PHYSICAL REVIEW VOLUME 172, NUMBER 5 25 AUGUST 1968

Final-State Interactions in the $K \to 3\pi$ and $\eta \to 3\pi$ Decays*

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(Received 19 April 1968)

We derive an integral equation which should describe, to a good approximation, the results of any theory of the K and η 3π decays. We show that present experimental information implies that one subtraction in this equation is not enough; instead, we find that two subtractions are necessary in order for the equation to describe experiment correctly. We then observe the striking fact that for the weak (K) and electromagnetic (η) decay processes leading to a 3π final state, the subtraction constants are identical, up to an over-all normalization. A short discussion of the possible significance of this is given, although no definite conclusions are reached. Our analysis confirms the suggestion that $\pi\pi$ final-state interactions will appear only in quadratic terms in the 3π final state.

I. INTRODUCTION

THERE have been numerous theoretical studies of the structure in the $K \to 3\pi$ and $\eta \to 3\pi$ decays. The most recent[1] have taken an approach via the algebra of currents, and while the results for the K decays are encouraging, the results for the η decays are somewhat puzzling. Furthermore, one can argue[2] that the current-algebra results for $\pi\pi$ scattering preclude the presence of significant final-state interactions in the $K \to 3\pi$ and $\eta \to 3\pi$ decays. Thus, if this contention is correct, the observed structure is due to weak (K-decay) or electromagnetic (η-decay) interaction itself.

This view is contrary to that of the older approach,[3] where it was assumed that the final-state interactions were the cause of structure observed. One of the arguments in favor of this view is the similarity of the K- and η-decay structure, as would be expected if the $\pi\pi$ final-state interactions were the dominant feature of the decay process. This would open a door to the determination of the low-energy $\pi\pi$ phase shifts, a door through which many have attempted to pass, with varying results.

In the light of the above, we propose to study these 3π decays in a very general way, making use of as much experimental and theoretical information as we can, in order to draw as many conclusions as we can from the present situation.

II. DEVELOPMENT OF AN INTEGRAL EQUATION

On very general theoretical grounds, we may claim that the integral equations proposed by Khuri and Treiman in 1960[1] should correctly describe the results of *any* theory of the K and η 3π decays which meets the requirements of Lorentz and isospin invariance, causality, smooth (analytic) behavior, etc.; in a word, all of the things most theoreticians would say they believed in, even though solid experimental proof of these things in the realm of high-energy physics is not really available, although it can be said that none of these assumptions has been proved wrong. It is also worth mentioning at this point that the Khuri-Treiman equations are three-body equations; thus rescattering effects are explicitly built into them.

So, granting the validity of the Khuri-Treiman equations, we have

$$\begin{aligned} A(s_1,s_2,s_3) = \frac{1}{\pi}\int_t^\infty \frac{ds'}{s'-s_1-i\epsilon}&\{\tilde{A}(s')f_0^*(s') \\ &+\tfrac{1}{3}[\tilde{B}(s')+\tilde{C}(s')][f_0^*(s')-f_2^*(s')]\} \\ &+\frac{1}{\pi}\int_t^\infty \frac{ds'}{s'-s_2-i\epsilon}\{\tfrac{1}{2}[\tilde{B}(s')+\tilde{C}(s')]f_2^*(s')\} \\ &+\frac{1}{\pi}\int_t^\infty \frac{ds'}{s'-s_3-i\epsilon}\{\tfrac{1}{2}[\tilde{B}(s')+\tilde{C}(s')]f_2^*(s')\} \\ &+\text{subtractions}. \qquad (1) \end{aligned}$$

There are two more equations just like the above in which A, B, and C are cyclicly permuted. The following definitions have been made:

$$\begin{aligned} \tilde{A}(s) &= \int \frac{d\Omega_{23}}{4\pi} A(s,s_2,s_3), \\ \tilde{B}(s) &= \int \frac{d\Omega_{23}}{4\pi} B(s,s_2,s_3), \qquad (2) \\ \tilde{C}(s) &= \int \frac{d\Omega_{23}}{4\pi} C(s,s_2,s_3), \end{aligned}$$

$$t=4m_\pi^2,\quad s_1=(p_2+p_3)^2,\quad s_2=(p_3+p_1)^2,$$
$$s_3=(p_1+p_2)^2,\quad f_0(s)=e^{i\delta_0(s)}\sin\delta_0(s),$$
$$f_2(s)=e^{i\delta_2(s)}\sin\delta_2(s),$$

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* Work supported by the U. S. Atomic Energy Commission.

