Reaction Amplitudes for Pseudoscalar Mesons on Spin-$\frac{1}{2}$ Fermions in Fourth-Order Perturbation Theory*

JOSEPH J. KUBIS
Texas A&M University, College Station, Texas and Cavendish Laboratory, Cambridge, England

AND

JOHN L. GAMMEL
Texas A&M University, College Station, Texas and Los Alamos Scientific Laboratory, Los Alamos, New Mexico

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The invariant reaction amplitudes for pseudoscalar mesons on spin-$\frac{1}{2}$ fermions are calculated in fourth-order perturbation theory. The masses of the incoming, outgoing, and internal lines of all the diagrams are arbitrary, subject to some tacit restrictions about anomalous thresholds. The amplitudes for the self-energy, vertex, and box graphs are calculated by dispersion-relation techniques. Formulas for absorptive parts and double spectral functions are explicitly given, and the self-energy and vertex graphs have been renormalized. The A amplitude for the box diagram is expressed as a single-variable dispersion relation plus a once-subtracted double dispersion relation. Numerical as well as analytic methods were used to check the expressions obtained.

I. INTRODUCTION

In this paper, we present a calculation of reaction amplitudes for pseudoscalar mesons and spin-$\frac{1}{2}$ fermions in fourth-order perturbation theory. This work was motivated by efforts to construct models for pion-nucleon scattering which used as input not only second-order, but also fourth-order, diagrams. We found to our surprise that calculations of the fourth-order amplitudes had never been published in the complete detail necessary for numerical work, and we have tried to fill that gap.

The earliest efforts to calculate fourth-order diagrams were those of Ashkin, Simon, and Marshak.[1] They gave analytic expressions for the self-energy, vertex, and box diagrams of pion-nucleon pseudoscalar-meson theory in the low-energy, or "Thomson" limit. Wyld[2] calculated numerically the amplitudes for the fourth-order pion-nucleon diagrams in the lowest power of the center-of-mass momentum near the threshold scattering energy, but he gave no analytic expressions. Schweber[3] briefly treated the renormalization problem in pseudoscalar-meson theory in a way analogous to that used in quantum electrodynamics,[4] but he did not give final results.

Our purpose is to present the calculation of fourth-order diagrams in more complete detail, explicit enough to use in numerical computations. Instead of restricting ourselves to pion-nucleon scattering, we consider more general combinations of masses, both on internal and external lines, so that we can treat reactions where the final meson-baryon state is not the same as the initial one.

Dispersion relations are the principal calculational technique. We write the two invariant amplitudes for meson-baryon reactions in dispersion-relation form and calculate the absorptive parts by using the Cutkosky rules.[5–7] When we do this we make a tacit assumption that the dispersion-relation form is valid even in certain cases where anomalous thresholds exist. Criteria for this are given by Mandelstam[6] and by Karplus, Sommerfield, and Wichmann.[8] We have not dealt with this problem any further. Because of the dispersion-relation form for the amplitude of a given diagram, we can easily implement crossing symmetry for scattering reactions, where the initial and final mesons are the same. Even in the general unequal-mass case we have formal relations between amplitudes of direct and crossed diagrams. The transformation from one type to the other is straightforward, and we thus can halve the amount of algebraic manipulation needed.

The fact of spin leads to some complications. When the Dirac algebra is done, we have polynomial expressions in the invariant variables replacing the simple constants that occur for the case of scalar particles. The renormalized self-energy and vertex amplitudes are expressed as single-variable dispersion relations, but the box diagram amplitude is more complicated. In the purely scalar case it would be described by a double dispersion relation. When spin-$\frac{1}{2}$ enters the picture, one of the invariant amplitudes has to be written as the sum of two terms, a double dispersion relation and an addi-

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[1] J. Ashkin, A. Simon, and R. Marshak, Progr. Theoret. Phys. (Kyoto) **5**, 634 (1950).

[2] H. W. Wyld, Jr., Phys. Rev. **96**, 1661 (1954).

[3] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961), pp. 575–579.

[4] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1952), pp. 178–202.

[5] R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960).

[6] S. Mandelstam, Phys. Rev. **115**, 1741 (1959).

[7] W. B. Rolnick, Phys. Rev. Letters **16**, 544 (1966). This paper notes mistakes of factors of 2 in the Cutkosky and Mandlestam papers cited in Refs. 5 and 6.

[8] R. Karplus, C. M. Sommerfield, and E. H. Wichmann, Phys. Rev. **114**, 376 (1959).

tional single-variable dispersion relation. The other invariant amplitude has only the double-dispersion-relation term. This situation was mentioned by Mandelstam.[9]

Our paper begins with a brief description in Sec. II of the kinematics and notation we use. We discuss the dispersion-relation approach in Sec. III, and devote the rest of the paper to the calculation of the different types of diagrams. We calculate the Born amplitudes in Sec. IV, the self-energy amplitudes in Sec. V, the vertex amplitudes in Sec. VI, and the box amplitudes in Sec. VII. A brief comment on the accuracy of our results in Sec. VIII concludes the paper.

II. KINEMATICS

The notation we use for the kinematics of meson-baryon reactions generally follows that of Frautschi and Walecka,[10] although we use the Feynman rules and Lorentz metric of Schweber.[11] We have four-momenta p_1 and k_1 for the incident baryon and meson, with masses m_1 and μ_1, respectively, and four-momenta p_2 and k_2 for the outgoing baryon and meson, with masses m_2 and μ_2, respectively (cf. Fig. 1). Then

$$p_1+k_1=p_2+k_2, \qquad (2.1)$$

and we call this the s channel. We define our invariants in the usual way:

$$\begin{aligned} s&=(p_1+k_1)^2,\\ t&=(p_1-p_2)^2,\\ u&=(p_1-k_2)^2, \end{aligned} \qquad (2.2)$$

with the relation

$$s+t+u=m_1^2+m_2^2+\mu_1^2+\mu_2^2. \qquad (2.3)$$

In the center-of-mass system for the s channel, we let W be the total energy. If we label the magnitudes of the three-momenta for incident and final states by q_1 and q_2, we have

$$\begin{aligned} s&=W^2,\\ q_i^2&=\tfrac{1}{4}[s-(m_i+\mu_i)^2][s-(m_i-\mu_i)^2]/s, \quad i=1,\,2. \end{aligned} \qquad (2.4)$$

Also,

$$\begin{aligned} p_1&=(p_{10},\mathbf{q}_1), & p_2&=(p_{20},\mathbf{q}_2),\\ k_1&=(k_{10},-\mathbf{q}_1), & k_2&=(k_{20},-\mathbf{q}_2), \end{aligned} \qquad (2.5)$$

with

$$\begin{aligned} p_{i0}&=\tfrac{1}{2}(s+m_i^2-\mu_i^2)/\sqrt{s},\\ k_{i0}&=\tfrac{1}{2}(s-m_i^2+\mu_i^2)/\sqrt{s}, \quad i=1,\,2. \end{aligned} \qquad (2.6)$$

The reaction angle ϑ is given by

$$t=m_1^2+m_2^2-2p_{10}p_{20}+2q_1q_2\cos\vartheta. \qquad (2.7)$$

[9] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[10] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

[11] S. S. Schweber, work cited in Ref. 3, pp. xiv, 478–479.

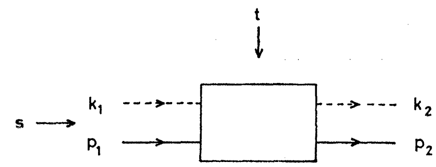


FIG. 1. Meson-baryon reaction in the s channel.

