and

$$\epsilon_0=\left[\frac{1}{2\xi}\int_0^\infty d\mu^2[\rho^{(\pi)}(\mu^2)+2\rho^{(K)}(\mu^2)-\tfrac{2}{3}\rho^{(\kappa)}(\mu^2)]\mu^2\right]^{1/2},$$

$$\epsilon_8=\left[\frac{4}{3\xi}\int_0^\infty d\mu^2\rho^{(\kappa)}(\mu^2)\mu^2\right]^{1/2}, \qquad (29)$$

$$\lambda_0=\xi\epsilon_0,$$

$$\lambda_8=\xi\epsilon_8.$$

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Conspirators and Daughters in Unequal-Mass, Nonzero-Spin Scattering*

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The problem of satisfying the all unequal-mass kinematic constraints at $s=0$ for helicity amplitudes for two-body scattering of particles with spin is analyzed in a Regge model. The Mandelstam-Sommerfeld-Watson transformation is used to obtain the form of the Regge contributions. It is shown that a sequence of daughter trajectories spaced at $\Delta\alpha=1$ is necessary to keep the amplitude analytic at $s=0$. For $\mu_m=\min(|\lambda_a-\lambda_b|, |\lambda_c-\lambda_d|)=0$, no further trajectories are needed; for $\mu_m\neq0$ and the reduced Regge residue finite at $s=0$, the kinematic constraint and analyticity require daughters *and* a conspirator sequence of opposite parity starting at $\alpha_c=\alpha_{\rm Regge}$. It is shown to be probable that, for a residue which vanishes as s^n $(n\geq\mu_m)$ at $s=0$, only the daughter trajectories are needed.

I. INTRODUCTION

THERE has been considerable interest in the requirements that one must impose on Regge contributions to scattering amplitudes to satisfy Mandelstam analyticity at $s=0$. For the case of equal-mass, spinless scattering, $z_s\sim1-t/2m^2$ near $s=0$, one finds a $t^{\alpha(0)}$ contribution to the amplitude from each Regge pole. Goldberger and Jones,[1] Freedman and Wang,[2] and Freedman, Jones, and Wang[3] have considered the $s=0$ behavior of unequal-mass, spinless scattering amplitudes. They found that the t^α dependence and analyticity in s within the region $|s|<s_{\rm forward}(t)|$ can be recovered, even though $|z_s|$ is bounded in this region, if there exists a sequence of Regge poles at positions $\alpha_l(0)=\alpha(0)-l$ with singular residues. Fearing[4] has given an expansion of the Regge amplitude that does satisfy Mandelstam analyticity at $s=0$ and does not require additional l-plane poles. However, he shows that the usual Regge expansion of l-plane poles does require daughter trajectories for $s=0$ analyticity, and we may conclude that his expansion has merely provided a procedure for explicitly summing the nonanalytic portions of the Regge sequence and showing their cancellation.

Equal-mass, nonzero-spin scattering has been investigated by Freedman and Wang,[5] who used the fact[6] that at precisely $s=0$ the scattering equations are invariant under the $O(4)$ group. A Regge trajectory can then be classified by its $O(4)$ quantum numbers at $s=0$. Since there are certain well-known kinematic constraints among the helicity amplitudes at $s=0$, the dynamics of their solution can be classified by $O(4)$ quantum numbers. Freedman and Wang then found that conspiracies[7] among various Regge trajectories must occur at $s=0$.

We have adapted the method of Freedman and Wang[2] to a consideration of the analyticity of Regge contributions to unequal-mass, nonzero-spin scattering. In Sec. II, we present this method and derive the usual residues of the daughter trajectories for spinless scattering. In Sec. III, we find that, for nonzero spin, in the case $\mu_m=\min(|\lambda|,|\mu|)=0$, where $\lambda=\lambda_a-\lambda_b$, $\mu=\lambda_c-\lambda_d$ for the reaction $a+b\rightarrow c+d$ (s channel), spin is a nonessential complication and daughter trajectories are sufficient to restore analyticity. For $\mu_m\neq0$ there exists a nontrivial

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[1] M. L. Goldberger and C. E. Jones, Phys. Rev. **150**, 1269 (1966).

[2] D. Z. Freedman and Jiunn-Ming Wang, Phys. Rev. **153**, 1596 (1967).

[3] D. Z. Freedman, C. E. Jones, and Jiunn-Ming Wang, Phys. Rev. **155**, 1645 (1967).

[4] H. W. Fearing, Stanford University Report (unpublished).

[5] D. Z. Freedman and Jiunn-Ming Wang, Phys. Rev. **160**, 1560 (1967).

[6] G. C. Wick, Phys. Rev. **96**, 1124 (1954).

[7] The problem of kinematic constraints and conspiracy at $s=0$ has been studied by a number of people, including: M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960); D. V. Volkov and V. M. Gribov, Zh. Eksperim. i Teor. Fiz. **44**, 1068 (1963) [English transl.: Soviet Phys.—JETP **17**, 720 (1963)]; M. Gell-Mann and E. Leader, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967); E. Abers and V. L. Teplitz, Phys. Rev. **158**, 1365 (1967); E. Leader, *ibid.* **166**, 1599 (1968); S. Frautschi and L. Jones, *ibid.* **167**, 1335 (1968).

kinematic constraint between the amplitudes $\bar{f}_{\lambda,\mu}$ and $\bar{f}_{\lambda,-\mu}$ at $s=0$ which cannot be satisfied, along with analyticity, by the daughters alone, unless the reduced residue vanishes. We find a solution to both of these requirements for the most divergent part of the residues of a Regge sequence α_l and a conspirator sequence α_l^c, with $\alpha_l^c=\alpha_l$. It appears probable, although we do not prove it, that a leading Regge pole which "evades" as s^{μ_m} at $s=0$ can satisfy the kinematic constraints and analyticity with only daughter trajectories. In Sec. IV, we comment on the connection between these solutions and those of Freedman and Wang for the $N\bar{N}$ case.

