Dressed-Electron Stationary States in Quantum Electrodynamics*

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The Rayleigh-Ritz procedure for functionalities is used to make a partial determination of the approximate stationary states associated with a free relativistic electron.

THE purpose of this paper is to report the generic form of a nonperturbative approximate solution to the Schrödinger equation $H\Phi=E\Phi$ for stationary states Φ associated with a single relativistic electron of bare mass m_0 and charge $-\epsilon$, dressed by electromagnetic radiation.[1] The one-electron-electromagnetic-radiation dynamical system is described by the Hamiltonian operator $H=H_{\text{el}}+H_{\text{rad}}$, where

$$H_{\text{el}}\equiv\boldsymbol{\alpha}\cdot(\mathbf{p}-\epsilon\mathbf{A}^{\text{tr}}(\mathbf{q}))+\beta m_0 \tag{1}$$

and

$$H_{\text{rad}}\equiv\int[\tfrac{1}{2}|\mathbf{E}(\mathbf{x})|^2+\tfrac{1}{2}|\nabla\times\mathbf{A}(\mathbf{x})|^2-u_{\text{vac}}]d^3x. \tag{2}$$

In Eq. (1), the three components of the transverse part of the electromagnetic potential appear as

$$A_j^{\text{tr}}(\mathbf{x})\equiv\int\delta_{jk}^{\text{tr}}(\mathbf{x}-\mathbf{y})A_k(\mathbf{y})d^3y\,,$$

and the quantity

$$u_{\text{vac}}\equiv\frac{1}{(2\pi)^3}\int|\mathbf{k}|d^3k$$

is a constant in Eq. (2). We satisfy the commutation relations for the dynamical variables in Eqs. (1) and (2), $[q_j,p_k]=i\delta_{jk}$ and $[A_j^{\text{tr}}(\mathbf{x}),E_k(\mathbf{y})]=i\delta_{jk}^{\text{tr}}(\mathbf{x}-\mathbf{y})$, by taking a representation in which the components of $\mathbf{q}$ and $\mathbf{A}^{\text{tr}}(\mathbf{x})$ are diagonal, so that

$$p_k=-i\partial/\partial q_k$$

and

$$E_k(\mathbf{y})=-i\delta/\delta A_k(\mathbf{y})\,.$$

With the latter functional differential operator representation for the electric radiation field, we have the state functional for the vacuum

$$\Omega=\exp\left[-\frac{1}{2}\int A_j^{\text{tr}}(\mathbf{x})(-\nabla^2)^{1/2}A_j^{\text{tr}}(\mathbf{x})d^3x\right] \tag{3}$$

satisfying the functional differential vacuum state equation

$$H_{\text{rad}}\Omega=\int\left[-\frac{1}{2}\frac{\delta^2}{\delta A_j(\mathbf{x})\delta A_j(\mathbf{x})}-\tfrac{1}{2}A_j^{\text{tr}}(\mathbf{x})\nabla^2A_j^{\text{tr}}(\mathbf{x})-u_{\text{vac}}\right]d^3x\Omega=0. \tag{4}$$

Thus, if we put $\Phi=\Omega\Psi$ with the four components of Ψ complex-valued functionals of $\mathbf{A}^{\text{tr}}(\mathbf{x})$ that depend on $\mathbf{q}$, the Schrödinger equation $H\Phi=E\Phi$ produces the functional differential-partial differential spinor equation for Ψ,

$$\left(H_{\text{el}}+\int\left[A_j^{\text{tr}}(\mathbf{x})(-\nabla^2)^{1/2}\frac{\delta}{\delta A_j(\mathbf{x})}-\frac{1}{2}\frac{\delta^2}{\delta A_j(\mathbf{x})\delta A_j(\mathbf{x})}\right]d^3x-E\right)\Psi=0. \tag{5}$$

The Rayleigh-Ritz procedure for functionalities[2] can be applied to establish a generic form for approximate solutions to Eq. (5). We introduce the *energy functionality*[2]

$$E=E\{\Psi,\Psi^\dagger\}\equiv\int\left[\langle\Psi^\dagger H_{\text{el}}\Psi\rangle+\left\langle\frac{1}{2}\int\frac{\delta\Psi^\dagger}{\delta A_j(\mathbf{x})}\frac{\delta\Psi}{\delta A_j(\mathbf{x})}d^3x\right\rangle\right]d^3q\Big/\int\langle\Psi^\dagger\Psi\rangle d^3q\,, \tag{6}$$

where functional integrals over all fields $\mathbf{A}^{\text{tr}}$ are denoted by

$$\langle\cdots\rangle\equiv\int\cdots\Omega^2\mathfrak{D}(\mathbf{A}^{\text{tr}})\,, \tag{7}$$

with the measure $\mathfrak{D}(\mathbf{A}^{\text{tr}})$ displacement-invariant and normalized to give $\langle 1\rangle=1$. The essential property of the energy functionality (6) is that it is stationary with respect to independent variations of Ψ and $\Psi^\dagger$ about solutions to Eq. (5), $\delta E=0$.

