Tree Graphs and Classical Fields

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(Received 1 April 1968)

The classical field produced by a prescribed external source is shown to be the generating functional of the tree-graph approximation to the corresponding quantum field theory.

The study of the tree-graph approximation to quantum field theory is of interest in connection with the phenomenological Lagrangian approach to current algebra. In particular, it has been shown[1] that the tree graphs represent the lowest-order contribution to each connected Green's function. Thus, since the current-algebra commutation relations and the conservation or partial conservation (PCAC) of the currents are respected order by order in perturbation theory, the set of tree graphs alone satisfies all the current-algebra constraints. Recently Nambu[2] has discussed the relationship of the tree-graph approximation to a formal semiclassical limit of the scattering matrix. It is the purpose of this paper to sharpen this correspondence: We shall show that the classical solution to the field equations in the presence of an arbitrary external source function is the generating functional for the connected Green's functions in the tree-graph approximation.

We consider a system described by a set of quantum fields $\phi_a(x)$ and a local Lagrange function $\mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x))$ constructed from these fields and their space-time gradients. The index a may label not only possible spin components but various internal degrees of freedom as well. Those components that refer to Fermi fields require special care, since they are intrinsically anticommutative. In particular, derivatives of Fermi fields must be defined in terms of anticommuting variations that are either consistently placed to the left (left derivatives) or to the right (right derivatives). We shall, however, not encumber the notation by making this distinction explicit. The Euler-Lagrange equations

$$\mathfrak{F}(\phi)=\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}-\frac{\partial\mathcal{L}}{\partial\phi_a}=0 \tag{1}$$

make no explicit reference to the quantum nature of the field variables. This is provided by the requirement that the conjugate fields

$$\pi_a(x)=\partial\mathcal{L}/\partial(\partial_0\phi_a(x)) \tag{2}$$

satisfy equal-time commutation $(-)$ or anticommutation $(+)$ relations:

$$[\phi_a(\mathbf{r},t),\pi_b(\mathbf{r}',t)]_\mp=i\hbar\delta_{ab}\delta(\mathbf{r}-\mathbf{r}')\,. \tag{3}$$

The vacuum expectation value of the time-ordered product of an exponential involving[3] a set of external sources $\zeta_a(x)$,

$$Z[\zeta]=\left\langle T\exp\left[\frac{i}{\hbar}\sum_a\int(dx)\,\zeta_a(x)\phi_a(x)\right]\right\rangle, \tag{4}$$

is the generating functional for all the Green's functions of the theory. Thus

$$\left\langle T\left(\phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)\exp\left[\frac{i}{\hbar}\sum_a\int(dx)\,\zeta_a(x)\phi_a(x)\right]\right)\right\rangle =\frac{\hbar}{i}\frac{\delta}{\delta\zeta_{a_1}(x_1)}\cdots\frac{\hbar}{i}\frac{\delta}{\delta\zeta_{a_n}(x_n)}Z[\zeta], \tag{5}$$

and the Green's functions are given by

$$\langle T(\phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n))\rangle =\frac{\hbar}{i}\frac{\delta}{\delta\zeta_{a_1}(x_1)}\cdots\frac{\hbar}{i}\frac{\delta}{\delta\zeta_{a_n}(x_n)}Z[\zeta]|_{\zeta=0}. \tag{6}$$

The representation of the quantum field in terms of a functional derivative provided by Eq. (5),

$$\phi_a(x)\leftrightarrow\frac{\hbar}{i}\frac{\delta}{\delta\zeta_a(x)},$$

* Supported in part by the U. S. Atomic Energy Commission under Contract No. RLO-138813.
† Research supported in part by the National Science Foundation.

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[1] B. W. Lee and H. T. Nieh, Phys. Rev. **166**, 1507 (1968).
[2] Y. Nambu, Phys. Letters **26B**, 626 (1968).