We define the T matrix by

$$S_{fi}=\delta_{fi}+i(2\pi)^4\delta(p_2+k_2-p_1-k_1)\times\left[\frac{m_1m_2}{4p_{10}p_{20}k_{10}k_{20}}\right]^{1/2}\bar{u}_2Tu_1, \qquad (2.8)$$

and T is written in invariant form

$$T=A(s,t,u)+\tfrac{1}{2}(k_1+k_2)B(s,t,u). \qquad (2.9)$$

We can write the reaction cross section in the center-of-mass system as

$$\frac{d\sigma_{fi}}{d\Omega}=\frac{q_2}{q_1}|f|^2, \qquad (2.10)$$

where

$$f=\chi_2^+[f_1+f_2(\boldsymbol{\sigma}\cdot\hat{p}_2)(\boldsymbol{\sigma}\cdot\hat{p}_1)]\chi_1 \qquad (2.11)$$

and is related to T by

$$f=\frac{1}{4\pi}\frac{(m_1m_2)^{1/2}}{W}\bar{u}_2Tu_1. \qquad (2.12)$$

The functions f_1 and f_2 are related to A and B by

$$f_1=\{[(p_{10}+m_1)(p_{20}+m_2)]^{1/2}/(8\pi W)\}\times\{A+[W-\tfrac{1}{2}(m_1+m_2)]B\}, \qquad (2.13)$$

$$f_2=\{[(p_{10}-m_1)(p_{20}-m_2)]^{1/2}/(8\pi W)\}\times\{-A+[W+\tfrac{1}{2}(m_1+m_2)]B\}. \qquad (2.14)$$

III. DISPERSION-RELATION METHODS

The diagrams which represent reactions of pseudoscalar mesons on spin-$\frac{1}{2}$ fermions are shown in Fig. 2. When we look at the meson-baryon reactions we see two major categories of graphs, the "direct" ones, labeled by the letter D, and the "crossed" ones, labeled by the letter C. The amplitudes for the direct graphs can be expressed in terms of the invariant variables s and t. The amplitudes for the crossed graphs are formally the same as those for the direct graphs upon replacing the variable s by the variable u and interchanging masses of internal and external lines. The details are given in the following sections.

The Born graphs, labeled BD and BC, have simple amplitudes which are poles in the variables s and u, respectively. The self-energy graphs, labeled SED and SEC, and the vertex graphs, labeled VD1, VD2, and VC1, VC2, have amplitudes which are expressible as

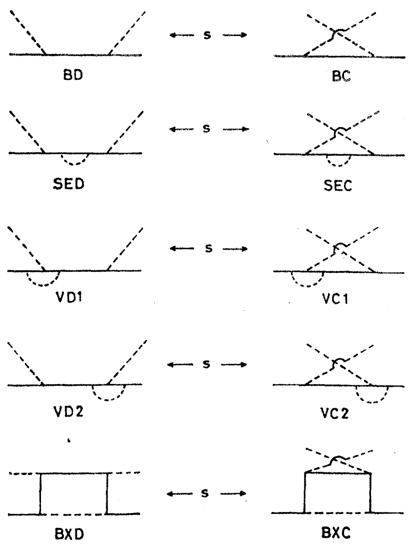


FIG. 2. Diagrams for meson-baryon reactions.

single-variable dispersion relations. The amplitudes must be renormalized and the infinite parts absorbed into the definition of the new coupling constants. This procedure is well known for quantum electrodynamics, and it is used here also. When we calculate the absorptive part of either the self-energy or the vertex graph and insert it into the dispersion relation, we automatically have the correct real part of the amplitude. All the finite parts of the amplitude are sorted out correctly. This method was recently used by Chou and Dresden[12] in their calculation of amplitudes in quantum electrodynamics. For the direct graphs the finite amplitudes have the form

$$F(s)=\frac{1}{\pi}\int_{s_0}^{\infty} ds'\frac{\operatorname{Im}F(s')}{s'-s-i\epsilon} \tag{3.1}$$

and for the crossed graphs the finite amplitudes have the form

$$F(u)=\frac{1}{\pi}\int_{u_0}^{\infty} du'\frac{\operatorname{Im}F(u')}{u-u'-i\epsilon}, \tag{3.2}$$

where F represents either the A or the B amplitude.

The box diagram is more complicated. If all particles were scalar, we know that we can write the amplitude as a double dispersion relation and obtain the spectral function by well-known rules. In the case of spin, we have not only a complicated numerator, but the dispersion relation for the A amplitude must be subtracted in the s variable. The subtraction term is finite. We have for the direct graph

$$A(s,t)=\frac{1}{\pi}\int_{t_0}^{\infty} dt'\frac{\operatorname{Im}A(s_o,t')}{(t'-t-i\epsilon)}+\frac{1}{\pi^2}(s-s_o)$$
$$\times\int_{s_1}^{\infty} ds'\int_{t_1(s')}^{\infty} dt'\frac{\rho_A(s',t')}{(s'-s_o)(s'-s-i\epsilon)(t'-t-i\epsilon)}, \tag{3.3}$$

$$B(s,t)=\frac{1}{\pi^2}\int_{s_1}^{\infty} ds'\int_{t_1(s')}^{\infty} dt'\frac{\rho_B(s',t')}{(s'-s-i\epsilon)(t'-t-i\epsilon)}. \tag{3.4}$$

For the crossed graph, the general form is the same, with u replacing s and some mass variables interchanged. The calculation of the double spectral functions ρ_A and ρ_B is done by standard application of the Cutkosky rules. In order to separate explicitly the portion of the box amplitude in the single-variable dispersion relation, we first calculate the absorptive part of the diagram in the t channel for fixed s. Then we evaluate the expression at that unphysical value $s=s_o$ which we have chosen as subtraction point, where the double dispersion term automatically vanishes. This then gives us $\operatorname{Im}A(s_o,t)$.

For all diagrams, we have assumed that the dispersion-relation forms are valid. We have not generalized to those cases where, because of the apparent arbitrariness in the mass variables, we might have complex singularities. Thus we have tacit restrictions upon our mass variables. Our formulas apply, nevertheless, to many physical systems, which have purely "normal" behavior.

IV. BORN DIAGRAMS

Using the notation for the Born diagrams listed in Fig. 3, we have, according to our Feynman rules,

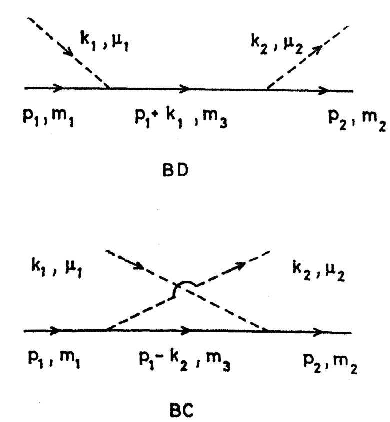


FIG. 3. Notation for Born diagrams.

$$F_{\mathrm{BD}}=-\bar{u}(p_2)\gamma_5[(p_1+k_1+m_3)/(s-m_3^2)]\gamma_5 u(p_1), \tag{4.1}$$

$$F_{\mathrm{BC}}=-\bar{u}(p_2)\gamma_5[(p_1-k_2+m_3)/(u-m_3^2)]\gamma_5 u(p_1). \tag{4.2}$$

Reduction to standard form gives us, for the direct Born graph,

$$A_{\mathrm{BD}}(s)=[m_3-\tfrac{1}{2}(m_1+m_2)]/(s-m_3^2), \tag{4.3}$$

$$B_{\mathrm{BD}}(s)=-1/(s-m_3^2), \tag{4.4}$$

[12] T. T. Chou and M. Dresden, Rev. Mod. Phys. 39, 143 (1967).

and for the crossed Born graph

$$A_{BC}(u)=[m_3-\tfrac{1}{2}(m_1+m_2)]/(u-m_3^2)\,, \qquad (4.5)$$

$$B_{BC}(u)=1/(u-m_3^2)\,. \qquad (4.6)$$

V. SELF-ENERGY DIAGRAMS

The self-energy diagrams, as shown in Fig. 4, are infinite and have to be renormalized. The calculation can be divided into two parts: (A) obtaining an expression for the invariant amplitudes in terms of the second-order renormalized self-energy insertion; (B) calculation of the finite part of this self-energy insertion.

