17] can similarly be done here for $U(6,6)$. Following the method of Freedman and Wang,[17] one first shows, for a certain unphysical range of s, that one may deal with the compact group structure $U(12)$ rather than $U(6,6)$ so far as partial-wave analysis and Reggeization are concerned, continuing back later to physical values of s. Denoting the $U(12)$ rotation functions by $d_{NN'}{}^{\mathfrak{N}}(\theta)$, where $\mathfrak{N}$ and N stand, respectively, for the set of $U(12)$ and $U(6)\otimes U(6)$ Casimirs, one can make the expansion at $P_\mu=0$,

$$\begin{aligned} &\langle N_3W_3,N_4W_4|T(\theta)|N_1W_1,N_2W_2\rangle \\ &=\sum_{NWN'}\langle N_3W_3,N_4W_4|N'W\rangle \\ &\qquad\times\langle N'W|T(\theta)|NW\rangle\langle NW|N_1W_1,N_2W_2\rangle \\ &=\sum_{NWN'\mathfrak{N}}\langle N_3W_3,N_4W_4|N'W\rangle T_{N'N}{}^{\mathfrak{N}} \\ &\qquad\times d_{N'WN}{}^{\mathfrak{N}}(\theta)\langle NW|N_1W_1,N_2W_2\rangle\,. \end{aligned} \tag{5.1}$$

In the case of $O(3,1)$ or $O(4)$ the appropriate rotation functions are known to be $C_N{}^1(\cos\theta)$ and their derivatives. *For the $U(12)$ or $U(6,6)$ degenerate series one can show that they are proportional to $C_N{}^{11/2}$ and their derivatives.*

The expansion (5.1) holds at $P_\mu=0$. It can, however, be extended to the case $t=P^2\neq 0$ for all W-spin-conserving amplitudes, since $d_{NWN'}{}^{\mathfrak{N}}(\theta)$ provide appropriate expansion functions for this case as well. This is analogous to the expansion of general flipless amplitudes for all momentum transfers[18] using $O(3,1)$ rotation. An extension to W-changing amplitudes is possible, analogous to the $O(3,1)$ expansion proposed recently by a number of authors[18] for spin-flip amplitudes. These expansions correctly incorporate threshold effects and at the same time have the merit of automatically building the Toller parent-daughter phenomenon into the formalism even for $t\neq 0$. Thus a trajectory in the $U(6,6)$

[16] It is worth pointing out that the factors $a_{J\kappa}$ are factorizable. This is a general consequence of completeness relations of the type

$$\begin{aligned} \langle NW'|e^{-i\theta J_2}|NW\rangle &= \sum_{\substack{J\kappa,J'\kappa' \\ (J+\kappa=J'+\kappa'=N)}}\langle NW'|J'\kappa'\rangle \\ &\qquad\times\langle J'\kappa'|e^{-i\theta J_2}|J\kappa\rangle\langle J\kappa|NW\rangle \\ &= \sum_{\substack{J\kappa \\ (J+\kappa=N)}}\langle NW'|J\kappa\rangle P_J(\cos\theta)\langle J\kappa|NW\rangle, \end{aligned}$$

where the vectors $|J\kappa\rangle$ denote a basis for the representation $\mathfrak{D}^N$ which diagonalizes the angular momentum $\mathbf{J}^2$.

[17] M. Toller, Nuovo Cimento, **53A**, 671 (1968); D. Freedman and J. M. Wang, Phys. Rev. **153**, 1596 (1967); A. Sciarrino and M. Toller, J. Math. Phys. 8, 1252 (1967).

[18] R. Delbourgo, Abdus Salam, and J. Strathdee, Phys. Letters **25B**, 230 (1967); R. F. Sawyer, Phys. Rev. **167**, 1372 (1968); G. Cosenza, A. Sciarrino, and M. Toller, University of Rome Report, 1968 (unpublished).