[1] For the η decay, see W. A. Bardeen, L. S. Brown, B. W. Lee, and H. T. Nieh, Phys. Rev. Letters **19**, 1170 (1967), and references therein. For the K decay, see Y. Hara and Y. Nambu, *ibid.* **16**, 875 (1966); D. K. Elias and J. C. Taylor, Nuovo Cimento **44A**, 528 (1966); **48A**, 814 (1967).

[2] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[3] I. M. Barbour and R. L. Schult, Phys. Rev. **155**, 1712 (1967); W. A. Dunn and R. Ramachandran, *ibid.* **153**, 1558 (1967); N. N. Khuri and S. B. Treiman, *ibid.* **119**, 1115 (1960); L. Brown and P. Singer, Phys. Rev. Letters **8**, 460 (1962); A. N. Mitra and Shubba Ray, Phys. Rev. **135**, B146 (1964). Other and earlier references may be traced from these; for this purpose, the first two above are most useful.

where Ω_{23} is the solid angle between particles 1 and 2 in the (23) c.m. system, and δ_I is the s-wave $\pi\pi$ phaseshift for isospin I. The $K(\eta)\rightarrow 3\pi$ amplitude is related to A, B, and C by

$$M_{\rho;\alpha\beta\gamma}=\delta_{\rho\alpha}\delta_{\beta\gamma}A(s_1,s_2,s_3)+\delta_{\rho\beta}\delta_{\gamma\alpha}B(s_1,s_2,s_3)+\delta_{\rho\gamma}\delta_{\alpha\beta}C(s_1,s_2,s_3),$$

where $\alpha,\beta,\gamma=1,2,3$ and indicate the charge state of a pion; ρ tells us the z component of the total isospin (which has been taken to be $I=1$) and thus tells us the charge of the decaying state.

In writing all the above, certain additional assumptions, besides the very general ones concerning causality, relativistic invariance, etc., have been made. We shall list and discuss these here:

(1) Only two-pion intermediate states are significant in the discontinuities which go into the dispersion integrals. This is a standard low-energy approximation, which should be a good one. However, the neglect of higher-cut contributions must be accommodated by *at least* one subtraction.

(2) Only s-wave $\pi\pi$ scattering is important. Again, this is a standard low-energy approximation, which should be good, especially since we have a free subtraction parameter to accommodate any (reasonably constant) error introduced.

(3) Only the $I=1$ state of the three pions is important. For the η decay this is hard to escape, since C invariance means only $I=1$ and $I=3$, and only a very unusual decay interaction could supply $\Delta I=3$. Such an interaction has never been seen. For the K decay, C is not conserved and we are stuck with the following argument: Since the kinetic energies involved are low, one would expect any pion to be in an s wave with respect to any other. In that case, we need a totally symmetric decay amplitude, which would come from the $I=1$ 3π final state.

Since all of the assumptions which we have mentioned are quite reasonable, the validity, to a close approximation, of Eq. (1) is very hard to dispute. It may be possible to improve it, but we shall not consider this problem here.

The following symmetry properties hold for A, B, and C:

$$\begin{aligned}
&A(s_1,s_2,s_3)=B(s_2,s_3,s_1)=C(s_3,s_1,s_2),\\
&A(s_1,s_2,s_3)=A(s_1,s_3,s_2),\\
&B(s_1,s_2,s_3)=B(s_1,s_3,s_2),\\
&C(s_1,s_2,s_3)=C(s_1,s_3,s_2).
\end{aligned}\tag{3}$$

These are the consequence of Bose statistics for the pions.

Note that for K^0 or $\eta\rightarrow\pi^+\pi^-\pi^0$, we have only $A(s_1,s_2,s_3)$ contributing to the decay matrix element. As nearly as can be told, $A(s_1,s_2,s_3)=f(s_1)$ only, i.e., only the odd-pion energy seems to show any variation.[4] This fact has some remarkable consequences, which we shall now begin to develop.