A. Expressions for Reaction Amplitudes

We have for the self-energy diagrams,

$$F_{SED}=-\bar{u}(p_2)\gamma_5\frac{(p_2+k_2+m_3)}{s-m_3^2}\times\Sigma(p_2+k_2)\frac{(p_1+k_1+m_3)}{s-m_3^2}\gamma_5 u(p_1) \qquad (5.1)$$

and

$$F_{SEC}=-\bar{u}(p_2)\gamma_5\frac{(p_2-k_1+m_3)}{u-m_3^2}\times\Sigma(p_1-k_2)\frac{(p_1-k_2+m_3)}{u-m_3^2}\gamma_5 u(p_1)\,, \qquad (5.2)$$

where $\Sigma(p)$ is the finite part of the second-order self-energy correction to the fermion propagator. We can write $\Sigma(p)$ in the form

$$\Sigma(p)=(p-M)[T_1(p^2)+pT_2(p^2)](p-M)\,, \qquad (5.3)$$

where p is the four-momentum of the fermion line entering the self-energy part and M is the mass of that line. When we insert this form for $\Sigma(p)$ into the amplitudes above and separate the invariant amplitudes, we get

$$\begin{aligned}A_{SED}(s)&=T_1(s)-\tfrac{1}{2}(m_1+m_2)T_2(s)\,,\\ B_{SED}(s)&=-T_2(s)\,,\end{aligned} \qquad (5.4)$$

and

$$\begin{aligned}A_{SEC}(u)&=T_1(u)-\tfrac{1}{2}(m_1+m_2)T_2(u)\,,\\ B_{SEC}(u)&=T_2(u)\,.\end{aligned} \qquad (5.5)$$

The form of the functions T_1 and T_2 is the same for both the direct and crossed self-energy diagrams. The only change is the replacement of s by u.

B. Calculation of the Second-Order Self-Energy Insertion

The second-order self-energy insertion in a fermion line can be written (cf. Fig. 5) as

$$\Sigma(p)=\frac{i}{(2\pi)^4}\gamma_5\int d^4k\,\frac{1}{k^2-\mu^2}\,\frac{p-k+M_2}{(p-k)^2-M_2^2}\gamma_5\,. \qquad (5.6)$$

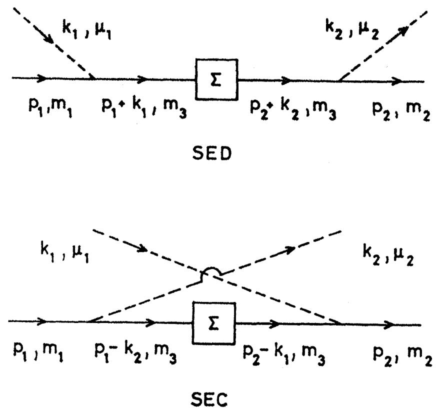


FIG. 4. Notation for self-energy diagrams.

This has a linear divergence, but the renormalization is done automatically through the dispersion relation. We calculate the absorptive part of $\Sigma(p)$:

$$\mathrm{Im}\Sigma(p)=-\frac{1}{2}\frac{1}{(2\pi)^2}\int d^4k\,\gamma_5(p-k+M_2)\gamma_5\times\delta(k^2-\mu^2)\delta[(p-k)^2-M_2^2]\,. \qquad (5.7)$$

The easiest way to evaluate this is in a special coordinate system, the center-of-mass system of the two vectors k and p-k, so that

$$\begin{aligned}p&=(p_0,0), & s&=p^2=p_0^2\,,\\ k&=(k_0,\mathbf{k}), & k_0&=\tfrac{1}{2}(s-M_2^2+\mu^2)/\sqrt{s}\,,\\ p-k&=(p_0-k_0,\,-\mathbf{k}).\end{aligned} \qquad (5.8)$$

Removing the δ functions, we get an expression

$$\mathrm{Im}\Sigma(p)=\frac{1}{64\pi^2}\frac{h(s)}{s}\int d\Omega_k(-p+k+M_2)\,, \qquad (5.9)$$

where

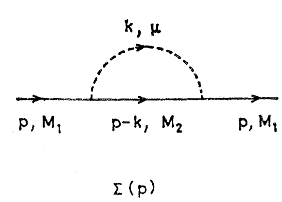
$$h(s)=[s-(M_2+\mu)^2]^{1/2}[s-(M_2-\mu)^2]^{1/2}. \qquad (5.10)$$

In order to do the angular integral, we expand k in components. We have, for example,

$$\int d\Omega_k k=4\pi\gamma_0 k_0-\int d\Omega_k \boldsymbol{\gamma}\cdot\mathbf{k}\,.$$

The integral over the spatial part of $\mathbf{k}$ vanishes, leaving only the first term. In our coordinate system, we have

FIG. 5. Second-order self-energy insertion in a fermion line.



$p=\gamma_0 p_0$ and

$$\gamma_0 k_0=\gamma_0 p_0(k_0/p_0)=\tfrac{1}{2}p(s-M_2^2+\mu^2)/s. \quad (5.11)$$

When we collect all terms together, we get

$$\mathrm{Im}\Sigma(p)=\mathrm{Im}C(s)+p\ \mathrm{Im}D(s) \quad (5.12)$$

with

$$\mathrm{Im}C(s)=\frac{1}{16\pi}m_2\frac{h(s)}{s}$$

and

$$\mathrm{Im}D(s)=-\frac{1}{16\pi}\frac{s+M_2^2-\mu^2}{2s}\frac{h(s)}{s}. \quad (5.13)$$

We equate the expressions of Eq. (5.12) to the imaginary part of the expression in Eq. (5.3), and we obtain

$$\mathrm{Im}T_1(s)=\frac{1}{16\pi}\frac{h(s)}{s(s-M_1^2)^2}\times[(M_2-M_1)(s-M_1M_2)+M_1\mu^2], \quad (5.14)$$

$$\mathrm{Im}T_2(s)=\frac{1}{32\pi}\frac{h(s)}{s^2(s-M_1^2)^2}[-(s-M_1^2)(s-M_2^2-\mu^2)-2(M_2-M_1)^2+2M_1^2\mu^2],$$

with $h(s)$ given by Eq. (5.10). These are the spectral functions for the single-variable dispersion relations, and the real parts of the amplitude can be calculated either analytically or numerically. With these results we have obtained the invariant amplitudes for the self-energy diagrams.

VI. VERTEX DIAGRAMS

The treatment of the vertex diagrams parallels that of the self-energy diagrams. The results are algebraically more complicated, and there are twice as many diagrams when initial and final particle states are not identical (cf. Fig. 6). Again we have to (a) express the invariant A and B amplitudes in terms of the renormalized vertex functions, and (b) calculate the finite part of the vertex functions.