[3] In the case of Fermi fields, the source functions both anticommute among themselves and anticommute with the Fermi fields; this anticommutativity can produce an extra factor of (-1) which we do not indicate explicitly. If the spin of the field is larger than $\frac{1}{2}$, some of its components are constraint variables and additional, noncovariant source terms may occur in the generating functional. Such additional terms are necessary if the Green's functions defined by Eq. (6) are to be covariant. They are also necessary for the validity of a covariant functional-differential equation for $Z[\zeta]$, Eq. (7), since they cancel commutators involving constraint variables.

may be used to construct the functional-differential equation satisfied by the generating functional. We have, without writing explicitly all indices and gradient terms,

$$\mathfrak{F}\left(\frac{\hbar}{i}\frac{\delta}{\delta\zeta(x)}\right)Z[\zeta]=\partial_\mu\left\langle T\left(\frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi(x))}\times\exp\left[\frac{i}{\hbar}\int(dx')\,\zeta(x')\phi(x')\right]\right)\right\rangle-\left\langle T\left(\frac{\partial\mathfrak{L}}{\partial\phi(x)}\exp\left[\frac{i}{\hbar}\int(dx')\,\zeta(x')\phi(x')\right]\right)\right\rangle.$$

By virtue of the Euler-Lagrange equations, the right-hand side of this equation would vanish if the divergence (∂_μ) occurred within the time ordering. Since the time-ordering process is discontinuous, an equal-time commutator appears, and using Eqs. (2) and (3), we obtain

$$\mathfrak{F}\left(\frac{\hbar}{i}\frac{\delta}{\delta\zeta(x)}\right)Z[\zeta]=\left\langle T\left(\left[\pi(\mathbf{r},t),\frac{i}{\hbar}\int(d\mathbf{r}')\,\zeta(\mathbf{r}',t)\phi(\mathbf{r}',t)\right]\times\exp\left[\frac{i}{\hbar}\int(dx')\,\zeta(x')\phi(x')\right]\right)\right\rangle=\zeta(x)Z[\zeta]. \quad (7)$$

It can be shown that the generating functional $W[\zeta]$ of the connected Green's functions $G_n^{(c)}(x_1,\cdots,x_n)$,

$$G_n^{(c)}(x_1,\cdots,x_n)=\hbar^{n-1}\frac{\delta}{\delta\zeta(x_1)}\cdots\frac{\delta}{\delta\zeta(x_n)}W[\zeta]\big|_{\zeta=0}, \quad (8)$$

is given by[4]

$$Z[\zeta]=\exp\left(\frac{i}{\hbar}W[\zeta]\right). \quad (9)$$

It follows from Eq. (7) that this generating functional satisfies

$$\zeta(x)=\exp\left(-\frac{i}{\hbar}W[\zeta]\right)\mathfrak{F}\left(\frac{\hbar}{i}\frac{\delta}{\delta\zeta(x)}\right)\exp\left(\frac{i}{\hbar}W[\zeta]\right)=\mathfrak{F}\left(e^{(i/\hbar)W}\frac{\hbar}{i}\frac{\delta}{\delta\zeta(x)}e^{(i/\hbar)W}\right)=\mathfrak{F}\left(\frac{\delta W[\zeta]}{\delta\zeta(x)}+\frac{\hbar}{i}\frac{\delta}{\delta\zeta(x)}\right). \quad (10)$$

We consider now the change induced in the generating functional $W[\zeta]$ by infinitesimal variations $\delta\hbar$ of Planck's constant. If we perform such a variation on Eq. (10), we obtain a relation between response terms involving $\delta_\hbar W[\zeta]$ and driving terms that have $\delta\hbar$ as a coefficient. Omitting possible space-time gradients and multiplicative factors, the response terms are of the form

$$\frac{\delta}{\delta\zeta(x)}\delta_\hbar W[\zeta],\qquad \frac{\hbar}{i}\frac{\delta}{\delta\zeta(x)}\frac{\delta}{\delta\zeta(x)}\delta_\hbar W[\zeta],\ \cdots,$$

while the driving terms appear as

$$-i\delta\hbar\frac{\delta^2W[\zeta]}{\delta\zeta(x)^2},\qquad \hbar\frac{\delta}{\delta\zeta(x)}\delta\hbar\frac{\delta^2W[\zeta]}{\delta\zeta(x)^2},\cdots.$$