Detailed consideration of the asymptotic character of Eq. (5) for large and small $\mathbf{A}^{\text{tr}}(\mathbf{x})$ shows that solutions

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[1] For an early attempt to solve this problem by applying perturbation theory, see P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, New York, 1949), pp. 306–309; for a more recent attempt, see P. A. M. Dirac, Nuovo Cimento Suppl. **6**, 322 (1957). Most recently, a new perturbation-theoretic approach to the problem has been reported by G. Frieder and A. Peres, Nucl. Phys. **4B**, 306 (1968). The latter work shows that the formal summation of all one-electron line rainbow diagrams may lead to a convergent approximate solution for the Dyson-Schwinger electron propagator.

[2] G. Rosen, Phys. Rev. Letters **16**, 704 (1966); Phys. Rev. **156**, 1517 (1967); **160**, 1278 (1967); **167**, 1395 (1968); J. Math. Phys. **9**, 996 (1968).

are described approximately by the form

$$\Psi=[\exp\boldsymbol{\alpha}\cdot\mathbf{T}(\mathbf{q})]\psi(\mathbf{q})$$
$$=\left(\cosh|\mathbf{T}(\mathbf{q})|+\frac{\boldsymbol{\alpha}\cdot\mathbf{T}(\mathbf{q})}{|\mathbf{T}(\mathbf{q})|}\sinh|\mathbf{T}(\mathbf{q})|\right)\psi(\mathbf{q}), \quad (8)$$

$$T_j(\mathbf{q})\equiv\epsilon E^{-1}\mathcal{F}(-E^{-2}\nabla^2)A_j^{\mathrm{tr}}(\mathbf{q})$$
$$=\frac{\epsilon E^{-1}}{(2\pi)^3}\int\mathcal{F}(E^{-2}|\mathbf{k}|^2)e^{i\mathbf{k}\cdot(\mathbf{q}-\mathbf{x})}A_j^{\mathrm{tr}}(\mathbf{x})d^3k d^3x, \quad (9)$$

where $\psi(\mathbf{q})$ is an ordinary Dirac spinor, E is the total energy of the state, and $\mathcal{F}(\xi)$ is a real scalar function. For application of the Rayleigh-Ritz procedure, we constrain $\mathcal{F}(\xi)$ to satisfy the conditions

$$\frac{\epsilon^2}{6\pi^2}\int_0^\infty\mathcal{F}(\xi)^2 d\xi\equiv\lambda, \quad (10)$$

λ being a fixed positive parameter, and

$$-(3+2\lambda)\int_0^\infty\mathcal{F}(\xi)d\xi+(\tfrac{1}{2}+\lambda+\lambda^{-1}-\lambda^{-1}e^{-\lambda})$$
$$\times\int_0^\infty\mathcal{F}(\xi)^2\xi^{1/2}d\xi$$
$$=(4\pi^2/\epsilon^2)(1+2\lambda-e^{-\lambda}). \quad (11)$$

The quantities $\psi(\mathbf{q})$, $\psi^\dagger(\mathbf{q})$, and $\mathcal{F}(\xi)$ [constrained by (10) and (11)] are to be varied independently to obtain Rayleigh-Ritz equations from an evaluated expression for the energy functionality (6) with the form (8). To accomplish the evaluation of (6) with (8), we first employ explicit functional integration or the functional integration by parts lemma[2] to derive the preliminary formulas

$$\langle A_i^{\mathrm{tr}}(\mathbf{x})A_j^{\mathrm{tr}}(\mathbf{y})\rangle=\tfrac{1}{2}(-\nabla^2)^{-1/2}\delta_{ij}^{\mathrm{tr}}(\mathbf{x}-\mathbf{y}), \quad (12)$$

$$\langle T_i(\mathbf{q})T_j(\mathbf{q})\rangle=\tfrac{1}{2}\lambda\delta_{ij}, \quad (13)$$

$$\langle T_{i_1}(\mathbf{q})\cdots T_{i_{2n}}(\mathbf{q})\rangle$$
$$=(\tfrac{1}{2}\lambda)^n\sum_{\substack{\text{perm.}\\ \text{distinct pairings}}}\delta_{i_1i_2}\cdots\delta_{i_{2n-1}i_{2n}}, \quad (14)$$

$$\langle T_i(\mathbf{q})T_j(\mathbf{q})|\mathbf{T}(\mathbf{q})|^{2n-2}\rangle=\frac{(2n+1)!}{n!}(\tfrac{1}{4}\lambda)^n\tfrac{1}{3}\delta_{ij}, \quad (15)$$