A. Expressions for Reaction Amplitudes

According to our Feynman rules we have

$$F_{\mathrm{VD1}}=-\bar{u}(p_2)\gamma_5[(p_1+k_1+m_3)/(s-m_3^2)]\times V_L(p_1,k_1)u(p_1), \quad (6.1)$$

$$F_{\mathrm{VD2}}=-\bar{u}(p_2)V_R(p_2,k_2)\times[(p_2+k_2+m_3)/(s-m_3^2)]\gamma_5 u(p_1), \quad (6.2)$$

$$F_{\mathrm{VC1}}=-\bar{u}(p_2)\gamma_5[(p_1-k_2+m_3)/(u-m_3^2)]\times V_L(p_1,-k_2)u(p_1), \quad (6.3)$$

$$F_{\mathrm{VC2}}=-\bar{u}(p_2)V_R(p_2,-k_1)\times[(p_2-k_1+m_3)/(u-m_3^2)]\gamma_5 u(p_1), \quad (6.4)$$

where V_R and V_L are the finite parts of the second-order vertex function. The arguments of V_R and V_L are the momenta of the two particles which are on the mass shell, and the sign convention is shown in Fig. 7. The remaining external line is always off the mass shell. For a scattering process, where initial and final particle states are the same, we have

$$F_{\mathrm{VD1}}=F_{\mathrm{VD2}},\quad F_{\mathrm{VC1}}=F_{\mathrm{VC2}}. \quad (6.5)$$

For the more general case of a reaction where the initial and final particle states are different, the choice of notation in Fig. 7 enables us to write the amplitudes for both kinds of vertex graphs in terms of similar functions. We

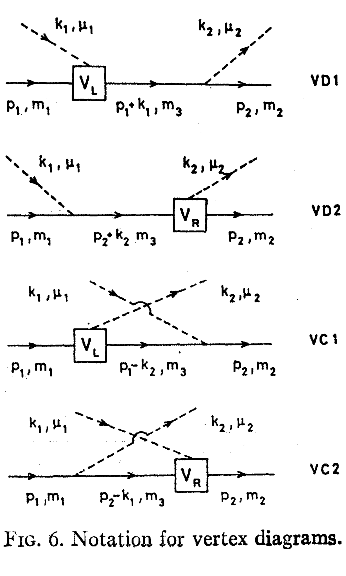


FIG. 6. Notation for vertex diagrams.

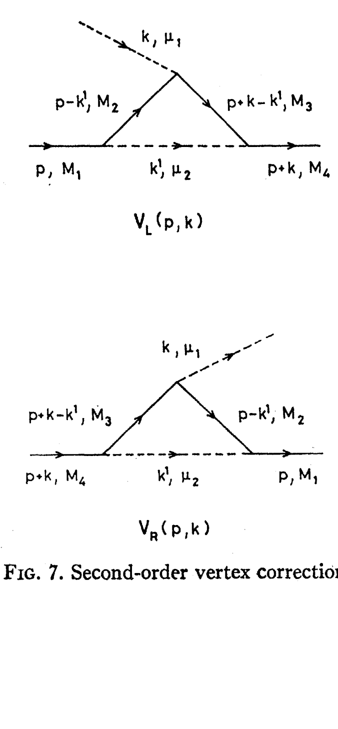


FIG. 7. Second-order vertex correction.

can write

$$V_L(p,k)=\gamma_5(s-M_4^2)V_1(s)+(\not p+\not k-M_4)\gamma_5 V_2(s), \quad (6.6)$$

$$V_R(p,k)=\gamma_5(s-M_4^2)V_1(s)+\gamma_5(\not p+\not k-M_4)V_2(s), \quad (6.7)$$

where $s=(p+k)^2$ and V_1 and V_2 are invariant scalar functions. Inserting this into our expressions for the F amplitudes, we have for the direct graphs

$$\begin{aligned}A_{\mathrm{VD}}(s)&=[-\tfrac{1}{2}(m_1+m_2)+m_3]V_1(s)+V_2(s),\\ B_{\mathrm{VD}}(s)&=-V_1(s),\end{aligned} \quad (6.8)$$

while for the crossed graphs we get

$$\begin{aligned}A_{\mathrm{VC}}(u)&=[-\tfrac{1}{2}(m_1+m_2)+m_3]V_1(u)+V_2(u),\\ B_{\mathrm{VC}}(u)&=V_1(u).\end{aligned} \quad (6.9)$$

These expressions apply to both types of vertex graphs, VD1 and VD2, VC1 and VC2. In the crossed vertex graphs the only changes from the direct graphs are that u replaces s and that the meson mass μ_2 is interchanged with the meson mass μ_1 (an automatic fact when the notation conventions of Fig. 7 are followed).

B. Calculation of the Second-Order Vertex Insertion

Because of the choice of notation in Fig. 7, the vertex functions V_1 and V_2 are used for all vertex graphs, regardless of the position of the vertex correction. This formal simplification means that we have much less algebraic manipulation to do. Accordingly, we exhibit the calculations for the diagram labeled VD1 in Fig. 6. The vertex functions for diagram VD2 are exactly the same.

The vertex function needed is $V_L(p,k)$, which is written as

$$V_L(p,k)=\frac{i}{(2\pi)^4}\int d^4k'\gamma_5\frac{\not W-\not k'+M_3}{(W-k')^2-M_3^2}\times\gamma_5\frac{\not p-\not k'+M_2}{(p-k')^2-M_2^2}\gamma_5\frac{1}{k'^2-\mu_2^2}, \quad (6.10)$$

where $W=p+k$, while p and k are free-particle momenta, and we assume a free-particle spinor $u(p)$ at the right of this operator expression. The logarithmic divergence is automatically removed by the use of dispersion relations. We can write the absorptive part in the channel where $s=(p+k)^2\geqslant(M_3+\mu_2)^2$ as follows, if there are no anomalous thresholds:

$$\mathrm{Im}V_L(p,k)=-\frac{1}{2}\frac{1}{(2\pi)^2}\int d^4k' N\delta(k'^2-\mu_2^2)\times\delta[(W-k')^2-M_3^2]\frac{1}{(p-k')^2-M_2^2} \quad (6.11)$$

with

$$N=\gamma_5(\not W-\not k'+M_3)\gamma_5(\not p-\not k'+M_2)\gamma_5. \quad (6.12)$$

Again the easiest way to evaluate this integral is in a special coordinate system, the center-of-mass system of p and k. We have

$$\begin{aligned}&p=(p_0,\mathbf{p}), \qquad p_0=\tfrac{1}{2}(s+M_1^2-\mu_1^2)/\sqrt{s},\\ &k=(k_0,-\mathbf{p}), \quad W=p+k=(\sqrt{s},0),\\ &|\mathbf{p}|^2=\tfrac{1}{4}[s-(M_1+\mu_1)^2][s-(M_1-\mu_1)^2]/s.\end{aligned} \quad (6.13)$$

Removing the δ functions and reducing some of the matrix algebra by the relation

$$(\not p-M_1)u(p)=0, \quad (6.14)$$

we have

$$\mathrm{Im}V_L(p,k)=\frac{1}{16\pi^2}\frac{|\mathbf{k}'|}{\sqrt{s}}\int d\Omega' N\frac{1}{(p-k')^2-M_2^2}, \quad (6.15)$$

where

$$\begin{aligned}&k_0'=\tfrac{1}{2}(s-M_3^2+\mu_2^2)/\sqrt{s},\\ &|\mathbf{k}'|^2=\tfrac{1}{4}[s-(M_3+\mu_2)^2][s-(M_3-\mu_2)^2]/s\end{aligned} \quad (6.16)$$

and

$$\begin{aligned}N&=N_1+N_2,\\ N_1&=-(\not W-M_4)\gamma_5[\not k'+(M_2-M_1)],\\ N_2&=\gamma_5[\not k'(M_1-M_2+M_3-M_4)\\ &\qquad+(M_3-M_4)(M_2-M_1)-\mu_2^2].\end{aligned} \quad (6.17)$$