First of all, from Eq. (1) we see that

$$A(s_1,s_2,s_3)=a(s_1)+b(s_2)+b(s_3).\tag{4}$$

Using the symmetry properties (3) while inserting (4) into (1), we find

$$a(s)+\tfrac{2}{3}b(s)=\frac{1}{\pi}\int_t^\infty\frac{ds'}{s'-s-i\epsilon}\int\frac{d\Omega_{23}}{4\pi}f_0^*(s')\times\{a(s')+\tfrac{1}{3}[a(s_2)+a(s_3)]+\tfrac{2}{3}b(s')+\tfrac{4}{3}[b(s_2)+b(s_3)]\}$$

and

$$b(s)=\frac{1}{\pi}\int_t^\infty\frac{ds'}{s'-s-i\epsilon}\int\frac{d\Omega_{23}}{4\pi}\tfrac{1}{2}[f_2^*(s')]\times[a(s_2)+a(s_3)+2b(s')+b(s_2)+b(s_3)].$$

Letting $u(s)=\frac{1}{2}[a(s)+b(s)]$ and $v(s)=\frac{1}{2}[a(s)-b(s)]$, and defining

$$C_a(s_1,s_2,s_3)=a(s_1)+a(s_2)+a(s_3)$$

and

$$C_b(s_1,s_2,s_3)=b(s_1)+b(s_2)+b(s_3),$$

we obtain

$$v(s)=\frac{1}{6\pi}\int_t^\infty\frac{ds'}{s'-s-i\epsilon}\{f_0^*(s')[\tilde{C}_a(s')+4\tilde{C}_b(s')]-\tfrac{5}{2}f_2^*(s')[\tilde{C}_a(s')+\tilde{C}_b(s')]+[4f_0^*(s')+5f_2^*(s')]v(s')\}$$

and

$$u(s)=\frac{1}{6\pi}\int_t^\infty\frac{ds'}{s'-s-i\epsilon}\{f_0^*(s')[\tilde{C}_a(s')+4\tilde{C}_b(s')]+\tfrac{1}{2}f_2^*(s')[\tilde{C}_a(s')+\tilde{C}_b(s')]+[4f_0^*(s')-f_2^*(s')]v(s')\},$$

where

$$\tilde{C}(s)=\int\frac{d\Omega_{23}}{4\pi}C(s,s_2,s_3)$$

and now

$$A(s_1,s_2,s_3)=C_b(s_1,s_2,s_3)+2v(s_1).$$

Thus, according to experiment, we have $C_b=\text{const}$.[4] This is because, as previously mentioned, there is no dependence on s_2 and s_3; therefore, since C_b is completely symmetric in s_1, s_2, and s_3, it must have no dependence on s_1 either. We shall take $C_b=1$ for convenience, for we cannot compute the absolute magni-

[4] For the K decay, see B. M. K. Nefkens *et al.*, Phys. Rev. **157**, 1233 (1967), and the references therein. For the η decay, see the Columbia-Berkeley-Purdue-Wisconsin-Yale Collaboration, Phys. Rev. **149**, 1044 (1967).

tude of the decay rate anyway. Finally, setting

$$C_a(s_1,s_2,s_3)=-2+\epsilon(s_1,s_2,s_3),\quad \zeta(s)=v(s)+\tfrac{1}{2},$$

and

$$\xi(s)=u(s)+\tfrac{1}{2},$$

we have

$$\xi(s)=\tfrac{1}{3}+\tfrac{1}{6}\epsilon_0+\frac{s-s_0}{6\pi}\int_t^\infty \frac{ds'}{(s'-s_0-i\epsilon)(s'-s-i\epsilon)}\times\{\bar{\epsilon}(s')[f_0^*(s')+\tfrac{1}{2}f_2^*(s')]+\zeta(s')[4f_0^*(s')-f_2^*(s')]\},\quad (5)$$