II. SCATTERING OF UNEQUAL-MASS, SPINLESS PARTICLES

For spinless, nonrelativistic potential scattering, Mandelstam[8] has shown that the standard Regge representation

$$A(s,z)=\frac{1}{2i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{dJ(2J+1)F(s,J)P_J(-z)}{\sin\pi J}+\sum_{\mathrm{Re}\,\alpha_i>-1/2}\frac{(2\alpha_i+1)\beta_i(s)P_{\alpha_i(s)}(-z)}{\sin\pi\alpha_i(s)}$$

can be converted into the form

$$A(s,z)=-\frac{1}{2\pi i}\int_{-M-i\infty}^{-M+i\infty}\frac{dJ(2J+1)F(s,J)Q_{-1-J}(-z)}{\cos\pi J}+\frac{1}{\pi}\sum_{n=N}^{\infty}(-1)^n 2nF(s,n-\tfrac{1}{2})Q_{n-1/2}(z)-\frac{1}{\pi}\sum_{\mathrm{Re}\,\alpha_i>-M}\frac{(2\alpha_i+1)\beta_i(s)Q_{-1-\alpha_i}(-z)}{\cos\pi\alpha_i},\quad(1)$$

with $M=+\frac{1}{2}$, $N=0$, using the identity

$$P_J(z)=\frac{\tan\pi J}{\pi}(Q_J(z)-Q_{-1-J}(z)),\quad(2)$$

before deforming the Sommerfeld-Watson contour integral. The first sum comes from the poles of $(\cos\pi J)^{-1}$ at the positive half odd integers. If we move the contour for the background integral left from $M=\frac{1}{2}$ to some fixed value M_0, the additional poles of $(\cos\pi J)^{-1}$ exactly cancel the first $N=[M_0+\frac{1}{2}]$[9] terms in this sum provided

$$F(s,J)=F(s,-1-J)$$

for $J=$half odd integers. We now include the additional poles of $F(s,J)$ from the region ($\mathrm{Re}J=-M_0$, $\mathrm{Re}J=-\frac{1}{2}$) in the Regge sum.

[8] S. Mandelstam, Ann. Phys. (N. Y.) 19, 254 (1959).
[9] $[N]$ means the greatest integer less than or equal to N.

In unequal-mass scattering

$$z_s=\frac{2st+s^2-s\Sigma+(m_a^2-m_b^2)(m_c^2-m_d^2)}{S_{ab}S_{cd}},$$

where

$$\Sigma=m_a^2+m_b^2+m_c^2+m_d^2,$$
$$S_{ab}^2=[s-(m_a-m_b)^2][s-(m_a+m_b)^2]=4sp_{ab}^2,$$
$$S_{cd}^2=[s-(m_c-m_d)^2][s-(m_c+m_d)^2]=4sp_{cd}^2.$$

Near $s=0$, z_s can be expanded as

$$z_s=1+\frac{2st}{\rho}\left(f_1(s)+\frac{1}{t}f_2(s)\times\frac{(m_a^2-m_b^2-m_c^2+m_d^2)(m_b^2m_c^2-m_a^2m_d^2)}{\rho}\right),\quad(3)$$

where $\rho=(m_a^2-m_b^2)(m_c^2-m_d^2)$, $f_1(s),f_2(s)$ are Taylor series in s, convergent in the region $|s|<s_m=\min((m_a-m_b)^2,(m_c-m_d)^2)$, with $f_1(0)=f_2(0)=1$. For $|z_s|\gg1$, the background integral and the first sum in (1) are bounded by z^{-M_0} and $z^{-N-1/2}$, respectively. Hence for fixed s they are bounded by t^{-M_0} and $t^{-N-1/2}$.

One Regge pole's contribution to the amplitude is

$$\frac{-(2\alpha+1)\beta(s)}{\pi\cos\pi\alpha(s)}Q_{-1-\alpha(s)}(-z_s),$$

where $-\beta(s)/\pi\cos\pi\alpha(s)$ is usually assumed to be of the form $\gamma_0(s)(p_{ab}p_{cd})^{\alpha(s)}$, and p_{ab}, p_{cd} are the initial and final s-channel, c.m. momenta. For fixed s this can be expanded in t as

$$(2\alpha+1)\gamma_0(s)\left(-\frac{S_{ab}S_{cd}}{4\rho}\right)^{\alpha(s)}\frac{[\Gamma(-\alpha(s))]^2}{2\Gamma(-2\alpha(s))}\Bigg\{t^\alpha+t^{\alpha-1}\left(\frac{\alpha\rho}{2s}+\text{terms less singular in } s \text{ at } s=0\right)+t^{\alpha-2}\left(\frac{\alpha(\alpha-1)^2}{(2\alpha-1)}\frac{\rho^2}{4s^2}+\text{terms less singular in } s \text{ at } s=0\right)+t^{\alpha-3}\left(\frac{\alpha(\alpha-1)(\alpha-2)^2}{3(2\alpha-1)}\frac{\rho^3}{8s^3}+\text{terms less singular}\right)+\cdots\Bigg\}.\quad(4)$$

From Mandelstam analyticity we expect $A(s,z)$ to have a Khuri-like expansion of the form

$$A(s,z)=A(s,t)=B(s,t)+\sum_{\mathrm{Re}\,\alpha_j>-M_0}g_j(s)t^{\alpha_j},$$

where $B(s,t)$ includes that portion of the function bounded by t^{-M_0}, and $g_j(s)$ are analytic and kinematically finite at $s=0$. Because of the bounds on the background integral and first sum of (1), we see that only Regge poles with $\mathrm{Re}\alpha > -M_0$ can contribute to the $g_j(s)$. However, the contributions of a single Regge pole, Eq. (4), explicitly violate this assumed analyticity. Therefore additional trajectories with $\alpha_l(0)=\alpha(0)-l$ and singular residues have the form

$$\gamma_1(s)=\gamma_0(s)(\rho/4s)(2\alpha+1)+\text{terms analytic at } s=0\,,$$
$$\gamma_2(s)=\gamma_0(s)(\alpha\rho^2/16s^2)(2\alpha+1)+\text{terms less singular at } s=0\,, \quad (5)$$

and similarly for $\gamma_l(s)$.