In order to evaluate the angular integral, we express $\not k'$ in components. We consider the set of γ matrices $(\gamma_1,\gamma_2,\gamma_3)$ formally as a fixed vector in, say, the xz plane so that

$$\begin{aligned}&\boldsymbol{\gamma}=\gamma(\sin\psi,\ 0,\ \cos\psi),\\ &\mathbf{k}'=k'(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta),\end{aligned} \quad (6.18)$$

and we set $\mathbf{p}$ as the polar axis. The angular integral over $\cos\phi$ is then trivial and we obtain the expression

$$\mathrm{Im}V_L(p,k)=(1/8\pi)(|\mathbf{k}'|/\sqrt{s})I, \quad (6.19)$$

where

$$I=\int_{-1}^{1}dz\,\frac{c_1+c_2z}{c_3+c_4z}$$

and

$$\begin{aligned}z&=\cos\theta,\\ c_1&=-(\not W-M_4)\gamma_5[\gamma_0k_0'+(M_2-M_1)]\\ &\quad+\gamma_5[\gamma_0k_0'(M_1-M_2+M_3-M_4)\\ &\qquad+(M_3-M_4)(M_2-M_1)-\mu_2^2],\\ c_2&=(\not W-M_4)\gamma_5\gamma k'\cos\psi-\gamma_5\gamma k'\\ &\qquad\times\cos\psi(M_1-M_2+M_3-M_4),\\ c_3&=M_1^2-M_2^2+\mu_2^2-2p_0k_0',\\ c_4&=2|\mathbf{p}||\mathbf{k}'|.\end{aligned} \quad (6.20)$$

We can reset these expressions in a more obviously covariant fashion by noting that

$$\gamma_0=\not W(1/\sqrt{s}) \quad (6.21)$$

and

$$\begin{aligned}\gamma\cos\psi&=(1/|\mathbf{p}|)\boldsymbol{\gamma}\cdot\mathbf{p}=(1/|\mathbf{p}|)(-\not p+\gamma_0p_0)\\ &=(1/|\mathbf{p}|)[-\not p+(p_0/\sqrt{s})\not W].\end{aligned} \quad (6.22)$$

The integral I is elementary and it is

$$I=2\frac{c_2}{c_4}+\left[\frac{c_1}{c_4}-\frac{c_2}{c_4}\frac{c_3}{c_4}\right]\ln\left[\frac{c_3+c_4}{c_3-c_4}\right]. \tag{6.23}$$

After a long series of manipulations with γ matrices, we can write

$$\operatorname{Im}V_L(p,k)=[\gamma_5(s-M_4^2)\operatorname{Im}V_1(s) +(W-M_4)\gamma_5\operatorname{Im}V_2(s)], \tag{6.24}$$

where

$$\operatorname{Im}V_1(s)=\frac{1}{16\pi}\frac{1}{|\mathbf{p}|\sqrt{s}} \times\left[2a_2+\left(a_1-a_2\frac{c_3}{c_4}\right)\ln L\right]\frac{1}{s-M_4^2}, \tag{6.25}$$

$$\operatorname{Im}V_2(s)=\frac{1}{16\pi}\frac{1}{|\mathbf{p}|\sqrt{s}}\left[2b_2+\left(b_1-b_2\frac{c_3}{c_4}\right)\ln L\right], \tag{6.26}$$

and

$$a_1=\frac{k_0'}{\sqrt{s}}[(s-M_4^2)-M_4(M_1-M_2+M_3-M_4)] -(M_1-M_2)(M_3-M_4)-\mu_2^2,$$

$$a_2=\frac{k'}{|\mathbf{p}|}\left[-\frac{p_0}{\sqrt{s}}(s-M_4^2) +(M_1-M_2+M_3-M_4)\left(M_1+M_4\frac{p_0}{\sqrt{s}}\right)\right], \tag{6.27}$$

$$b_1=-\frac{k_0'}{\sqrt{s}}(M_1-M_2+M_3)+(M_1-M_2),$$

$$b_2=\frac{k'}{|\mathbf{p}|}\left[-M_1+\frac{p_0}{\sqrt{s}}(M_1-M_2+M_3)\right],$$

$$L=|(c_3+c_4)/(c_3-c_4)|,$$

with all other expressions as defined in this section. The functions $\operatorname{Im}V_1(s)$ and $\operatorname{Im}V_2(s)$ are the spectral functions for the single-variable dispersion relations, through which we may calculate the real parts. With these results we have obtained the invariant amplitudes for the vertex diagrams.

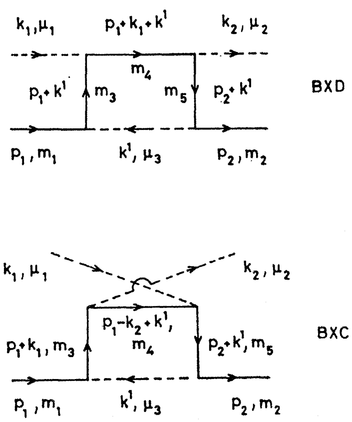


FIG. 8. Notation for box diagrams.

VII. BOX DIAGRAMS

As we mentioned in Sec. III, the amplitudes for the box diagrams have two terms, one the double-dispersion-relation form, analogous to that in the scalar particle case obtained by Mandelstam,[6] and the other a single-variable dispersion relation, needed because the amplitude for the case of spin is not as convergent at infinite momentum as the purely scalar one. We will break up the calculation into two parts: (a) the double-spectral function; and (b) the absorptive part of the subtraction term.

According to the Feynman rules in the notation of Fig. 8, we have

$$F_{\mathrm{BXD}}=-\frac{i}{(2\pi)^4}\bar{u}(p_2)\int d^4k'\frac{1}{k'^2-\mu_2^2}\gamma_5\frac{\mathbf{p}_2+\mathbf{k}'+m_5}{(p_2+k')^2-m_5^2} \times\gamma_5\frac{\mathbf{p}_1+\mathbf{k}_1+\mathbf{k}'+m_4}{(p_1+k_1+k')^2-m_4^2}\gamma_5\frac{\mathbf{p}_1+\mathbf{k}'+m_3}{(p_1+k')^2-m_3^2}\gamma_5u(p_1) \tag{7.1}$$

and

$$F_{\mathrm{BXC}}=-\frac{i}{(2\pi)^4}\bar{u}(p_2)\int d^4k'\frac{1}{k'^2-\mu_2^2}\gamma_5\frac{\mathbf{p}_2+\mathbf{k}'+m_5}{(p_2+k')^2-m_5^2} \times\gamma_5\frac{\mathbf{p}_1-\mathbf{k}_2+\mathbf{k}'+m_4}{(p_1-k_2+k')^2-m_4^2}\gamma_5\frac{\mathbf{p}_1+\mathbf{k}'+m_3}{(p_1+k')^2-m_3^2}\gamma_5u(p_1). \tag{7.2}$$

We can write the invariant amplitudes, by virtue of simila.ity in formal structure, as

$$A_{\mathrm{BXD}}=A(s,t;\mu_1,\mu_2),\quad B_{\mathrm{BXD}}=B(s,t;\mu_1,\mu_2) \tag{7.3}$$

and

$$A_{\mathrm{BXC}}=A(u,t;\mu_2,\mu_1),\quad B_{\mathrm{BXC}}=-B(u,t;\mu_2,\mu_1), \tag{7.4}$$

where the only difference between crossed and direct amplitudes is that u replaces s and μ_1 and μ_2 are interchanged.