$$\langle|\mathbf{T}(\mathbf{q})|^{2n}\rangle=\frac{(2n+1)!}{n!}(\tfrac{1}{4}\lambda)^n, \quad (16)$$

$$\langle\mathbf{A}^{\mathrm{tr}}(\mathbf{q})\cdot\mathbf{T}(\mathbf{q})|\mathbf{T}(\mathbf{q})|^{2n-2}\rangle$$
$$=\frac{(2n+1)!}{n!}(\tfrac{1}{4}\lambda)^{n-1}\frac{\epsilon E}{24\pi^2}\int_0^\infty\mathcal{F}(\xi)d\xi. \quad (17)$$

Then, by using the additional formula

$$\int\frac{\delta T_i(\mathbf{q})}{\delta A_k(\mathbf{x})}\frac{\delta T_j(\mathbf{q})}{\delta A_k(\mathbf{x})}d^3x=\frac{\epsilon^2E}{6\pi^2}\int_0^\infty\mathcal{F}(\xi)^2\xi^{1/2}d\xi\delta_{ij}, \quad (18)$$

performing some elementary algebraic manipulations, and summing up transcendental power series in λ, we evaluate the functional integrals in (6) as

$$\langle\Psi^\dagger H_{\mathrm{el}}\Psi\rangle=\tfrac{1}{3}[2+(1+2\lambda)e^\lambda]\psi^\dagger(\mathbf{q})\boldsymbol{\alpha}\cdot\mathbf{p}\psi(\mathbf{q})$$
$$-(1+\tfrac{2}{3}\lambda)e^\lambda\frac{\epsilon^2E}{2\pi^2}\int_0^\infty\mathcal{F}(\xi)d\xi\,\psi^\dagger(\mathbf{q})\psi(\mathbf{q})$$
$$+m_0\psi^\dagger(\mathbf{q})\beta\psi(\mathbf{q}), \quad (19)$$

$$\left\langle\frac{1}{2}\int\frac{\delta\Psi^\dagger}{\delta A_j(\mathbf{x})}\frac{\delta\Psi}{\delta A_j(\mathbf{x})}d^3x\right\rangle$$
$$=[(\tfrac{1}{2}+\lambda+\lambda^{-1})e^\lambda-\lambda^{-1}]\frac{\epsilon^2E}{6\pi^2}$$
$$\times\int_0^\infty\mathcal{F}(\xi)^2\xi^{1/2}d\xi\,\psi^\dagger(\mathbf{q})\psi(\mathbf{q}), \quad (20)$$

$$\langle\Psi^\dagger\Psi\rangle=(1+2\lambda)e^\lambda\psi^\dagger(\mathbf{q})\psi(\mathbf{q}). \quad (21)$$

Hence, the energy functionality is given by

$$E=E[\psi,\psi^\dagger,\mathcal{F}]=\left\{\frac{1}{3}[2(1+2\lambda)^{-1}e^{-\lambda}+1]\right.$$
$$\times\int\psi^\dagger(\mathbf{q})\boldsymbol{\alpha}\cdot\mathbf{p}\psi(\mathbf{q})d^3q+(1+2\lambda)^{-1}e^{-\lambda}m_0$$
$$\left.\times\int\psi^\dagger(\mathbf{q})\beta\psi(\mathbf{q})d^3q\right\}\Big/\int\psi^\dagger(\mathbf{q})\psi(\mathbf{q})d^3q$$
$$+\tfrac{2}{3}E[1-(1+2\lambda)^{-1}e^{-\lambda}], \quad (22)$$

where condition (11) has been used to combine terms proportional to E. Thus, we have $\delta E\equiv 0$ for a variation of $\mathcal{F}(\xi)$ which preserves conditions (10) and (11). For independent variations of $\psi(\mathbf{q})$ and $\psi^\dagger(\mathbf{q})$, $\delta E=0$ produces the proper relativistic wave equation

$$(\boldsymbol{\alpha}\cdot\mathbf{p}+m\beta-E)\psi(\mathbf{q})=0 \quad (23)$$

and its complex conjugate for an electron of positive observable energy E greater or equal to the dressed mass

$$m\equiv 3[(1+2\lambda)e^\lambda+2]^{-1}m_0. \quad (24)$$

Hence, the Rayleigh-Ritz procedure indicates that a generic form for approximate solutions to Eq. (5) is given by (8) with $\mathcal{F}(\xi)$ constrained by (10) and (11), $\psi(\mathbf{q})$ satisfying (23). However, the Rayleigh-Ritz procedure does not admit a complete determination of $\mathcal{F}(\xi)$