$$\zeta(s)=\tfrac{1}{6}\epsilon_0+\frac{s-s_0}{6\pi}\int_t^\infty \frac{ds'}{(s'-s_0-i\epsilon)(s'-s-i\epsilon)}\times\{\bar{\epsilon}(s')[f_0^*(s')-\tfrac{5}{2}f_2^*(s')]+\zeta(s')[4f_0^*(s')+5f_2^*(s')]\},\quad (6)$$

where

$$\bar{\epsilon}(s)=\int\frac{d\Omega_{23}}{4\pi}\epsilon(s,s_2,s_3),\ \zeta(s_1)+\zeta(s_2)+\zeta(s_3)=\tfrac{1}{2}\epsilon(s_1,s_2,s_2);$$

we also define

$$s_1+s_2+s_3=S=3s_0=P^2+3m_\pi^2,$$
$$P=p_1+p_2+p_3,\quad\text{and}\quad \epsilon_0=\epsilon(s_0,s_0,s_0).$$

We have now explicitly made one subtraction at the symmetric point $s_1=s_2=s_3=s_0$, where all three pions have the same energy. And we now have simply $A(s_1,s_2,s_3)=\zeta(s_1)$.

Note that Eq. (5) does not enter into the decay process at all, and Eq. (6) has a solution if and only if

$$4f_0^*(s')+5f_2^*(s')+[\bar{\epsilon}(s')/\zeta(s')]\times[f_0^*(s')-\tfrac{5}{2}f_2^*(s')]=6e^{-i\delta(s')}\sin\delta(s'),\quad (7)$$

where $\delta(s')$ is some real function of s'. If Eq. (7) is true, then

$$\zeta(s)=\exp\left[\frac{s-s_0}{\pi}\int_t^\infty\frac{\delta(s')ds'}{(s'-s_0-i\epsilon)(s'-s-i\epsilon)}\right]\quad (8)$$

is a solution to (6). And thus we finally obtain

$$f_0(s)+I(s)[f_0(s)-\tfrac{5}{2}f_2(s)]-e^{i\delta(s)}\sin\delta(s),\quad (9)$$

where

$$I(s)=\frac{1}{3R(s)}\int_{-R(s)}^{+R(s)}dx\exp\left[\frac{y+x-s}{\pi}\times\int_t^\infty\frac{\delta(s')ds'}{(s'-s+i\epsilon)(s'-y-x+i\epsilon)}\right],$$

$$y=\tfrac{1}{2}(s_2+s_3)=\tfrac{1}{2}(S-s),\quad x=\tfrac{1}{2}(s_2-s_3),$$

$$R(s)=\left[\frac{\lambda(P^2,s,m_\pi^2)}{4s}\frac{s-t}{4}\right]^{1/2},$$

and

$$\lambda(x,y,z)=x^2+y^2+z^2-2(xy+yz+zx).$$

$I(s)$ is the explicit form of the integral over $d\Omega_{23}$, which we have changed to an integral over x. We have thus managed to convert the linear, multivariable, coupled integral equations of Khuri and Treiman into a single, nonlinear, one-variable integral equation. As the price for obtaining this simplification, we have some nonlinearity. At first sight, this may seem like a high price to pay, since Eq. (9) is far from easy to solve for $\delta(s)$, given an $f_0(s)$ and an $f_2(s)$. However, we shall see that there is much to be learned from it anyway. Besides, we do not really know $f_0(s)$ and $f_2(s)$ anyway.

III. CONSEQUENCES OF THE INTEGRAL EQUATION (9)

The first point to note is that, since $I(4m_\pi^2)$ will be, in general, a complex number with nonvanishing imaginary part, when we take the limit $s\to 4m_\pi^2$ in Eq. (9), we find we must have

$$a_2=\tfrac{2}{5}a_0,\quad (10)$$

where a_I is the s-wave $\pi\pi$ scattering length for isospin I. For, as $s\to 4m_\pi^2$, we have from Eq. (9) that $a_0+I(4m_\pi^2)(a_0-\frac{5}{2}a_2)=a_s$, where a_s is the scattering length for $\delta(s)$. Since all the a's are real, Eq. (10) follows. We also see that

$$a_0=a_s.\quad (11)$$

Equation (10) is the famous Chew-Mandelstam relation and seems to come mainly from crossing symmetry.[5] That we get it is not too surprising, since crossing symmetry is built into Eq. (1). So Eq. (9) passes its first test by yielding a (perhaps wrong but at least) familiar result.