A. Double-Spectral Function

For the direct box diagram we have

$$\rho(s,t)=\int d^4k'\delta(k'^2-\mu_3^2)\delta[(p_2+k')^2-m_5^2] \times\delta[(p_1+k_1+k')^2-m_4^2]\delta[(p_1+k')^2-m_3^2]\times N \tag{7.5}$$

and

$$N=\bar{u}(p_2)\gamma_5(\mathbf{p}_2+\mathbf{k}'+m_5)\gamma_5(\mathbf{p}_1+\mathbf{k}_1+\mathbf{k}'+m_4) \times\gamma_5(\mathbf{p}_1+\mathbf{k}'+m_3)\gamma_5u(p_1) \tag{7.6}$$

by the usual application of Cutkosky's rules. We can write

$$\rho(s,t)=\sigma(s,t)\bar{u}(p_2)\times[N_A(s,t)+\tfrac{1}{2}(k_1+k_2)N_B(s,t)]u(p_1)\,, \quad (7.7)$$

where $\sigma(s,t)$ is the double spectral function for the scalar particle case, and $N_A(s,t)$ and $N_B(s,t)$ contain the spin-dependent parts. For completeness and correct normalization we include formulas for $\sigma(s,t)$ as follows, following Mandelstam[6]:

$$\sigma(s,t)=\{32q_1q_2q_I\sqrt{s}[k(z,z_{IE},z_{IO})]^{1/2}\}^{-1} \quad \text{for } (m_4+\mu_3)^2\leqslant s<\infty \quad\text{and}\quad z_o\leqslant z<\infty\,, \quad (7.8)$$

where q_1 and q_2 are the magnitudes of the incident and final three-momenta in the s channel, as in Eq. (2.4), and

$$q_I^2=\tfrac{1}{4}[s-(m_4+\mu_3)^2][s-(m_4-\mu_3)^2]/s\,, \quad (7.9)$$

$$z=(1/2q_1q_2)[q_1^2+q_2^2+t-(1/4s)\times(m_1^2-m_2^2-\mu_1^2+\mu_2^2)^2]\,, \quad (7.10)$$

$$z_{IE}=(1/2q_1q_I)[q_1^2+q_I^2+m_3^2-(1/4s)\times(m_1^2+m_4^2-\mu_1^2-\mu_3^2)^2]\,, \quad (7.11)$$

$$z_{IO}=(1/2q_2q_I)[q_2^2+q_I^2+m_5^2-(1/4s)\times(m_2^2+m_4^2-\mu_2^2-\mu_3^2)^2]\,, \quad (7.12)$$

$$k(z,z_{IE},z_{IO})=z^2-2zz_{IE}z_{IO}+z_{IE}^2+z_{IO}^2-1\,, \quad (7.13)$$

and

$$z_o=z_{IE}z_{IO}+(z_{IE}^2-1)^{1/2}(z_{IO}^2-1)^{1/2}. \quad (7.14)$$

We will also call

$$\rho_A(s,t)=\sigma(s,t)N_A(s,t)\,,\qquad \rho_B(s,t)=\sigma(s,t)N_B(s,t)\,. \quad (7.15)$$

The calculation of the numerator functions is strictly algebraic. The four δ functions completely determine the intermediate four-momentum vector k' and we can write it as

$$k'=a_1p_1+a_2k_1+a_3p_2+a_4r\,,$$

where r is a vector perpendicular to the hyperplane of p_1, k_1, and p_2. The vector r has two possible orientations relative to the hyperplane; this leads to a factor 2 which has been taken into account in the definition of $\sigma(s,t)$. Otherwise, the vector r never enters the calculation. We get the coefficients a_1, a_2, a_3 by taking scalar products successively with p_1, k_1, and p_2 and solving the set of three scalar equations. Let us define column vectors x_1, x_2, x_3, x_c of three components each by

$$x_1=\begin{pmatrix} m_1^2 \\ \tfrac{1}{2}(s-m_1^2-\mu_1^2) \\ -\tfrac{1}{2}(t-m_1^2-m_2^2)\end{pmatrix},\quad x_2=\begin{pmatrix} \tfrac{1}{2}(s-m_1^2-\mu_1^2) \\ \mu_1^2 \\ -\tfrac{1}{2}(u-m_2^2-\mu_1^2)\end{pmatrix},\quad x_3=\begin{pmatrix} -\tfrac{1}{2}(t-m_1^2-m_2^2) \\ -\tfrac{1}{2}(u-m_2^2-\mu_1^2) \\ m_2^2\end{pmatrix},\quad x_c=\begin{pmatrix} \tfrac{1}{2}(m_3^2-m_1^2-\mu_3^2) \\ -\tfrac{1}{2}(s-m_1^2+m_3^2-m_4^2) \\ \tfrac{1}{2}(m_5^2-m_2^2-\mu_3^2)\end{pmatrix}, \quad (7.16)$$

and determinants

$$d=\det(x_1,x_2,x_3)\,,\quad d_1=\det(x_c,x_2,x_3)\,,\quad d_2=\det(x_1,x_c,x_3)\,,\quad d_3=\det(x_1,x_2,x_c)\,. \quad (7.17)$$

Our coefficients are, very simply,

$$a_i=d_i/d,\quad i=1,\,2,\,3. \quad (7.18)$$

The reduction of the numerator is an elementary algebraic procedure which is tedious but offers no complications. We merely state the results. Let us define

$$\begin{aligned} h_1&=(m_1-m_3)(m_2-m_5)\,,\\ h_2&=m_1-m_3+m_2-m_5\,,\\ h_3&=\tfrac{1}{2}(m_2-m_1)(m_2-m_5)\,,\\ h_4&=\tfrac{1}{2}(m_2-m_1)(m_1-m_3)\,,\\ h_5&=(s-m_1^2-\mu_1^2)-\tfrac{1}{2}m_1(m_2-m_1)\,. \end{aligned} \quad (7.19)$$

Then we can write

$$\begin{aligned} N_A(s,t)&=h_1[\tfrac{1}{2}(m_1+m_2)+m_4]+\mu_2^2[h_2+m_4-\tfrac{1}{2}(m_1+m_2)]+[m_1a_1+\tfrac{1}{2}(m_2-m_1)a_2+m_2a_3]\\ &\quad\times[h_1+m_4h_2-(s-m_4^2)]+a_1[m_1^2h_2+m_1h_3+(m_1-m_3)h_5]+a_2[\mu_1^2h_2+m_1h_4+(m_2-m_5)h_5]\\ &\quad+a_3[\tfrac{1}{2}(m_1-m_3)m_2(m_1+m_2)+(m_2-m_5)(s-\mu_2^2)+m_2h_3] \end{aligned} \quad (7.20)$$

and

$$N_B(s,t)=h_1-\mu_3^2+[m_1(a_1-a_2)-m_2a_3]\times(-m_1+m_3+m_2-m_5)+a_2[h_1+m_4h_2-(s-m_4^2)]. \quad (7.21)$$

This completely defines the double-spectral function for the direct box graph. For the crossed box graph we use the same form but with u replacing s and masses μ_1 and μ_2 interchanged.

B. Subtraction Term

Because there is a part of the box diagram amplitude A that is expressible only as a single-variable dispersion relation in t, we must calculate the absorptive part of

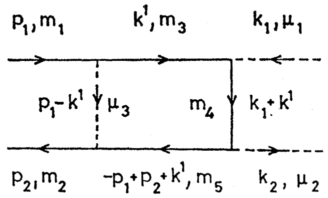


FIG. 9. Notation for calculation of subtraction part of box diagram.

the amplitude in the t channel, where s is the momentum transfer variable. For this it is convenient to use a slightly different notation, given by Fig. 9. Again we will calculate for the direct box graph; the change for the crossed box graph is as before.