III. NONZERO-SPIN, UNEQUAL-MASS SCATTERING

The Reggeization of nonzero-spin, two-body scattering amplitudes has been considered by Calogero, Charap, and Squires[10] and by Gell-Mann, Goldberger Low, Marx, and Zachariasen.[11] An extension of these amplitudes to the Mandelstam-Sommerfeld-Watson transformation has been done by several people.[11–13] We follow here the method of Drechsler.[13] The method involves the conversion, analogous to $P_l \to Q_{-1-l}$, of the spin rotation functions $d_{\lambda\mu}{}^J$ to rotation functions of the second kind, $e_{-\lambda-\mu}{}^{-1-J}$. The relevant formula is

$$d_{\lambda\mu}{}^J(z)=\frac{\tan\pi(J-\lambda)}{\pi}\left[e_{\lambda\mu}{}^J(z)-e_{-\lambda,-\mu}{}^{-1-J}(z)\right]. \quad (6)$$

The definition and useful properties of $e_{\lambda\mu}{}^J(z)$ are given in the Appendix. These e's are not to be confused with the e's of Ref. 11, which are simply the d's without the half-angle factors.

Applying the Mandelstam-Sommerfeld-Watson transformation and again neglecting the contributions of the arc at infinity and cuts, Drechsler obtains

$$f_{\lambda_c\lambda_d\lambda_a\lambda_b}(s,z_s)=f_h(s,z_s)=\frac{1}{2\pi i}\int_{-M_0-i\infty}^{-M_0+i\infty}\frac{dJ(2J+1)}{\cos\pi(J-\lambda)}\{F_h{}^{(+)}(s,J)e_{-\lambda,\mu}{}^{\epsilon+}(-z,-1-J)+F_h{}^{(-)}(s,J)e_{-\lambda,\mu}{}^{\epsilon-}(-z,-1-J)\}$$
$$+(-1)^{\lambda_m-\lambda}\frac{1}{\pi}\sum_{n=N}^{\infty}(-1)^n2(n+\lambda_m)\{F_h{}^{(+)}(s,n+\lambda_m-\tfrac12)e_{\lambda-\mu}{}^{+}(-z,n+\lambda_m-\tfrac12)+F_h{}^{(-)}e_{\lambda,-\mu}{}^{-}\}$$
$$+\sum_{\mathrm{Re}\alpha_i>-M_0}\frac{(2\alpha_i+1)\beta_h{}^i(s)}{\cos\pi(\alpha_i-\lambda)}e_{-\lambda,\mu}{}^{\epsilon\pm}(-z,-1-\alpha_i)\,, \quad (7)$$

where

$$h=\{\lambda_c\lambda_d,\lambda_a\lambda_b\}\,,$$
$$\lambda_m=\max(|\lambda|,|\mu|)\,,$$
$$N=[M_0+\tfrac12-\lambda_m]\,,$$
$$\epsilon\pm=\pm\ \lambda,\mu \text{ integer}$$
$$=\mp\ \lambda,\mu \text{ half odd integer}\,,$$
$$F^{(\pm)}=\text{signatured amplitudes (not parity)}\,,$$
$$e_{\lambda\mu}{}^{\pm}(z,J)=\tfrac12[e_{\lambda\mu}{}^J(z)\pm(-1)^{\lambda-v}e_{\lambda,-\mu}{}^J(-z)] \text{ (signature combinations)}\,,$$
$$v=\tfrac12,\quad BF\to BF$$
$$=0,\quad \text{otherwise}.$$

For $M_0>\frac12$ the generalized Mandelstam symmetry $F_h{}^{(\pm)}(s,J)=(-1)^{\lambda-\mu}F_h{}^{(\epsilon\pm)}(s,-1-J)$ for $(J-\lambda)$ a half odd integer has been used. For large z the background integral is bounded by z^{-M_0} and the leading term in the first sum behaves like $z^{-(N+\lambda_m+1/2)}<z^{-M_0}$.

Since we shall eventually be concerned with those portions of the amplitude coming only from the Regge sum in (7), we note that

$$e_{\lambda\mu}{}^{\pm}(z,-1-\alpha)=\tfrac12(1\pm e^{-i\pi(\alpha+v)})e_{\lambda\mu}(z,-1-\alpha)\,,\quad \mathrm{Im}z>0.$$

We shall absorb the signature factor $\frac12(1\pm e^{-i\pi(\alpha+v)})$ into the residue function.