We now discuss the kind of solutions to Eq. (9) for $\delta(s)$ which can explain the 3π decay spectrum for the K and η decays. Restricting ourselves to simple analytic forms, we immediately note the following two:

Case (a): The scattering-length approximation (discovered by Khuri and Treiman in their 1960 paper[3]).

Case (b): The pseudoresonance approximation (inspired by Brown and Singer's σ meson[3]). These two seem to exhaust the class of easily imaginable, simple functions for $\delta(s)$ which can explain the data.

Case (a). We assume $(k/E)\cot\delta(s)=-1/a_s$, where $k^2=\frac{1}{4}s-m_\pi^2$ and $E=\sqrt{(\frac{1}{4}s)}$; a_s is a dimensionless scattering "length," which is equivalent to the usual nonrelativistic scattering length measured in units of $m_\pi^{-1}\approx\sqrt{2}f$. We know that the decay data for both the K_2^0 and η decays are well described by $A(s_1,s_2,s_3)=1-0.2(2T_1-T_{\max})M/m_\pi^2$,[4] where M is the mass of the decaying particle, $s_1=(M-m_\pi)^2-2MT_1$, and $T_{\max}$ occurs when $s_1=4m_\pi^2$. If a_s is small, then

[5] G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

$\delta(s)\approx-(k/E)a_s$. Thus

$$\ln\zeta(s)\approx\frac{-a_s(s-s_0)}{\pi}\int_t^\infty\left(\frac{s'-t}{s'}\right)^{1/2}\frac{ds'}{(s'-s_0-i\epsilon)(s'-s-i\epsilon)}$$
$$=(-a_s)\left\{\frac{1}{\pi}\left(\frac{s-t}{s}\right)^{1/2}\ln\frac{1-[(s-t)/s]^{1/2}}{1+[(s-t)/s]^{1/2}}-\frac{1}{\pi}\left(\frac{s_0-t}{s_0}\right)^{1/2}\ln\frac{1-[(s_0-t)/s_0]^{1/2}}{1+[(s_0-t)/s_0]^{1/2}}+i\left[\left(\frac{s-t}{s}\right)^{1/2}-\left(\frac{s_0-t}{s_0}\right)^{1/2}\right]\right\}.$$

The imaginary part becomes just a phase in $\zeta(s)$ and does not concern us here. To get an idea of how this behaves, let us assume that $s-t\ll s$; we then find

$$\zeta(s)\approx1+\frac{2}{\pi}(-a_s)\left(-\frac{s-t}{s}+\frac{s_0-t}{s_0}\right)$$
$$\approx1+(2a_s/\pi s_0)(s-s_0).$$

Experiment tells us to take $a_s\approx+1$. This means that $a_0\approx+1$, i.e., that $\delta_0(s)<0$ for $s=t+\epsilon$, where $\epsilon\ll t$. And it can be seen that $\delta_0(s)$ not only starts out going negative, but with $\delta(s)=-k/E$, it will *continue* to go negative. This behavior for δ_0 seems to be ruled out by the K_{e4}-decay data, which strongly suggest $\delta_0>0$ in the low-energy region.[6] This seems to eliminate the scattering-length approximation.

Case (*b*). We assume here the following form for $\delta(s)$:

$$\cot\left(\frac{\delta(s)}{\Delta}\right)=\frac{m_R^2-s}{m_R\Gamma_R}\left[\frac{s}{s-t}\left(\frac{m_R^2-t}{m_R^2}\right)\right]^{1/2},\qquad(12)$$

where $\Delta\leq1$, m_R is the mass of the "resonance," and Γ_R is its width. For $\Delta=1$, the above is the normal relativistic Breit-Wigner resonance approximation. If we now take the small-width approximation $\Gamma_R\ll1$, we find, again neglecting over-all phase and normalization factors, that