By the Cutkosky rules, the absorptive part in the t channel is

$$\mathrm{Im}F_t(s,t)=+\frac{1}{2}\frac{1}{(2\pi)^2}\int d^4k'N\delta(k'^2-m_3^2)$$
$$\times\delta[(k'-p_1+p_2)^2-m_5^2][(p_1-k')^2-\mu_3^2]^{-1}$$
$$\times[(k_1+k')^2-m_4^2]^{-1}, \quad (7.22)$$

where

$$N=\bar{u}(p_2)\gamma_5(\boldsymbol{k}'-\boldsymbol{p}_1+\boldsymbol{p}_2+m_5)\gamma_5(\boldsymbol{k}'+\boldsymbol{k}_1+m_4)$$
$$\times\gamma_5(\boldsymbol{k}'+m_3)\gamma_5u(p_1). \quad (7.23)$$

We will work in the center-of-mass system for the t channel, where the kinematics are defined by

$$\begin{aligned} p_1&=(p_{10},\mathbf{p}), & k_1&=(k_{10},\mathbf{k}),\\ p_2&=(p_{20},\mathbf{p}), & k_2&=(k_{20},\mathbf{k}), \end{aligned} \quad (7.24)$$

with

$$t=(p_{10}-p_{20})^2$$

and

$$\begin{aligned} |\mathbf{p}|^2&=\tfrac{1}{4}[t-(m_1+m_2)^2][t-(m_1-m_2)^2]/t,\\ |\mathbf{k}|^2&=\tfrac{1}{4}[t-(\mu_1+\mu_2)^2][t-(\mu_1-\mu_2)^2]/t,\\ p_{10}&=\tfrac{1}{2}(t+m_1^2-m_2^2)/\sqrt{t},\\ p_{20}&=-\tfrac{1}{2}(t-m_1^2+m_2^2)/\sqrt{t},\\ k_{10}&=-\tfrac{1}{2}(t+\mu_1^2-\mu_2^2)/\sqrt{t},\\ k_{20}&=\tfrac{1}{2}(t-\mu_1^2+\mu_2^2)/\sqrt{t}. \end{aligned} \quad (7.25)$$

We can remove the δ functions to get

$$\mathrm{Im}F_t(s,t)=\frac{1}{32\pi^2}\frac{|\mathbf{k}'|}{\sqrt{t}}\int d\Omega'N$$
$$\times\frac{1}{(p_1-k')^2-\mu_3^2}\frac{1}{(k_1+k')^2-m_4^2} \quad (7.26)$$

with

$$\begin{aligned} k_0'&=\tfrac{1}{2}(t+m_3^2-m_5^2)/\sqrt{t},\\ |\mathbf{k}'|^2&=\tfrac{1}{4}[t-(m_3+m_5)^2][t-(m_3-m_5)^2]/t. \end{aligned} \quad (7.27)$$

In order to evaluate this integral we define

$$\begin{aligned} \tau_1&=(1/2|\mathbf{p}||\mathbf{k}'|)[2p_{10}k_0'-(m_1^2+m_3^2-\mu_3^2)],\\ \tau_2&=(1/2|\mathbf{k}||\mathbf{k}'|)[-2k_{10}k_0'-(m_3^2-m_4^2+\mu_1^2)],\\ z_1&=\hat{p}\cdot\hat{k}',\\ z_2&=\hat{k}\cdot\hat{k}', \end{aligned} \quad (7.28)$$

so that

$$\begin{aligned} (p_1-k')^2-\mu_3^2&=2|\mathbf{p}||\mathbf{k}'|(-\tau_1+z_1),\\ (k_1+k')^2-m_4^2&=-2|\mathbf{k}||\mathbf{k}'|(\tau_2+z_2). \end{aligned} \quad (7.29)$$

The two denominators can be combined by the relation

$$(ab)^{-1}=\int_0^1 dx\,[ax+b(1-x)]^{-2}$$

to give

$$\mathrm{Im}F_t(s,t)=\frac{1}{128\pi^2}\frac{1}{|\mathbf{p}||\mathbf{k}||\mathbf{k}'|\sqrt{t}}$$
$$\times\int d\Omega'\int_0^1 dx\frac{N}{[\tau_1x+\tau_2(1-x)-\mathbf{V}\cdot\hat{k}']^2}, \quad (7.30)$$

where

$$\mathbf{V}=\hat{p}x-\hat{k}(1-x)$$

will be the polar axis about which we will integrate over the angles of $\mathbf{k}'$. In contrast to the purely scalar case, we have a numerator which does depend on the angles of $\mathbf{k}'$. We can set up a special coordinate system, where vectors $\mathbf{p}$ and $\mathbf{k}$ define the xz plane (cf. Fig. 10). We let the three spatial γ matrices form an arbitrary vector

$$\boldsymbol{\gamma}=(\gamma_1,\gamma_2,\gamma_3)$$

and

$$\mathbf{k}=(k_x,0,k_z) \quad (7.31)$$

with

$$\begin{aligned} k_x&=|\mathbf{k}|\sin\theta,\\ k_z&=-|\mathbf{k}|\cos\theta, \end{aligned}$$

and

$$\mathbf{k}'=|\mathbf{k}'|(\sin\theta'\cos\phi',\sin\theta'\sin\phi',\cos\theta'),$$

the coordinates taken relative to $\mathbf{V}$. The magnitude of $\mathbf{k}'$ is fixed; we integrate over the angles. We can write the numerator N in powers of the four-vector k' as follows:

$$N=\bar{u}(p_2)[N_1+N_2\boldsymbol{k}'+\boldsymbol{k}'N_3+2(k_1\cdot k')\boldsymbol{k}']u(p_1) \quad (7.32)$$

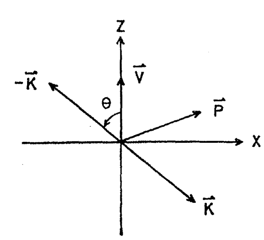


FIG. 10. Coordinate system for calculation of subtraction part of box diagram.

with

$$N_1=m_3(p_1-p_2+m_5)[-(m_3-m_4)+k_1]-m_3^2(m_3-m_4)-m_3^2k_1, \quad (7.33)$$

$$N_2=(p_1-p_2+m_5)(m_3-m_4-k_1)+m_3^2,$$

$$N_3=-m_3(k_1+m_4).$$

Our next step is to substitute the coordinate representation for $\mathbf{k}'$. By our choice of axes, only the numerator contains terms in the angle ϕ'. We interchange the order of integration over x and Ω'. The integration over ϕ' is trivial and we obtain the following integrals in the variable $y=\cos\theta'$:

$$\mathrm{Im}F_t(s,t)=\frac{1}{64\pi}\frac{\bar{u}(p_2)}{|\mathbf{p}||\mathbf{k}||\mathbf{k}'|\sqrt{t}}\times\int_0^1 dx\,(C_0J_0+C_1J_1+C_2J_2)u(p_1), \quad (7.34)$$

where

$$J_i=\int_{-1}^1 dy\frac{y^i}{(D-|\mathbf{V}|y)^2}, \quad \text{for} \quad i=1,\,2,\,3 \quad (7.35)$$

with

$$D=\tau_1x+\tau_2(1-x)$$

and

$$C_0=N_1+N_2\gamma_0k_0'+\gamma_0k_0'N_3+2k_{10}k_0'^2\gamma_0+\gamma_1k_x|\mathbf{k}'|^2,$$

$$C_1=-|\mathbf{k}'|(+N_2\gamma_3+\gamma_3N_3+2\gamma_0k_zk_0'+2\gamma_3k_{10}k_0'), \quad (7.36)$$

$$C_2=|\mathbf{k}'|^2(2\gamma_3k_z-\gamma_1k_x).$$

We can bring these expressions into a more obviously covariant form by some manipulations. The first is to note that

$$\gamma_0=\gamma_0(p_{10}-p_{20})/\sqrt{t}=(p_1-p_2)/\sqrt{t}. \quad (7.37)$$

The next is to define the four-vector analog of the three-vector $\mathbf{V}$ as

$$V=(1/|\mathbf{p}|)p_1x-(1/|\mathbf{k}|)k_1(1-x). \quad (7.38)$$

This obviously contains $\mathbf{V}$ as the spatial part, and we now have a 0-component

$$V_0=(p_{10}/|\mathbf{p}|)x-(k_{10}/|\mathbf{k}|)(1-x).$$

The operator γ_3 can be rewritten by noting that, with the definition of $\mathbf{V}$ as polar axis, we have

$$\gamma_3=\boldsymbol{\gamma}\cdot\mathbf{V}/|\mathbf{V}|=(-V+\gamma_0V_0)/|\mathbf{V}|=[-V+(V_0/\sqrt{t})(p_1-p_2)]/|\mathbf{V}|. \quad (7.39)$$