Following Gell-Mann *et al.*,[11] we consider the following combinations of helicity amplitudes:

$$\bar f_h{}^{\pm}\equiv\bar f_h\pm(-1)^{\lambda+\lambda_m}\eta_c\eta_d(-1)^{s_c+s_d-v}\bar f_{h_-}\,,$$

where

$$\bar f_h=(\sqrt2\cos\tfrac12\theta_s)^{-|\lambda+\mu|}(\sqrt2\sin\tfrac12\theta_s)^{-|\lambda-\mu|}f_h\,,$$
$$h_-=\{-\lambda_c-\lambda_d,\ \lambda_a\lambda_b\}\,.$$

When parity is conserved, the transition matrix element

$$F_h{}^{J\pm}\equiv(\sqrt{\tfrac12})[\langle\lambda_c\lambda_d,JM|\pm\eta_c\eta_d(-1)^{s_c+s_d-v}\times\langle-\lambda_c-\lambda_d,JM|]\,T\,(\sqrt{\tfrac12})[|JM,\lambda_a\lambda_b\rangle\pm\eta_a\eta_b(-1)^{s_a+s_b-v}|JM,-\lambda_a,-\lambda_b\rangle]$$

contains contributions from intermediate states of parity $\pm(-1)^J$ only. It follows that

$$F_{h_-}{}^{J\pm}=\pm\eta_c\eta_d(-1)^{s_c+s_d-v}F_h{}^{J\pm}\,.$$

Since $\beta_h{}^{\pm}(s)$ is the residue of $F_h{}^{J\pm}$, it satisfies the same symmetry property. Regge-pole contributions to $\bar f_h{}^{\pm}$ can

[10] F. Calogero, J. M. Charap, and E. J. Squires, Ann. Phys. (N. Y.) **25**, 325 (1963).
[11] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).
[12] R. L. Thews, Phys. Rev. **155**, 1624 (1967).
[13] W. Drechsler, Nuovo Cimento **53**, 115 (1968).

then be written as

$$\bar{f}_h{}^{\pm}=\sum_{\mathrm{Re}\alpha_i{}^+>-M_0}\left\{\frac{(2\alpha_i{}^++1)\beta_h{}^{i+}}{\cos\pi(\alpha_i{}^+-\lambda)}E_{-\lambda,\mu}{}^{\pm}(-z,\alpha_i{}^+)\right\}+\sum_{\mathrm{Re}\alpha_i{}^->-M_0}\left\{\frac{(2\alpha_i{}^-+1)\beta_h{}^{i-}}{\cos\pi(\alpha_i{}^--\lambda)}E_{-\lambda,\mu}{}^{\mp}(-z,\alpha_i{}^-)\right\},\quad(8)$$

where $\beta_h{}^{\pm}=\frac{1}{2}[\beta_h\pm\eta_c\eta_d(-1)^{s_c+s_d-v}\beta_{\bar{h}}]$ is the residue of a trajectory $\alpha_i{}^{\pm}(s)$ with parity $\pm(-1)^J$ and

$$E_{\lambda\mu}{}^{\pm}(z,J)=\frac{e_{\lambda\mu}(z,-1-J)}{(\sqrt{2}\cos\frac{1}{2}\theta_s)^{|\lambda+\mu|}(\sqrt{2}\sin\frac{1}{2}\theta_s)^{|\lambda-\mu|}}\pm(-1)^{\lambda+\lambda_m}\frac{e_{\lambda,-\mu}(z,-1-J)}{(\sqrt{2}\cos\frac{1}{2}\theta_s)^{|\lambda-\mu|}(\sqrt{2}\sin\frac{1}{2}\theta_s)^{|\lambda+\mu|}}.\quad(9)$$

Properties of the $E_{\lambda\mu}{}^{\pm}(z,J)$ are discussed in the Appendix.[14] As in the spinless case, we expand $\bar{f}^{\eta}$ in the region $|t|>|\rho/s|$, fixed s, as

$$\bar{f}_h{}^{\eta}=B_h{}^{\eta}(s,t)+\sum_{\mathrm{Re}\alpha_j>-M_0}g_{h,j}{}^{\eta}(s)t^{\alpha_j}.\quad(10)$$

The terms in the sum of Eq. (10) come entirely from the Regge-pole terms in Eq. (7), while $B_h(s,t)$ includes terms from the background integral, the first sum, and lower-order terms in Eq. (7). $B_h(s,t)$ is bounded for large t by $t^{-M_0-\lambda_m}$.

A single Regge pole of parity η contributes to the following amplitudes:

$$\bar{f}_h{}^{\eta}=\frac{2\alpha+1}{\cos\pi(\alpha-\lambda)}\beta_h{}^{\eta}(s)E_{-\lambda,\mu}{}^{+}(-z,\alpha)$$

and

$$\bar{f}_h{}^{-\eta}=\frac{2\alpha+1}{\cos\pi(\alpha-\lambda)}\beta_h{}^{\eta}(s)E_{-\lambda,\mu}{}^{-}(-z,\alpha).\quad(11)$$

Expanding these amplitudes as in (10), we obtain

$$\bar{f}_h{}^{\eta}=\frac{\gamma_h{}^0(s)}{(\sqrt{s})^{|\lambda|+|\mu|}}\left(\frac{S_{ab}S_{cd}}{2\rho}\right)^{\alpha(s)-\lambda_m}C(\alpha,\lambda,\mu)\left\{t^{\alpha-\lambda_m}+t^{\alpha-\lambda_m-1}\left(\frac{(\alpha-\lambda_m)\rho}{2s}+\text{terms less singular in } s \text{ at } s=0\right)+t^{\alpha-\lambda_m-2}\left(\frac{(\alpha-\lambda_m)(\alpha-\lambda_m-1)(\alpha^2-\alpha+\mu_m{}^2)}{\alpha(2\alpha-1)}\frac{\rho^2}{4s^2}+\text{terms less singular in } s \text{ at } s=0\right)+\cdots\right\},\quad(12a)$$