$$\zeta(s)\approx1/(s-m_R^2-im_R\Gamma_R)^\Delta.$$

Examples of choices of Δ, m_R, and Γ_R which can accommodate the K-decay data are shown in Fig. 1. The $\Delta=1$ case, of course, is just the explanation offered by Brown and Singer[3]—a real resonance at a mass $\sim$400 MeV. However, the analysis shows here that all we really need is a step-function-like behavior: $\delta(s)\sim\alpha\theta(s-s_R)$, where α does *not* have to have the value π. Using the simple Watson form[7] of the final-state–interaction theorem $[A\sim e^{i\delta_0(s)}\sin\delta_0(s)]$, one would not be led to the form that we obtain here. In other words, Eq. (9), which includes three-body effects, gives us much more general expressions for the decay amplitude, expressions which may differ widely from what we would otherwise expect. At this point, it would be good to emphasize that we are not seriously proposing a cut on the second sheet (when $0<\Delta<1$) for $\zeta(s)$ as Eq. (13) might suggest. Exactly what singularity structure would cause the behavior for $\zeta(s)$ that we have obtained here is not clear; nevertheless, for s on the real axis, $\zeta(s)$ is approximately given by (13).

If now, using Eq. (12) for $\delta(s)$, we solve Eq. (9) for δ_0 and δ_2, which can be done easily and directly, we shall have some $\pi\pi$ phase shifts which can then be reused in Eq. (9) with $S=m_\eta^2+3m_\pi^2$ to predict the η-decay amplitudes. [The only change in Eq. (9) between the η and K decays is to let $P^2=m_\eta^2$ and $S=m_\eta^2+3m_\pi^2$ instead of $P^2=m_K^2$ and $S=m_K^2+3m_\pi^2$.] The $\pi\pi$ phase shifts extracted from Eq. (9) are shown in Fig. 2. In order to find an approximate solution to Eq. (9) for the η decay using the phase shifts of Fig. 2, we do the following: Parametrize $\delta(s)$ by Eq. (12), with m_R and Γ_R to be adjusted so that the left and right sides of Eq. (9) stay reasonably close to one another over the range $4m_\pi^2\leq s\lesssim16m_\pi^2$. We shall in this way generate an approximate solution to Eq. (9) for the η decay. We expect this procedure to be workable, since $m_\eta^2\approx m_K^2$. Having obtained m_R and Γ_R in this way, Eq. (13) yields the η-decay amplitude. Doing this, we find that the η-decay amplitudes using the phase shifts of Fig. 2 in Eq. (9) are in reasonable agreement with experiment. One can quickly convince oneself that this has to be so by remembering again that $m_\eta^2\approx m_K^2$, which means that Eq. (9) does not change much.

We have succeeded, therefore, in explaining the K- and η-decay data, with a positive s-wave $I=0$ $\pi\pi$ phase shift. But we still have problems, for the K_{e4}-decay data also show no structure as would be expected from $\delta_0(s)\approx\text{const}\times\theta(s-s_R)$. In fact, the K_{e4} data are almost consistent with straight phase space. So we must also eliminate this possibility of explaining the $K\to3\pi$ and $\eta\to3\pi$ decays.

IV. DISCUSSION OF RESULTS

These two forms for $\delta(s)$ seem to be the most general simple kinds (i.e., without many oscillations) that can explain the $K\to3\pi$ and $\eta\to3\pi$ decay data. Since they both contradict the $\delta_0(s)$ inferred from the K_{e4}-decay data, we are forced to the following conclusion: Provided that there are no problems hitherto unrecognized in the present interpretation of the results of the K_{e4} experiment, we must have, not one, but *two* subtractions in the Khuri-Treiman equations, or, equivalently, we must

[6] R. W. Birge *et al.*, Phys. Rev. 139, B1600 (1965); in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967).

[7] K. M. Watson, Phys. Rev. 88, 1163 (1952).