Finally,

$$\gamma_1k_x=-k_1+\gamma_0k_{10}-\gamma_3k_z=-k_1+(k_{10}/\sqrt{t})(p_1-p_2)-(k_z/|\mathbf{V}|)[-V+(V_0/\sqrt{t})(p_1-p_2)]. \quad (7.40)$$

The substitution of these expressions into the equations for C_0, C_1, and C_2 and the subsequent algebraic manipulation is long, tedious, but elementary. We state the results below. In these expressions we have not yet contracted any matrix operators into the free spinors. As a simplification of notation we write

$$p=|\mathbf{p}|,\quad k=|\mathbf{k}|,\quad k'=|\mathbf{k}'|,\quad V=|\mathbf{V}|. \quad (7.41)$$

We have, then,

$$\begin{aligned}C_0=&-m_3(m_3-m_4)(m_3+m_5)+m_3(s-m_1^2-\mu_1^2)\\&+(k_0'/\sqrt{t})[-m_3(s-m_1^2-\mu_1^2)+(m_3-m_4)t-m_5(u-m_2^2-\mu_1^2)]+(k'^2k_z/pV)xp_1\\&+(p_1-p_2)\{-m_3(m_3-m_4)+(k_0'/\sqrt{t})[(m_3-m_4)(m_3+m_5)+(t+\mu_1^2-\mu_2^2)+2k_{10}k_0']\\&+(k'^2/\sqrt{t})[k_{10}-k_zV_0/V]\}+k_1[-m_3(m_3-m_5)+(k_0'/\sqrt{t})t-k'^2-(k'^2k_z/Vk)(1-x)]\\&+(k_1p_1+p_2k_2)[-m_3+(k_0'/\sqrt{t})(m_3-m_4)], \quad (7.42)\end{aligned}$$

$$\begin{aligned}C_1=&(k'/V)\{(x/p)[(m_3-m_4)(m_1^2-p_1p_2)-m_3(s-m_1^2-\mu_1^2)\\&+p_1[(m_3-m_4)(m_3+m_5)-(s-m_1^2-\mu_1^2)+2k_0'k_{10}]+m_1^2k_1+p_2k_1p_1+(m_3-m_5)k_1p_1]\\&+[(1-x)/k][-(m_3-m_4)(s-m_1^2-\mu_1^2)+\mu_1^2(m_3+m_5)+\mu_1^2(p_1-p_2)+k_1[-(m_3-m_4)(m_3+m_5)-2k_0'k_{10}]\\&+(m_3-m_4)(k_1p_1+p_2k_1)]+(V_0/\sqrt{t})[m_3(s-m_1^2-\mu_1^2)-(m_3-m_4)t+m_5(u-m_2^2-\mu_1^2)\\&+(p_1-p_2)[-(m_3-m_4)(m_3+m_5)-(t+\mu_1^2-\mu_2^2)-2k_0'k_{10}]\\&-k_1t-(m_3-m_5)(k_1p_1+p_2k_1)]-2k_0'k_z(V/\sqrt{t})(p_1-p_2)\}, \quad (7.43)\end{aligned}$$

$$C_2=k'^2\left\{-\frac{3k_z}{V}\frac{x}{p}p_1+\frac{1}{\sqrt{t}}(p_1-p_2)\left(3k_z\frac{V_0}{V}-k_{10}\right)+k_1\left[1+\frac{3k_z}{V}\frac{(1-x)}{k}\right]\right\}. \quad (7.44)$$

The final step in our calculation is to separate the parts of the amplitudes C_0, C_1, C_2 into the invariant pieces. We can write

$$C_i = A_i + \tfrac{1}{2}(k_1+k_2)B_i, \quad i=0, 1, 2. \tag{7.45}$$

In our case, we really do not need the amplitudes B_i, because the B amplitude is already described completely by a double dispersion relation. Rather than rewrite the amplitudes of Eqs. (7.42)–(7.44) in invariant form, we give a series of rules to extract them. The amplitudes A_i are obtained from the C_i by replacing

$$\begin{aligned} &p_1, \text{ on the right, } && \text{by} \quad m_1\,, \\ &p_2, \text{ on the left, } && \text{by} \quad m_2\,, \\ &k_1, && \text{by} \quad -\tfrac{1}{2}(m_1-m_2)\,, \\ &k_1p_1+p_2k_1, && \text{by} \quad -\tfrac{1}{2}(m_1^2-m_2^2)\,, \end{aligned} \tag{7.46}$$

whenever they occur in the expressions for C_i.

The integrals J_i are elementary:

$$\begin{aligned} J_0 &= 2/(D^2-V^2)\,, \\ J_1 &= 2\frac{D}{V}\frac{1}{D^2-V^2}+\frac{1}{V^2}\ln\frac{D-V}{D+V}\,, \\ J_2 &= \frac{2}{V^2}\left[1+\frac{D^2}{D^2-V^2}+\frac{D}{V}\ln\frac{D-V}{D+V}\right]. \end{aligned} \tag{7.47}$$

If we write

$$\mathrm{Im}F_t(s,t) = \bar{u}(p_2)[\mathrm{Im}A_t(s,t)+\tfrac{1}{2}(k_1+k_2) \times \mathrm{Im}B_t(s,t)]u(p_1)\,, \tag{7.48}$$

then we have

$$\mathrm{Im}A_t(s,t) = \frac{1}{64\pi}\frac{1}{pkk'\sqrt{t}} \times \int_0^1 dx\,(A_0J_0+A_1J_1+A_2J_2)\,, \tag{7.49}$$

which is in a form suitable for computational purposes. When we use this expression for $\mathrm{Im}A_t(s,t)$ for values of t greater than the physical threshold in the t channel, then s is the momentum transfer variable and must lie within certain physically defined limits. For convenience we can evaluate our direct box amplitude at

$$s_o = -m_1m_2+(m_2\mu_1^2+m_1\mu_2^2)/(m_1+m_2) \tag{7.50}$$

and our crossed box amplitude at

$$u_o = -m_1m_2+(m_1\mu_1^2+m_2\mu_2^2)/(m_1+m_2)\,. \tag{7.51}$$

When t is an energy variable, these points s_o and u_o lie in the proper range of the momentum transfer variables, and our expressions for $\mathrm{Im}A_t(s_o,t)$ and $\mathrm{Im}A_t(u_o,t)$ are legitimate.

VIII. ACCURACY OF RESULTS

Because this work was intended for numerical computation, the procedure for checking the accuracy of the formulas was both analytic and numerical. We chose the sign conventions for the S matrix to make the sign of the Born amplitude agree with the commonly accepted one. The signs of amplitudes relative to the Born term were checked by use of the unitarity condition. The imaginary parts of fourth-order amplitudes, as computed by Cutkosky's rules, were also checked by unitarity against the square of the Born term in pion-nucleon scattering. The self-energy and vertex imaginary amplitudes, which are algebraic expressions, checked exactly, while the box imaginary amplitude, which requires a single integration, checked against the square of the Born term to a precision that suggested correctness at least to order $(\mu/m)^2$. The real parts of amplitudes were checked numerically against those of Wyld[2] for pion-nucleon scattering at the threshold energy. They agreed very well. The crossed box amplitude for the pion-nucleon case was computed by Feynman parametric techniques[13] at several energies. The results agreed with the dispersion-relation approach used in this paper. The combination of these numerical checks with a careful reading of our formulas gives us confidence in our results.

[13] G. M. Hale, M.S. thesis, Physics Department, Texas A&M University, 1967 (unpublished).