nor a determination of the value of the fixed parameter defined by (10). Suitable additional relations derived from Eq. (5) are required to fix $\mathfrak{F}(\xi)$ completely. To obtain such additional relations, we note that the direct substitution of (8) into the left side of Eq. (5) produces the expression

$$[D_0+D_1+(\text{higher-order terms in } \mathbf{A}^{\text{tr}})]\psi(\mathbf{q})\,, \quad (25)$$

where

$$D_0=\boldsymbol{\alpha}\cdot\mathbf{p}+m_0\beta-(\mu+1)E\,, \quad (26)$$

$$D_1=\{\boldsymbol{\alpha}\cdot\mathbf{p}+m_0\beta-[(5/9)\mu+1]E\}\boldsymbol{\alpha}\cdot\mathbf{T}(\mathbf{q}) -\epsilon\boldsymbol{\alpha}\cdot\mathbf{A}^{\text{tr}}(\mathbf{q})+\boldsymbol{\alpha}\cdot[(-\nabla^2)^{1/2}\mathbf{T}(\mathbf{q})]\,, \quad (27)$$

and

$$\mu\equiv\frac{\epsilon^2}{4\pi^2}\int_0^\infty \mathfrak{F}(\xi)^2\xi^{1/2}d\xi\,. \quad (28)$$

Now, if (8) were an exact solution to (5), we would have

$$D_0\psi(\mathbf{q})=0 \quad\text{and}\quad D_1\psi(\mathbf{q})=0\,, \quad (29)$$

with (25) vanishing identically for all $\mathbf{A}^{\text{tr}}$ if (8) were an exact solution. The simplest relations compatible with the form of (8) that are implied by conditions (29), therefore logical relations to be imposed on the approximate solution (8), are

$$\int\psi^\dagger(\mathbf{q})\beta D_0\psi(\mathbf{q})d^3q =[m_0-(\mu+1)m]\int\psi^\dagger(\mathbf{q})\psi(\mathbf{q})d^3q=0 \quad (30)$$

and

$$\int\psi^\dagger(\mathbf{q})[\boldsymbol{\alpha}\cdot\mathbf{T}(\mathbf{q})D_0+D_1]\psi(\mathbf{q})d^3q =\int\psi^\dagger(\mathbf{q})\boldsymbol{\alpha}\cdot\{-(14/9)\mu E\mathbf{T}(\mathbf{q})-\epsilon\mathbf{A}^{\text{tr}}(\mathbf{q}) +[(-\nabla^2)^{1/2}\mathbf{T}(\mathbf{q})]\}\psi(\mathbf{q})d^3q=0\,, \quad (31)$$

where Eq. (23) and the associated integral relation

$$\int\psi^\dagger(\mathbf{q})\beta\psi(\mathbf{q})d^3q=\left(\frac{m}{E}\right)\int\psi^\dagger(\mathbf{q})\psi(\mathbf{q})d^3q$$

have been used to get the second members in Eqs. (30) and (31). Equation (30) and the definition (28) produce

$$\frac{m_0}{m}-1=\frac{\epsilon^2}{4\pi^2}\int_0^\infty \mathfrak{F}(\xi)^2\xi^{1/2}d\xi\,, \quad (32)$$

while Eq. (31) and definition (9) yield

$$\mathfrak{F}(\xi)=[\xi^{1/2}-(14/9)\mu]^{-1}=\left[\xi^{1/2}-\frac{14}{9}\left(\frac{m_0}{m}-1\right)\right]^{-1}. \quad (33)$$

The latter expression for $\mathfrak{F}(\xi)$ confirms the general form of the solution prescribed by (8) and (9), but (33) gives divergent values for the integrals in (10) and (11); therefore, (33) cannot be accurate for $\xi^{1/2}\cong(14/9)\times[(m_0/m)-1]$ or for very large values of ξ. If, however, the main qualitative feature of the square of Eq. (33) is accepted, that is, a sharply peaked form for $\mathfrak{F}(\xi)^2$ about $\xi^{1/2}=(14/9)[(m_0/m)-1]$, we would expect to have

$$\int_0^\infty \mathfrak{F}(\xi)^2\xi^{1/2}d\xi\cong\frac{14}{9}\left(\frac{m_0}{m}-1\right)\int_0^\infty \mathfrak{F}(\xi)^2d\xi\,. \quad (34)$$

Then it follows immediately from (32), (10), and (24) that $\lambda\cong3/7$ and $m\cong0.62m_0$.

In conventional terminology, the analysis presented here gives an approximate solution for the cloud of virtual photons around an electron. However, a theory based on the one-electron Hamiltonian (1) does not allow for electron-positron pairs in the cloud, as predicted by full-blown quantum electrodynamics with a second-quantized electron field. It is hoped that a generalization of the nonperturbative functional integration approximate method presented here can be worked out for the complete theory of the electron.