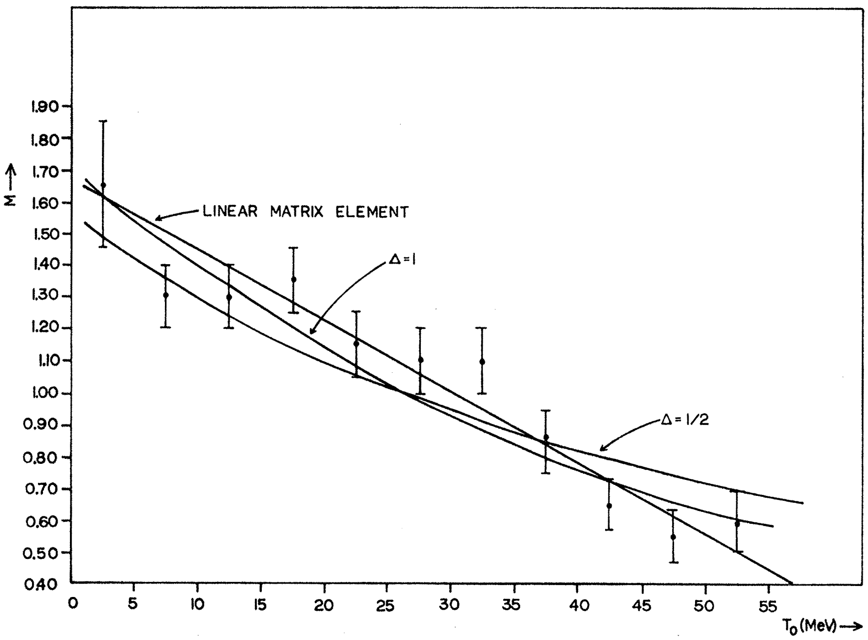

FIG. 1. Predicted K-decay spectrum compared with the experimental values of Ref. 9, and also the linear matrix element which describes the spectrum well.

take

$$\zeta(s)=\exp\left[\beta(s-s_0)+\left(\frac{s-s_0}{\pi}\right)^2 \times\int_t^\infty \frac{\delta(s')ds'}{(s'-s_0-i\epsilon)^2(s'-s-i\epsilon)}\right]. \quad (13)$$

Clearly, with the parameter β we can easily explain the K and $\eta \to 3\pi$ data; as has been suggested before,[8] only quadratic effects $\propto (s-s_0)^2$ on the 3π Dalitz plot will show the influence of the $\pi\pi$ final-state interactions.

V. IMPLICATIONS OF THE TWO-SUBTRACTION HYPOTHESIS

First of all, we note that when the parameter $\beta=0.2\text{m}_\pi^{-2}$, Eq. (13) yields

$$\zeta(s)\approx 1+0.2(s-s_0)/m_\pi^2 \approx 1-0.4(T-\tfrac{1}{2}T_{\max})M/m_\pi^{-2}, \quad (14)$$

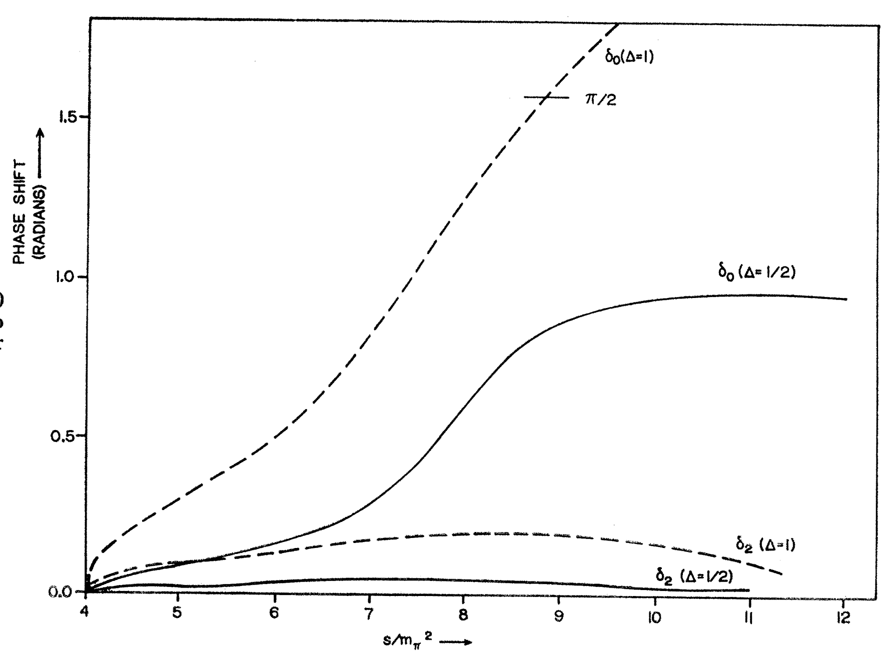

FIG. 2. δ_0 and δ_2, the s-wave $I=0$ and $I=2$ $\pi\pi$ phase shifts for the two cases $\Delta=\frac{1}{2}$ and $\Delta=1$. See text for discussion.