$$\bar{f}_h{}^{-\eta}=\frac{-\gamma_h{}^0(s)}{(\sqrt{s})^{|\lambda|+|\mu|}}\left(\frac{S_{ab}S_{cd}}{2\rho}\right)^{\alpha-\lambda_m}C(\alpha,\lambda,\mu)\mu_m\,\mathrm{sign}(\lambda\mu)\frac{\alpha-\lambda_m}{\alpha}\left\{t^{\alpha-\lambda_m-1}\left(\frac{\rho}{2s}+\text{terms less singular in } s \text{ at } s=0\right)+t^{\alpha-\lambda_m-2}\left(\frac{\rho^2(\alpha-\lambda_m-1)}{4s^2}+\text{terms less singular in } s \text{ at } s=0\right)+\cdots\right\},\quad(12b)$$

where

$$\frac{\beta_h(s)}{\cos\pi(\alpha-\lambda)}=\gamma_h{}^0(s)(p_{ab}p_{cd})^{\alpha-\lambda_m}(\sqrt{s})^{-|\lambda|-|\mu|},$$

$$C(\alpha,\lambda,\mu)=\epsilon(-1)^{\lambda_m-\lambda+\alpha}\mathrm{sgn}(\lambda,\mu)\frac{1}{2^{\alpha}}\frac{(2\alpha+1)}{\Gamma(-2\alpha)}\{\Gamma(\lambda_m-\alpha)\Gamma(-\lambda_m-\alpha)\Gamma(\mu_m-\alpha)\Gamma(-\mu_m-\alpha)\}^{1/2},$$

$\mu_m=\min(|\lambda|,|\mu|)$, and $\mathrm{sgn}(\lambda,\mu)$ is the sign[13] of $e_{\lambda\mu}{}^J(z)$ and is given in the Appendix.

Mandelstam analyticity implies that $f_h(s,t)$ is analytic at $s=0$. From the half-angle factors both[15]

$$\bar{f}_h{}^{\pm}\underset{s\to0}{\longrightarrow}(\sqrt{s})^{-|\lambda|-|\mu|}\quad(13)$$

and

$$s^a\bar{f}_h{}^{+}+\mathrm{sgn}(\lambda\mu\rho)s^a\bar{f}_h{}^{-}=0\quad(14)$$

and its first μ_m-1 derivatives are zero, where $a=\frac{1}{2}(\lambda_m+\mu_m)$. In order for the expansion in (12a) or (12b) to satisfy (13), we must again include daughter trajectories, spaced at $\alpha(0)-1$, $\alpha(0)-2,\cdots$. However, the daughters are not enough to satisfy (13) and (14) simultaneously.

For (12a) to satisfy (13), the residues of the daughters are

$$\gamma_h{}^1=\gamma_h{}^0\frac{\rho}{4s}\left(\frac{\alpha-\lambda_m}{\alpha+\lambda_m}\right)^{1/2}\frac{\alpha}{(\alpha^2-\mu_m{}^2)^{1/2}}\times(2\alpha+1),\quad(15)$$

$$\gamma_h{}^2=\gamma_h{}^0\frac{\rho^2}{16s^2}\left(\frac{\alpha-\lambda_m}{\alpha+\lambda_m}\right)^{1/2}\left(\frac{\alpha-\lambda_m-1}{\alpha+\lambda_m-1}\right)^{1/2}\times\frac{(\alpha^2-\mu_m{}^2)^{1/2}(\alpha-1)}{[(\alpha-1)^2-\mu_m{}^2]^{1/2}}\times(2\alpha+1),$$

and similarly for $\gamma_h{}^i$. Substituting these values into

[14] The $E_{\lambda\mu}{}^{\pm}(z,J)$ that we have defined in Eq. (9) are proportional to $E_{\lambda\mu}{}^{J\pm}(z)$ of Ref. 11. Their equivalence is demonstrated in the Appendix.

[15] Equation (14) does not hold for $BF\to B'F'$. In what follows, we do not consider these reactions.

(12b), we have

$$\bar{f}_h^{-\eta}=\frac{-\mu_m\gamma_h{}^0}{(\sqrt{s})^{|\lambda|+|\mu|}}\left(\frac{S_{ab}S_{cd}}{2\rho}\right)^{\alpha-\lambda_m}C(\alpha,\lambda,\mu)\ \mathrm{sgn}(\lambda\mu)\times\frac{\alpha-\lambda_m}{\alpha}\left\{\frac{\rho}{2s}t^{\alpha-\lambda_m-1}-\left(\frac{\alpha-\lambda_m-1}{\alpha-1}\right)\frac{\rho^2}{4s^2}t^{\alpha-\lambda_m-2}+\cdots\right\}.$$

This violates (13) unless $\mu_m=0$. To restore the analyticity at $s=0$ (if $\mu_m\neq 0$), additional trajectories of opposite parity (conspirators)[7] are required, with $\alpha_l{}^c(0)=\alpha_l(0)$, whose residues satisfy

$$\gamma_h{}^{c,l}=-\mathrm{sign}(\lambda\mu\rho)\gamma_h{}^l$$

and the new values of the residues of the daughters satisfy

$$\gamma_h{}^1=\gamma_h{}^0\frac{\rho}{4s}\left(\frac{\alpha-\lambda_m}{\alpha+\lambda_m}\right)^{1/2}\left(\frac{\alpha+\mu_m}{\alpha-\mu_m}\right)^{1/2}\times(2\alpha+1),$$

$$\gamma_h{}^2=\gamma_h{}^0\frac{\rho^2}{16s^2}\left(\frac{\alpha-\lambda_m}{\alpha+\lambda_m}\right)^{1/2}\left(\frac{\alpha+\mu_m}{\alpha-\mu_m}\right)^{1/2}\left(\frac{\alpha-\lambda_m-1}{\alpha+\lambda_m-1}\right)^{1/2}\times\left(\frac{\alpha+\mu_m-1}{\alpha-\mu_m-1}\right)^{1/2}\alpha(2\alpha+1).$$