A formal solution of this relation may be obtained that gives $(\delta/\delta\zeta)\delta_\hbar W$ as an integral operator [which depends upon $(\hbar/i)\delta/\delta\zeta$] acting on the driving terms. The driving terms involve $\delta^2W/\delta\zeta(x)^2$ and its local functional derivatives, and are related to connected Green's functions containing $\phi(x)^2$ and higher powers of the field operator at a single space-time point. They therefore correspond to Feynman graphs that contain at least one closed loop. Thus, variations of $\hbar$ alter only graphs that contain closed loops but do not change tree graphs which, by definition, contain no closed loops, and the limit $\hbar=0$ suffices for the construction of the generating functional in the tree-graph approximation. The functional equation now reduces to

$$\mathfrak{F}(\phi'[x;\zeta])=\partial_\mu\frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi'(x))}-\frac{\partial\mathfrak{L}}{\partial\phi'(x)}=\zeta(x), \quad (11)$$

with

$$\phi'[x;\zeta]=\lim_{\hbar\to0}\frac{\delta W[\zeta]}{\delta\zeta(x)}. \quad (12)$$

This is precisely the Euler-Lagrange equation for the classical field ϕ'. It is a simple matter to check directly that its perturbative solution in coupling-constant strengths corresponds only to connected tree graphs. Hence, the classical field generated by an arbitrary external source function serves as the generating functional for the tree-graph approximation to the connected Green's functions. An alternative derivation of the $\hbar\to0$ limit which employs the Feynman path-integral formulation is presented in the Appendix.

This general discussion can be illustrated by the simple example of a neutral scalar field with a cubic self-interaction. In this case the field equation reads

$$\mathfrak{F}(\phi(x))=(-\partial^2+\kappa^2)\phi(x)-g\phi(x)^2=0, \quad (13)$$

and, on introducing an integral operator notation with

$$G=[-\partial^2+\kappa^2-i\epsilon]^{-1}, \quad (14)$$

[4] The number of factors of $\hbar$ in Eqs. (8) and (9) is determined by the requirement that $G_n^{(c)}$ have the same dimension as the n-point Green's function (the dimension of ϕ^n), and that $W[\zeta]$ have the dimension of action.

the functional-derivative Eq. (10) becomes

$$\frac{\delta W}{\delta\zeta}=G\left\{\zeta+g\left[\left(\frac{\delta W}{\delta\zeta}\right)^2+\frac{\hbar}{i}\frac{\delta^2 W}{\delta\zeta^2}\right]\right\}. \tag{15}$$

The variation of this equation with a change in $\hbar$ yields

$$\frac{\delta}{\delta\zeta}(\delta_\hbar W)=\left(1-2gG\frac{\delta W}{\delta\zeta}-gG\frac{\hbar}{i}\frac{\delta}{\delta\zeta}\right)^{-1}\frac{\delta\hbar}{i}gG\frac{\delta^2 W}{\delta\zeta^2}, \tag{16}$$

which explicitly relates the response $(\delta/\delta\zeta)(\delta_\hbar W)$ to $\delta^2W/\delta\zeta^2$, a quantity that involves at least one closed loop. Thus, with the neglect of closed loops we may set $\hbar=0$ and obtain

$$\phi'=G\{\zeta+g\phi'^2\}. \tag{17}$$

The iterative solution

$$\phi'=G\zeta+gG(G\zeta)^2+2g^2G[(G\zeta)G(G\zeta)^2]+\cdots, \tag{18a}$$

or, more explicitly,

$$\phi'(x)=\int(dx_1)\,G(x-x_1)\zeta(x_1)+g\int(dy)\,(dx_1)(dx_2)G(x-y)G(y-x_1)\zeta(x_1)\times G(y-x_2)\zeta(x_2)+\cdots \tag{18b}$$

displays the decomposition of ϕ' into tree-graph structures of increasing order, shown in Fig. 1.