[8] G. Barton and C. Kacser, Phys. Rev. Letters 8, 226 (1962).

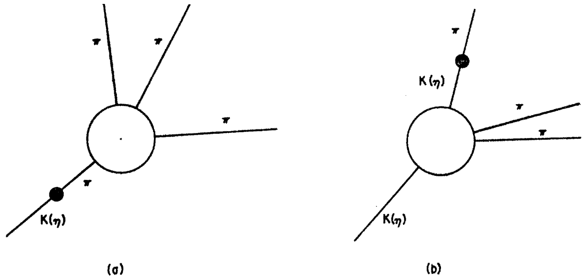

FIG. 3. Pion-pole model for 3π decays.

so that we obtain the usual experimental fits to both the K- and η-decay data.[9] Thus β is a constant independent of *whether the decay is weak or electromagnetic.* This is indeed a striking result. In the current-algebra calculations,[1] the same thing occurs, although not without some effort and extra, almost *ad hoc* assumptions for the η decay. It would be fair to say that current algebra does not immediately supply a strong reason to expect $\beta=$const.

Why, then, is this so? One possible explanation is shown in Fig. 3(a), the pion-pole model.[10] One assumes that the $K\rightarrow\pi$ and $\eta\rightarrow\pi$ transitions are the dominant weak and electromagnetic processes. Figure 3(a) assures $\beta_\eta=\beta_K$. On the other hand, why should we not include Fig. 3(b)? And Fig. 3(b) does *not* yield $\beta_\eta=\beta_K$. Furthermore, note that $SU(3)$ symmetry predicts that the $\eta\rightarrow 3\pi$ amplitude given by Figs. 3(a) and 3(b) is identically zero.[11] What all this means is not immediately clear.

There are no doubt other possible explanations for the fact that $\beta_\eta=\beta_K$, although the author has not managed to discover any simple ones. So, at this point, we shall leave the explanation of this phenomenon as a very suggestive but open question.

VI. CONCLUSIONS

On very general grounds, we have discussed the K and $\eta\rightarrow 3\pi$ decays. An accounting of three-body effects was included in the dispersion analysis made, and the following results were obtained:

(1) $a_0=\frac{5}{2}a_2$, which is the famous Chew-Mandelstam s-wave $I=0,2$ scattering-length relation.

(2) *Two* subtractions are needed in the dispersion equations. This indicates increasing high-energy behavior of the discontinuity.

(3) The subtraction constants are equal for both the K and η decays. This is particularly striking, since the decay interactions are not the same. No convincing explanation of this phenomenon seems to be available.

(4) $\pi\pi$ final-state–interaction effects can first be observed only in quadratic terms on the Dalitz plot. That this might be so has been emphasized before.[8]

ACKNOWLEDGMENTS

The author would like to express his thanks to E. C. G. Sudarshan, J. M. Schechter, and N. Papastamatiou for useful discussions; to T. Kalogeropoulos for communication of the results on the η decay presented at the Heidelberg conference; to C. Baltay for a discussion of his results on the η-decay branching ratios; to S. H. Patil, whose interest and work with the author on the now nonexistent η-decay branching-ratio difficulty was a great stimulus to this work; and particularly to Nina Byers, whose emphasis on the importance of three-body effects led directly to the investigation presented here.

[9] See, for example, B. M. K. Nefkens *et al.*, Phys. Rev. **157**, 1233 (1967), where the linear fit given by Eq. (14) is compared with both the K and η decays.

[10] G. Barton and S. P. Rosen, Phys. Rev. Letters **8**, 414 (1962); C. Kacser, Phys. Rev. **130**, 355 (1963).

[11] I am indebted to J. M. Schechter for informing me of this remark due to S. Okubo.