The above analysis has assumed that $\gamma_h{}^0(s)$ is analytic and nonzero at $s=0$; other forms, proportional to s^n, $n=1, 2, \cdots$, are possible. These evasive solutions are frequently introduced to satisfy the kinematic constraint (14) trivially. For

$$\gamma_h{}^0(s)=s\gamma_h{}^E(s),\quad [\gamma_h{}^E(0)\text{ nonzero}]$$

we find that a solution to (13) and (14) exists with no conspirator required above $\alpha_c=\alpha(0)-4$, the extent of our calculation, for $\mu_m=0$ or 1, with the residues of the daughters given by

$$\gamma_h{}^1(s)=\gamma_h{}^E(s)\left(\frac{\alpha-\lambda_m}{\alpha+\lambda_m}\right)^{1/2}\left(\frac{\alpha-\mu_m}{\alpha+\mu_m}\right)^{1/2}\tfrac{1}{4}\rho\times(2\alpha+1),$$

$$\gamma_h{}^2(s)=-\gamma_h{}^E(s)\left(\frac{\alpha-\lambda_m}{\alpha+\lambda_m}\right)^{1/2}\left(\frac{\alpha-\lambda_m-1}{\alpha+\lambda_m-1}\right)^{1/2}\frac{(\alpha-2\mu_m)}{(\alpha^2-\mu_m{}^2)^{1/2}}\times\frac{\alpha}{[(\alpha-1)^2-\mu_m{}^2]^{1/2}}\frac{\rho^2}{16s}(\alpha-1)(2\alpha+1).$$

For $\gamma_h{}^0(s)=s^2\gamma_h{}^E(s)$, we find again that no conspirator is required above $\alpha_c=\alpha(0)-4$ for $\mu_m=0, 1, 2$, with, now,

$$\gamma_h{}^1(s)=\gamma_h{}^E(s)\left(\frac{\alpha-\lambda_m}{\alpha+\lambda_m}\right)^{1/2}\left(\frac{\alpha-\mu_m}{\alpha+\mu_m}\right)^{1/2}\tfrac{1}{4}\rho s\times(2\alpha+1),$$

$$\gamma_h{}^2(s)=-\gamma_h{}^E(s)\left(\frac{\alpha-\lambda_m}{\alpha+\lambda_m}\right)^{1/2}\left(\frac{\alpha-\lambda_m-1}{\alpha+\lambda_m-1}\right)^{1/2}\left(\frac{\alpha-\mu_m}{\alpha+\mu_m}\right)^{1/2}\times\left(\frac{\alpha-\mu_m-1}{\alpha+\mu_m-1}\right)^{1/2}\times\tfrac{1}{16}\alpha\rho^2(2\alpha-3).$$

For $\gamma_h{}^0(s)\to s^n\gamma_h{}^E(s)$ and $n<\mu_m$, conspirators are required, with $\alpha_c(0)=\alpha(0)$. We speculate that, in the cases above, with $n=1$, $\mu_m=0, 1$ and $n=2$, $\mu_m=0, 1, 2$, conspirators are not required to any order and that, in general, for $n\geq\mu_m$ a solution to (13) and (14) exists without conspiracy.

IV. SUMMARY

Freedman and Wang[2] have shown that for unequal-mass, zero-spin scattering the reduced Regge residue $\gamma(s)$ is analytic at $s=0$. Assuming that their result remains true for the nonzero-spin case, we have investigated in the Regge model the consistency of $s=0$ kinematic constraints and analyticity. We find that both daughters and conspirators are necessary unless the Regge residue evades sufficiently.

The problem of equal-mass $N_1\bar{N}_1\to N_2\bar{N}_2$ scattering has been investigated by Freedman and Wang,[5] using the $O(4)$ symmetry of the helicity amplitudes for $s=0$, $(m_1-m_2)^2<t<(m_1+m_2)^2$. Although their results are not strictly applicable to the unequal-mass scattering in the large-t region and our results are not applicable when either the initial or final masses are equal, we might expect the general features of the two cases to be similar. Freedman and Wang found three distinct solutions of the $s=0$, equal-mass constraint

$$f_1-f_3-z_sf_4\underset{s\to 0}{\longrightarrow}s,\qquad(16)$$

where

$$f_1=f_{++,++}{}^s-f_{++,--}{}^s,$$
$$f_2=f_{++,++}{}^s+f_{++,--}{}^s,$$
$$f_3=(1+z)^{-1}f_{+-,+-}{}^s-(1-z)^{-1}f_{+-,-+}{}^s,$$
$$f_4=(1+z)^{-1}f_{+-,+-}{}^s+(1-z)^{-1}f_{+-,-+}{}^s,$$
$$f_5=(1-z^2)^{-1/2}f_{++,+-}{}^s.$$

Their type-I solution corresponds to the vanishing of the residues of Regge poles in f_1, f_3, f_4 at $s=0$ (evasion) and a nonvanishing natural-parity contribution in f_2. This corresponds to an evasive solution for those amplitudes with $\mu_m\neq 0$, f_3 and f_4, and a finite contribution to an amplitude with $\mu_m=0$, f_2. This is consistent with our results.