Casimir $\mathfrak{N}$ plane gives rise to a series of parent and daughter trajectories in the $U(6)\otimes U(6)$ N plane—all of these daughters unfortunately being parallel to the parent.[19]

To see the complexion of these daughters, take the Feynman meson trajectory in $U(6,6)$ which, for this degenerate series, passes through the $U(12)$ representations **1**, **143**, **5940**, $\cdots$. From the $U(6)\otimes U(6)$ reduction of these multiplets

$$\begin{aligned} 1 &= (1,1)\,, \\ 143 &= (6,\bar{6})+(\bar{6},6)+(1,1)+(1,35)+(35,1)\,, \\ 5940 &= (21,\overline{21})+(\overline{21},21)+(6,\bar{6})+(\bar{6},6)+(6,\overline{120}) \\ &\quad +(\overline{120},6)+(\bar{6},120)+(120,\bar{6})+(1,1)+(1,35) \\ &\quad +(35,1)+(35,35)+(405,1)+(1,405)\,, \end{aligned}$$

one is led to sets of $U(6)\otimes U(6)$ trajectories, among which is the master meson trajectory considered earlier. The important relativistic aspects which emerge are as follows.

A. Gribov Doubling

The reduction of $U(12)$ multiplets into $U(6)\times U(6)$ multiplets produces pairs of the variety $(A,B)\oplus(B,A)$.[20] For example $(6,\bar{6})$ is accompanied by $(\bar{6},6)$; likewise $(35,1)$ by $(1,35)$.

To characterize this doubling, one may say that the states are populated equally by composites of quarks $(6,1)$ and pseudoquarks $(1,6)$ [also by antiquarks $(1,\bar{6})$ and antipseudoquarks $(\bar{6},1)$]. Even apart from Tollerization, this particular doubling should have been expected from the Gribov-Pomeranchuk-Okun phenomenon which even in conventional Reggeization schemes would lead one to expect that a Reggeized quark state should be accompanied by a pseudoquark state from MacDowell symmetry.[21] If composites of quarks exist, one should expect composites of pseudoquarks also to exist.

The important point to note about the Gribov doubling is that whereas for fermions it always leads to *parity doubling* [$(56,1)\rightarrow(1,56)$] (the two states have opposite parity) this is not necessarily the case for mesons [consider $(6,\bar{6})\rightarrow(\bar{6},6)$; parity of the two states is the same].

B. Parity Doubling for Mesons

In addition to Gribov doubling (which, as remarked above, does not lead to parity doubling for mesons), another peculiarly Toller-like phenomenon of parity doubling for mesons does take place. This is the doubling implied, for example, for the **143** by $(6,\bar{6})+(\bar{6},6)\Rightarrow(1,35)+(35,1)$. This is analogous (but not the same) as the parity-doubling phenomenon for Toller's theory of $SL(2,C)$ when for mesons one may expect parity degeneracy whenever the Lorentz quantum number M in Toller's notation does not equal zero. Perhaps one way to understand this new doubling is to remark that the chiral subgroup $U(6)\times U(6)|_{\gamma_5}$ is as equally a subgroup of $U(12)$ as nonchiral $U(6)\times U(6)|_{\gamma_0}$. As we have seen above, *for Reggeization* $U(6,6)$ and $U(12)$ possess completely interchangeable roles; one may start with either group and pass to the other by continuations in s and t variables. One may expect the theory therefore to exhibit doubling associated both with the chiral as well as nonchiral subgroups.

All this is not too clear at present. What we seem to have is that whithin an S-matrix approach, at the point $P_\mu=0$, one can resolve the old dilemma of chiral $U(6)\times U(6)$ being a symmetry at the same time as well as $U(6)\times U(6)$ nonchiral.

6. OUTLOOK

It must be admitted that it needs trepidation and courage to propose a theory of the type suggested here where the expectation is that higher symmetries may exhibit themselves best in giving a coherent description of Regge residues.[22] This is because, on superficial evidence, the major necessary condition for the theory—the existence of a string of higher supermultiplets lying on the master trajectories—seems unfulfilled. Unfortunately the situation in this regard may remain unchanged for a number of years.

The higher supermultiplets of $U(6)\times U(6)$ contain vast numbers of particles. The present rate of resonance identification, notwithstanding the heroic efforts of experimental colleagues, is slow. The situation is complicated further because, as has been shown by Horn, Lipkin, and Meshkov and Abramsky and King,[23] firstly, the higher $SU(3)$ multiplets contained in these supermultiplets are hard to produce in normal meson-baryon and baryon-baryon collisions and, secondly, most of these resonances do not possess two-body decays. Also, there is a great amount of mixing going on when resonances have the same quantum numbers. Indeed much theoretical work needs to be done to identify experi-

[19] This parallelism is of course one shortcoming of the formalism. It is important to distinguish the $U(6)\otimes U(6)$ satellites and the $U(6,6)$ daughters. The first are a consequence of the supermultiplet symmetry, the second a consequence of its relativistic enlargement.