The assertion that the tree-graph approximation is all that survives in the $\hbar\to 0$ limit is easily confirmed in perturbation theory. It follows from the commutation relation [Eq. (3)] that the free-particle propagator is of order $\hbar$. Since perturbation theory involves an expansion of

$$T\exp\left(\frac{i}{\hbar}\int(dx)\,\mathcal{L}_{\rm int}\right),$$

there is a factor $\hbar^{-1}$ associated with each power of a coupling constant. Now, a tree graph with n external lines and m vertices (m coupling constants) has $m-1$ internal lines and is of order

$$(\hbar)^n(1/\hbar)^m(\hbar)^{m-1}=\hbar^{n-1}. \tag{19}$$

Hence, according to Eq. (8), a tree graph gives a contribution of zeroth order in $\hbar$ to the connected generating functional $W[\zeta]$ and to the field ϕ' defined by Eq. (12). Any diagram containing closed loops may be constructed from tree graphs by contracting external lines, a process that omits a propagator for each pair of lines that are contracted. If l such contractions are performed, l propagators are omitted, and this incurs a factor $\hbar^{-l}$. However, the number of external lines is reduced by $2l$, and according to the order shown in Eq. (19), this gives a relative factor of $\hbar^{2l}$. Therefore, l contractions produce an over-all factor of $\hbar^l$, and the corresponding closed-loop diagrams vanish as $\hbar^l$ relative to tree graphs in the limit $\hbar\to 0$.

Our formal considerations have direct physical significance only when all the fields become classical in the $\hbar\to 0$ limit. The situation is very different when some of the fields are taken to create particles in this limit, and the techniques that we have used are certainly inadequate for its discussion. In this general case, coupling constants can contain explicit factors of $\hbar$, particle masses rather than Compton wavelengths must be taken to be independent of $\hbar$, and some effects of radiative corrections (loops) survive the classical limit.

We have enjoyed conversations with S. Deser.

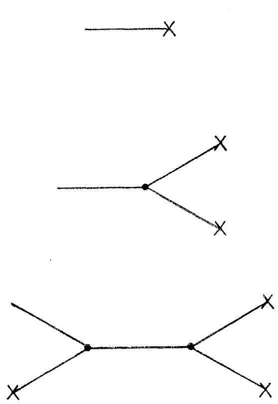

FIG. 1. Tree graphs representing the classical field given in Eq. (18). The crosses represent the external source while the line without a cross corresponds to the classical field.

APPENDIX

The generating functional may be written as a Feynman path integral[5]

$$Z[\zeta]=\int[d\phi']\exp\left(\frac{i}{\hbar}\int(dx')\,[\mathcal{L}(\phi'(x'),\partial_\mu{}'\phi'(x'))+\zeta(x')\phi'(x')]\right), \tag{A1}$$

where the functional volume element $[d\phi']$ is normalized such that

$$Z[0]=1. \tag{A2}$$

Since

$$\begin{aligned}0&=\int[d\phi']\frac{\hbar}{i}\frac{\delta}{\delta\phi'(x)}\exp\left(\frac{i}{\hbar}\int(dx')\,(\mathcal{L}+\zeta\phi')\right)\\&=\int[d\phi']\{\zeta(x)-\mathfrak{F}(\phi'(x))\}\exp\left(\frac{i}{\hbar}\int(dx')\,(\mathcal{L}+\zeta\phi')\right)\\&=\left[\zeta(x)-\mathfrak{F}\left(\frac{\hbar}{i}\frac{\delta}{\delta\zeta(x)}\right)\right]\int[d\phi']\times\exp\left(\frac{i}{\hbar}\int(dx')\,(\mathcal{L}+\zeta\phi')\right),\end{aligned} \tag{A3}$$

[5] A general discussion of such integrals is given in R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill Book Co., Inc., New York, 1965).