The type-II solution consists of unnatural-parity Regge poles in amplitudes f_1 at $\alpha-1$ and f_3 at α and no nonvanishing natural-parity contributions to the other amplitudes. For this to be consistent in the unequal-mass case, we require

$$\beta_{f_3}(s)\underset{s\to 0}{\longrightarrow}\frac{s\gamma(s)(p_{ab}p_{cd})^{\alpha-1}}{(\sqrt{s})^{\lambda_m+\mu_m}}.$$

This corresponds to an evasive solution of the unequal-mass, $s=0$ constraint,

$$f_3+f_4\underset{s\to 0}{\longrightarrow}\text{finite}.\qquad(17)$$

The type-III solution has unnatural-parity poles in f_1 at α and in f_3 at $\alpha-1$, and a natural-parity pole in f_4 at α. For unequal-mass scattering the particle which corresponds to the pole at α in f_1 will contribute to the amplitude f_3 with residue

$$\beta_{f_3}^{-}(s)\sim(m_1-m_2)(m_3-m_4)\beta_{f_3}'^{-}(s).$$

We then have two Regge poles at α of opposite parity contributing to the amplitudes f_3 and f_4 which may conspire to satisfy Eq. (17). In the equal-mass case, the residue of the pole at α in f_4 does not have the factor $(m_1-m_2)(m_3-m_4)$. In the unequal-mass case, its residue is expected to pick up just such a factor because

$$f_4\to[s-(m_1-m_2)^2]^{1/2} \text{ near } s=(m_1-m_2)^2$$

and

$$f_4\to[s-(m_3-m_4)^2]^{1/2} \text{ near } s=(m_3-m_4)^2,$$

while f_3 is analytic near these points. Near $s=0$ these factors provide the required $(m_1-m_2)(m_3-m_4)$, so that the equality of the residues implied by (17) does not contain an additional factor of ρ.

In our method, since we have not explicitly used a dispersion integral for the $g_{h,j}{}^{\eta}(s)$ in Eq. (10), we can calculate the most divergent term in $g_{h,j}{}^{\eta}(s)$ but cannot determine the less singular portions. Hence we have not shown the *necessity of* daughters at spacings of $\Delta\alpha=2$ in the $\lim \rho\to 0$, which have been obtained by Freedman and Wang,[2] since these residues are less singluar than the ones which we have found for $\rho\neq 0$.

We have assumed that branch cuts and essential singularities are sufficiently far to the left in the J plane that they do not invalidate moving the contour integral to $\mathrm{Re}J=-M_0$, or if they are not sufficiently far removed, their contributions to the amplitudes are bounded at $s=0$.

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APPENDIX I: JACOBI FUNCTIONS

The two linearly independent solutions of Jacobi's differential equation[16]

$$\left\{(1-z^2)\frac{d^2}{dz^2}-[a-b+(a+b+2)z]\frac{d}{dz}+n(n+a+b+1)\right\}y=0$$

are

$$P_n{}^{a,b}(z)=\frac{\Gamma(n+a+1)}{\Gamma(n+1)\Gamma(a+1)}\times F(-n,n+a+b+1,a+1;\tfrac{1}{2}(1-z)), \quad \text{(A1)}$$

$$Q_n{}^{a,b}(z)=\frac{2^{n+a+b}\Gamma(n+a+1)\Gamma(n+b+1)}{\Gamma(2n+a+b+2)(z-1)^{n+a+1}(z+1)^b}\times F(n+1,n+a+1,2n+a+b+2;2/(1-z)). \quad \text{(A2)}$$

Since $P_n{}^{a,b}$, $Q_n{}^{a,b}$, and $Q_{-n-a-b-1}{}^{a,b}$ are all solutions of the differential equation for the same values of a, b, and n, there must exist a linear relation between them. Using hypergeometric function identities, we find

$$P_n{}^{a,b}(z)=\frac{\tan\pi n}{\pi}\left\{(-1)^a Q_n{}^{a,b}(z)-\frac{\Gamma(n+b+1)\Gamma(n+a+1)}{\Gamma(n+1)\Gamma(n+a+b+1)}Q_{-n-a-b-1}{}^{a,b}(z)\right\}. \quad \text{(A3)}$$

Continuing to use the hypergeometric function identities and (A3), we obtain the following reflection formulas:

$$Q_n{}^{b,a}(-z)=e^{-i\pi(n+a+b+1)}Q_n{}^{a,b}(z),$$

$$P_n{}^{b,a}(-z)=e^{i\pi n}P_n{}^{a,b}(z)-2(-1)^b\frac{\sin\pi n}{\pi}Q_n{}^{a,b}(z), \quad \mathrm{Im}z>0. \quad \text{(A4)}$$

Another expansion of $Q_n{}^{a,b}(z)$ involving only $(z-1)$ which can be obtained from (A2) by hypergeometric function identities is

$$Q_n{}^{a,b}(z)=2^{n+a+b}\frac{\Gamma(n+a+1)\Gamma(n+b+1)}{\Gamma(2n+a+b+2)}\frac{1}{(z-1)^{n+a+b+1}}\times F(n+a+b+1,n+b+1,2n+a+b+2;-2/(z-1)). \quad \text{(A5)}$$

This expansion is particularly useful since, near $s=0$, $z-1\sim 2st/\rho$.

APPENDIX II: ROTATION FUNCTIONS

In terms of these functions the rotation functions of the first and second kinds, $d_{\lambda\mu}{}^J(z)$ and $e_{\lambda\mu}{}^J(z)$, are defined as

$$d_{\lambda\mu}{}^J(z)=\mathrm{sgn}(\lambda,\mu)2^{-\lambda_m}\left\{\frac{\Gamma(J+\lambda_m+1)\Gamma(J-\lambda_m+1)}{\Gamma(J+\mu_m+1)\Gamma(J-\mu_m+1)}\right\}^{1/2}\times(1-z)^{\frac{1}{2}|\lambda-\mu|}(1+z)^{\frac{1}{2}|\lambda+\mu|}P_{J-\lambda_m}{}^{|\lambda-\mu|,|\lambda+\mu|}(z),$$

$$e_{\lambda\mu}{}^J(z)=(-1)^{\lambda-\mu}\mathrm{sgn}(\lambda,\mu)2^{-\lambda_m}\times\left\{\frac{\Gamma(J+\lambda_m+1)\Gamma(J-\lambda_m+1)}{\Gamma(J+\mu_m+1)\Gamma(J-\mu_m+1)}\right\}^{1/2}\times(1-z)^{\frac{1}{2}|\lambda-\mu|}(1+z)^{\frac{1}{2}|\lambda+\mu|}Q_{J-\lambda_m}{}^{|\lambda-\mu|,|\lambda+\mu|}(z), \quad \text{(A6)}$$