[20] V. Gribov, L. Okun, and I. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. **45**, 1114 (1963) [English transl.: Soviet Phys.—JETP **18**, 769 (1964)].

[21] S. MacDowell, Phys. Rev. **116**, 774 (1959).

[22] This is the point of view which has consistently been emphasized from the first when the symmetries were suggested rather that their use without qualification. Thus, for the S matrix, the outlook was stated in Ref. 5 as follows: "with the effective baryon-meson and meson-meson vertices available, it is a trivial step to write pole approximations for the strong-interaction four-point processes. With this approximation as the starting point, all S-matrix techniques (like Mandlestam representation, Reggeization, analytic continuation both in angular momentum and unitarity spin) are available for determining the complete $\tilde{U}(12)$ S-matrix theory. This is so because, as a rule, all that the S-matrix theory requires are Born approximations as the "input."

[23] D. Horn, H. Lipkin, and S. Meshkov, Phys. Rev. Letters **17**, 1200 (1966); J. Abramsky and R. King, **20** 1408 (1968).

mental situations where there is most chance of observing these higher multiplets.

In practical applications of the theory, one difficulty has been noted in Sec. 4. This is the difficulty associated with symmetry-breaking effects in mass formulas and the trajectory shifts these can produce, so that the trajectory parameters must at present be taken from experiment. A second difficulty is connected with the general Reggeization program. The Regge-pole model, even with its large number of parameters, has spectacular failures as well as successes. The failures have been attributed to kinematic effects, imperfectly understood so far, and to the fact that pion-exchange effects (perhaps on account of their exceptionally long range) appear less amenable to a Regge treatment and more to absorption or coherent-droplet models. The Reggeization scheme presented in the present paper will inherit the conventional kinematical structure. To be sure, though, there will be new features, like the threshold factor $(|q'||q)^N$ rather than the conventional factor $(|q'||q|)^J$, mentioned in Sec. 4, and the new zeros contained in $a_{N\kappa}$ of Eq. (4.1) as well as the new features which will arise from a consideration of sense and nonsense phenomena anew in the present case.[24]

It is possible that a Toller-like program may provide here, as in conventional Regge theory, one way to define singularity-free amplitude. It is perhaps worth remarking that something mathematically similar to a Toller expansion of conventional amplitudes in terms of $O(4)$ rotation functions[18] is automatically included in our formalism, through the $U(2)\times U(2)$ subgroup of $U(6)$ $U(6)$. Even though $U(2)\times U(2)$ has a completely different physical significance from $O(3,1)$, the rotation functions for the two cases are identical. Whether this feature is enough to take care of all kinematic singularities automatically, we do not know. Only experience with the formalism can tell.

To expand on this point, it has been stressed before[3] that the $U(6,6)$ theory has two relatively disconnected features: First, the obvious one, it includes the internal symmetry $SU(3)$; second, and unfortunately the less emphasized but in our view the more important feature, it includes the extension of the space-time Lorentz structure $SL(2,C)$ to the bigger (perhaps conformal) structure $U(2,2)$. This extension $U(2,2)\approx O(4,2)$ increases the number of "space-time" Casimirs from the two well-known ones of $SL(2,C)$ to three of $U(2,2)$. It was pointed out in Ref. 3 that the empirically well-established proportionality of electric and magnetic form factors of the proton is a direct consequence of this particular extension of $SL(2,C)$ space-time group to the $U(2,2)$ group. Thus, even if $SU(3)$ was a badly broken symmetry or if it was conclusively established that all hadron resonances make up only the **8**'s and the **10**'s of $SU(3)$ and never any other multiplet, it would still, in our view, make better dynamical sense for Reggeization ideas to make a partial-wave analysis using the $U(2,2)$ extension of space time structure[25] [in practice in terms of functions $C_N{}^1$ and their derivatives corresponding to the little group $U(2)\times U(2)$]. Thus the first logical step in Reggeization of higher symmetries is to consider Reggeization of $U(2)\times U(2)$; this will give baryon and meson trajectories with content similar to those derived from noncompact groups by Barut and Kleinert[26]; next, one may include isospin and extend the symmetry to $U(4)\times U(4)$ and, finally, with the inclusion of $SU(3)$ to $U(6)\times U(6)$. The kinematic factors $a_{NJ}{}^{\nu}$ arising from the decomposition of the relevant $C_N{}^{\nu}$ to $C_J{}^{\frac{1}{2}}=a_{NJ}{}^{\nu}P_J$ for each assumed symmetry would be different. [For $U(\nu)\otimes U(\nu)$ symmetry the rotation functions are $C_N{}^{\frac{1}{2}\nu}\times(\cos\theta)$; for $U(2\nu)$ they are $C_N{}^{\nu-1}(\cos\theta)$; and for $O(\nu)$ they are $C_N{}^{\frac{1}{2}\nu-1}(\cos\theta)$.]