this functional integral representation indeed satisfies the functional differential equation (7) for the generating functional, and the volume-element normalization has been chosen to ensure that the boundary value at $\zeta=0$ is obeyed. In the limit $\hbar\to 0$ only the stationary phase path contributes, and we obtain

$$(\hbar\to 0)\qquad Z[\zeta]=\exp\left(\frac{i}{\hbar}W_{cl}[\zeta]\right), \tag{A4}$$

in which the action

$$W_{cl}[\zeta]=\int (dx)\,\{\mathcal{L}(\phi'[\zeta])+\zeta\phi'[\zeta]\} \tag{A5}$$

is evaluated with the classical field $\phi'[\zeta]$ produced by the source function. Since the action is stationary with respect to field variations about the classical path,

$$\delta_{\phi'}\int (dx)\,\{\mathcal{L}+\zeta\phi'\}=0\,, \tag{A6}$$

we have

$$\delta W_{cl}[\zeta]/\delta\zeta(x)=\phi'[x;\zeta]\,, \tag{A7}$$

and we recover the result stated in the text that the classical field produced by an external source is the generating functional for the semiclassical limit of the connected Green's functions. We also find that the quantum-field functional $W[\zeta]$ reduces to the classical action in the limit $\hbar\to 0$.

Note added in proof. Previous discussions of tree graphs have been given by R. P. Feynman, Acta Phys. Polon. **24**, 697 (1963); H. M. Fried, J. Math. Phys. **3**, 1107 (1962); B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach Science Publishers, Inc., New York, 1965); B. S. DeWitt, Phys. Rev. **162**, 1218 (1967). We would like to thank H. J. Schnitzer, H. M. Fried, and B. S. DeWitt for bringing this work to our attention.

Feynman has indicated how the corrections of order $\hbar$ to the semiclassical limit may be computed. According to the discussion in the text, these terms correspond to an arbitrary number of tree graphs attached to single closed loops (or ring graphs). The calculation of this correction with the external source technique is straightforward. We translate the functional integration variable ϕ' by the solution to the classical field equation $\phi'[\zeta]$,

$$\phi'=\phi'[\zeta]+\chi'\,,$$

which preserves the functional volume element

$$[d\phi']=[d\chi'].$$

The exponential occurring in the fundamental integral (A1) can be expanded in powers of χ', and the quadratic terms give the correction of order $\hbar$. Since the action is stationary with respect to deviations from the classical solution [Eq. (A6)], the linear terms in χ' vanish, and we have, in a matrix notation,

$$\int (dx)[\mathcal{L}+\zeta\phi']\cong W_{cl}[\zeta]+\chi' G^{-1}[\phi'[\zeta]]\chi'.$$

In the absence of interaction, the Lagrangian is a quadratic form, and thus G^{-1} must become the free-particle inverse Green's function in this limit,

$$G^{-1}\to G_0^{-1}.$$

Hence G^{-1} has the general structure

$$G^{-1}=G_0^{-1}-f(\phi'[\zeta])\,,$$

where f is a local function of $\phi'[\zeta]$, and its derivatives, which plays the role of an external field. Now the functional integral of an exponential of a quadratic form is simply the Jacobian of the linear transformation that diagonalizes this form, and we obtain

$$\begin{aligned} Z[\zeta]&=\exp\{(i/\hbar)W_{cl}[\zeta]\}\int [d\chi']\exp\{(i/\hbar)\chi' G^{-1}\chi'\}\\ &=\exp\{(i/\hbar)W_{cl}[\zeta]\}\,\mathrm{Det}^{-1/2}[G_0G^{-1}]\\ &=\exp\{(i/\hbar)W_{cl}[\zeta]\}\,\mathrm{Det}^{-1/2}[1-G_0f(\phi'[\zeta])].\end{aligned}$$

The presence of the free-particle Green's function G_0 is necessary to ensure that $Z[\zeta]$ reduce to unity when the external sources ζ vanish. If we write

$$Z[\zeta]=\exp\{(i/\hbar)W[\zeta]\}$$

and expand the determinant, we obtain

$$W[\zeta]=W_{cl}[\zeta]-\tfrac{1}{2}i\hbar\sum_{n=0}^{\infty}\frac{1}{n}\mathrm{Tr}\{G_0f(\phi'[\zeta])\}^n\,,$$

which exhibits the order $\hbar$ correction to the connected generating functional as a series of single loop diagrams with vertices $f(\phi'[\zeta])$ that are connected to tree graphs. These terms, of course, contain the usual infinities of perturbation theory and must be renormalized in the standard way.