[16] A number of the equations and relations in the Appendix have been taken from *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. II, and *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (U. S. Department of Commerce, National Bureau of Standards, Washington, D. C., 1964), Appl. Math. Ser. 55.

where λ_m, μ_m = max, min($|\lambda|,|\mu|$), respectively, and

$$\mathrm{sgn}(\lambda,\mu)=(-1)^{(\lambda-\mu)\theta(\mu-\lambda)}=(-1)^{\frac{1}{2}[\lambda-\mu-\lambda_m+\mathrm{sgn}(\lambda\mu)\mu_m]}.$$

The function $\mathrm{sgn}(\lambda,\mu)$ has the following symmetry properties:

$$\mathrm{sgn}(\lambda,-\mu)=(-1)^{\lambda_m-\lambda}\,\mathrm{sgn}(\lambda,\mu),$$
$$\mathrm{sgn}(-\lambda,\mu)=(-1)^{\lambda_m+\mu}\,\mathrm{sgn}(\lambda,\mu),$$
$$\mathrm{sgn}(-\lambda,-\mu)=(-1)^{\lambda-\mu}\,\mathrm{sgn}(\lambda,\mu),$$
$$\mathrm{sgn}(\mu,\lambda)=(-1)^{\lambda-\mu}\,\mathrm{sgn}(\lambda,\mu).$$

From (A3) and (A6) we then obtain (6):

$$d_{\lambda\mu}{}^J(z)=\frac{\tan\pi(J-\lambda)}{\pi}[e_{\lambda\mu}{}^J(z)-e_{-\lambda,-\mu}{}^{-1-J}(z)].\quad(6)$$

The asymptotic form of $e_{\lambda\mu}{}^J(z)$ for large z can be obtained from (A5) and (A6).

APPENDIX III: *E* FUNCTIONS

The functions $E_{\lambda\mu}{}^{\pm}(z,J)$ can be written conveniently as

$$E_{\lambda\mu}{}^{\pm}(z,J)=\frac{(-1)^{\lambda-\mu}\mathrm{sgn}(\lambda,\mu)}{2^{\lambda_m}}\times\left\{\frac{\Gamma(J+\mu_m+1)\Gamma(J-\mu_m+1)}{\Gamma(J+\lambda_m+1)\Gamma(J-\lambda_m+1)}\right\}^{1/2}\times\left(Q_{-1-J-\lambda_m}{}^{|\lambda-\mu|,|\lambda+\mu|}(z)\pm e^{-\mathrm{sign}\,\mathrm{Im}(z)i\pi(J-\lambda_m)}\times Q_{-1-J-\lambda_m}{}^{|\lambda-\mu|,|\lambda+\mu|}(-z)\right).$$

This follows from Eqs. (9), (A4), and (A6). $E_{\lambda\mu}{}^{\pm}(z,J)$ have the symmetry property

$$E_{\lambda\mu}{}^{\pm}(z,J)=\pm e^{i\pi(J-\lambda_m)}E_{\lambda\mu}{}^{\pm}(-z,J)$$

for Im$z>0$.

For large z, $E_{\lambda\mu}{}^{\pm}(z,J)$ behave as follows:

$$E_{\lambda\mu}{}^{+}(z,J)\underset{z\to\infty}{\longrightarrow}z^{J-\lambda_m}P_1(1/z^2),$$
$$E_{\lambda\mu}{}^{-}(z,J)\underset{z\to\infty}{\longrightarrow}z^{J-\lambda_m-1}P_2(1/z^2),$$

where P_1 and P_2 are power series convergent in $|1/z^2|<1$.

Our functions $E_{\lambda\mu}{}^{\pm}(z,J)$ and the functions $E_{\lambda\mu}{}^{J\pm}(z)$ of Gell-Mann *et al.*[11] are related by

$$E_{\lambda\mu}{}^{\pm}(z,J)=\frac{-\pi}{\tan\pi(J-\lambda_m)}E_{\lambda\mu}{}^{J\pm}(z).\quad(\mathrm{A7})$$

For $\lambda=\mu=0$ this follows from the definition of the rotation function of the second kind, $e_{00}{}^J(z)$, and their definition of $\mathcal{P}_J(z)$ [Eq. (B5) of Ref. 11]. For λ, $\mu\neq0$ their definition of $E_{\lambda\mu}{}^{J\pm}(z)$ involves use of the "stepping" operators

$$S_1P_n{}^{a,b}(z)=\frac{d}{dz}P_n{}^{a,b}(z)=\tfrac{1}{2}(n+a+b+1)P_{n-1}{}^{a+1,b+1}(z)$$

and

$$S_2P_n{}^{a,b}(z)=\left((1-z)\frac{d}{dz}-a\right)P_n{}^{a,b}(z)=-(n+a)P_n{}^{a-1,b+1}(z).$$

Operating on $Q_n{}^{a,b}(z)$, S_1 and S_2 can be shown to yield $-\frac{1}{2}(n+a+b+1)Q_{n-1}{}^{a+1,b+1}(z)$ and $+(n+a)Q_n{}^{a-1,b+1}(z)$, respectively. Therefore Eq. (A7) is a consequence of our definition (A6) and Eq. (9) and their definition given by (A9) and the paragraph above (B11) of Ref. 11.