Hopefully, experiment may distinguish between the various possibilities which correspond to the successive chains of symmetry breaking. One of the important parameters relevant to this distinction is the F/D ratio[27]; it is a mathematically fascinating problem to compute the F/D ratio along the $SU(3)$ **8**-projection of the Feynman trajectory. Other problems are a better understanding of the mathematical expansion theorem for the case of less degnerate series, a simpler procedure for computing the relevant d^N functions, and, most critical of all, a reliable mass formula for use in (4.3).

APPENDIX

There are at least four formulations of $U(6,6)$ and its subgroup symmetries known to the authors.[3,28] Though their relative merits are hotly debated,[29] all of them unfortunately suffer from one shortcoming or another. All approaches do at least agree on the subgroup hierarchy of Sec. 2 as representing the maximal possible invariance attainable. To describe the approaches and their interrelations, let us briefly recall the group structure they use in order to explain the detailed differences. To begin with, there is the $U(6,6)$ algebra which is isomorphic to the algebra generated by the 16 Dirac matrices γ multiplied into nine $SU(3)$ matrices T^i:

$$(1,\boldsymbol{\sigma},\gamma_5,\gamma_5\boldsymbol{\sigma},\gamma_0,\gamma_0\boldsymbol{\sigma},\gamma_0\gamma_5,\gamma_0\gamma_5\boldsymbol{\sigma})\times T^i.$$

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[24] These latter (not studied so far) present fascinating problems; the mysterious vanishing of a number of residues in conventional theory may possibly find a *kinematical* explanation in the present formalism. This may not be surprising if one remembers that the extended kinematics of this formalism is an expression of the dynamics of hadron physics.

[25] We summarize here the results on rotation functions; for the most degenerate representations the rotation functions correspond to derivatives of $C_N{}^{\nu-1/2}$ for $U(2\nu)$ and $C_N{}^{2/\nu}$ for $U(\nu)\times U(\nu)$ groups. We conjecture that the same Gegenbauer polynomials and their derivatives occur for all other representations of the relevant groups.

[26] A. O. Barut and H. Kleinert, Phys. Rev. **160**, 1149 (1967).

[27] This has recently been emphasized by P. N. Dobson, Jr., Phys. Rev. **163**, 1619 (1967).

[28] S. Coleman, Phys. Rev. **138**, B1262 (1965), has listed a still larger number of variants.

[29] See, for example, Y. Ne'eman, *Algebraic Theory of Particle Physics* (W. A. Benjamin, Inc., New York, 1967) and P. T. Matthews, Nature **217**, 197 (1968), review of this book.

(The Lorentz subalgebra is generated by $\boldsymbol{\sigma}$ and $\gamma_5\boldsymbol{\sigma}$.) Four translations P_μ are adjoined to $U(6,6)$, whose commutation property is obtained through the isomorphism $P_\mu \doteqdot \gamma_\mu$. For processes involving one (timelike) vector $P_0 \doteqdot \gamma_0$, the subgroup of $U(6,6)$ which commutes with γ_0 is the "little" group $U(6)\otimes U(6)$ which consist of $(1,\gamma_0,\boldsymbol{\sigma},\gamma_0\boldsymbol{\sigma})T^i$. Collinear processes confined to the 0-3 plane require the "lesser" group which commutes with the pair of vectors γ_0 and γ_3; this is $U(6)_W$ and consists of $(1,\sigma_3,\gamma_0\sigma_1,\gamma_0\sigma_2)T^i$ [for the Lorentz case the analogous subgroups are $SU(2)_J$, consisting of $\boldsymbol{\sigma}$ and the helicity group $U(1)$ consisting of σ_3 alone]. *W-spin is thus the generalized helicity of U(6,6)*. Finally, there are the coplanar processes confined to the 013 subspace whose "least" group is $U(3)\otimes U(3)$ made up of $(1,\gamma_0\sigma_2)T^i$; this has no analog in the Lorentz group case.

So much is common ground. However, the four approaches differ in the concrete realizations which they give to the generators of $U(6,6)$ and the way the translations P_μ are handled.

(1) First, there is the simple field-theoretic approach[30] based on a Lagrangian formulation of $U(6)\otimes U(6)$ multiplets, e.g., the quark Lagrangian $\mathfrak{L}=\bar{\psi}(i\gamma\cdot\partial-m)\times\psi+g(\bar{\psi}\psi)^2$ or a more complicated Lagrangian constructed from the $U(6,6)$ multispinors. In this formulation the mass and interaction terms are $U(6,6)$-invariant whereas the kinetic energy terms of the type $\bar{\psi}i\partial\psi$ are not.[6] Evidently open diagrams and their sums do possess the hierarchy of little group symmetries (even if derivative interactions are included), whereas closed loops are not likely to preserve these. If a Regge pole is pictured as an infinite sum of pole diagrams[31] the hierarchy of symmetries survives. However, inclusion of two-particle or more intermediate states, i.e., imposition of unitarity, breaks the chain through the symmetry breaking introduced by closed loops.

(2) The second approach was suggested by a number of authors[11] and developed in particular by Fronsdal and his collaborators. Here the full noncompact $U(6,6)$ may be taken as a rest symmetry with the consequence that there must exist an infinity of particle states corresponding to representations of $U(6)\otimes U(6)$ all *having the same mass*. The subgroup hierarchy provides *exact* invariance groups for the relevant processes; unitarity also is exactly satisfied but only in the mass degenerate limit—as soon as mass differences are introduced between different particle states unitarity disappears. It is clear that Reggeization of approach (1) and its interpretation as a summation over an infinity of particle states brings closer together approaches (1) and (2).

(3) The third approach is based on current algebras[6] and is wide enough to encompass either (1) or (2). Unhappily, there exists no model, however idealized, for which the charges defined from the full set of $U(6)\otimes U(6)$ currents are conserved.

(4) The last approach is the inhomogeneous $U(6,6)$ theory of Bell and Rüegg and Charap, Matthews, and Streater,[32] which adjoins 143 momenta to $U(6,6)$. Before specializing to four physical momenta the subgroup hierarchy, as well as unitarity in a generalized partial-wave expansion, emerge as *exact* consequences of the theory; also, one may write equations of motion for the nonunitary finite-dimensional representations of $U(6,6)$ since one is dealing with a 143-dimensional Poincaré group. In the physical limit of four-momenta surviving from among the 143, the equations of motion of approach (1) are obtained. One could write if one wished Majorana type equations in the (144) space for infinite-dimensional representations of $U(6,6)$ to give a physical particle spectrum. The unresolved difficulty of this approach is the definition of a sensible (stereographic) limit whereby the 143-dimensional space maps onto physical four dimensions.

In Sec. 2 we have tried to formulate yet another viewpoint by accepting the subgroup hierarchy as empirical input. We have worked with just the conventional S-matrix setup in the physical space of four dimensions. We have made a partial-wave analysis based on the existence of a complete set of functions in terms of which $S(\theta)$ can be expanded—a purely mathematical procedure which must always succeed provided the weak statement of the input hierarchy of subgroups is guaranteed by the choice of the expansion functions. The full symmetry of the S matrix under the higher group is not needed.

[30] R. Delbourgo, M. A. Rashid, Abdus Salam, and J. Strathdee, Proc. Roy. Soc. (London) **285A**, 312 (1965).

[31] L. Van Hove, Phys. Letters **24B**, 183 (1967).

[32] J. Bell and H. Rüegg, Nuovo Cimento **39**, 1166 (1965); J. Charap, P. T. Matthews, and R. F. Streater, Proc. Roy. Soc. (London) **290A**, 24 (1966).