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# Spectral-Function Sum Rules with Symmetry Breaking*

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The symmetry breaking based on Okubo's ansatz is considered to modify Weinberg's spectral-function sum rules (particularly the second). We then apply the modified sum rules to the chiral $U(3)\times U(3)$ and obtain physically interesting results for the mass relation and mixing angles of the vector mesons, a nonet assignment of the axial-vector mesons, and the decay constants of the pseudoscalar mesons.

Since Weinberg's successful derivation[1] of the mass relation between $\rho$ and $A_1$ mesons from the spectral-function sum rules of the chiral $SU(2)\otimes SU(2)$ symmetry, the sum-rule approach has been extended to the higher symmetries by several authors.[2] It has been recognized, however, for some time that[3,4] the first and the second sum rules of Weinberg together as applied to the higher symmetries give degenerate masses for the particles and that the original form of the second sum rule should be modified somehow by taking into account the symmetry-breaking effects.

In the present paper, we modify the second sum rule by introducing a scheme of symmetry-breaking suggested by Okubo[5] and apply this modification together with the first sum rule for the chiral $U(3)\otimes U(3)$ symmetry to obtain some physically interesting results for the masses or the decay constants of mesons.

We begin by defining the propagator functions

$$\Delta_{\mu\nu}{}^{\alpha\beta}(q)=\int d^4x\, e^{-iq\cdot x}\langle 0|T(J_\mu{}^\alpha(x)J_\nu{}^\beta(0))|0\rangle$$

$$=-i\int_0^\infty dm^2(q^2+m^2)^{-1}$$

$$\times\left[\left(g_{\mu\nu}+\frac{q_\mu q_\nu}{m^2}\right)\rho_{\alpha\beta}{}^{(1)}(m^2)+q_\mu q_\nu\rho_{\alpha\beta}{}^{(0)}(m^2)\right]$$

$$-ig_{\mu0}g_{\nu0}\int_0^\infty dm^2[\rho_{\alpha\beta}{}^{(1)}(m^2)/m^2+\rho_{\alpha\beta}{}^{(0)}(m^2)],\quad(1)$$

where $J_\mu{}^\alpha(x)$ are vector or axial-vector currents normalized to $\bar{q}\gamma_\mu\frac{1}{2}\lambda_\alpha q$ or to $\bar{q}i\gamma_5\gamma_\mu\frac{1}{2}\lambda_\alpha q$ in terms of the quark field $q$ and $\alpha,\beta=0, 1, 2,\cdots, 8$. From invariance under the $CPT$ operation, one can show that[6] the spectral functions $\rho_{\alpha\beta}{}^{(1)}(m^2)$ and $\rho_{\alpha\beta}{}^{(0)}(m^2)$ are symmetric in the indices $\alpha$ and $\beta$. The first and second sum rules of Weinberg contain the following integrals, respectively:

$$J_{\alpha\beta}\equiv\int_0^\infty dm^2[\rho_{\alpha\beta}{}^{(1)}(m^2)/m^2+\rho_{\alpha\beta}{}^{(0)}(m^2)],\quad(2)$$

$$K_{\alpha\beta}\equiv\int_0^\infty dm^2\rho_{\alpha\beta}{}^{(1)}(m^2).\quad(3)$$

To get a simple $\alpha$, $\beta$ dependence of these integrals in the presence of the symmetry breaking, we would like to adopt the Okubo ansatz that has been used to explain the mass relation among the vector mesons.[5] Just as in the case of the Gell-Mann–Okubo mass formula, we assume that $SU(3)$ symmetry is broken by the eighth components of octet currents. Considering the symmetry breaking due to the interaction which behaves as the $T_3{}^3$ component of a second-rank tensor $T_i{}^j$ to the first-order effect, the integrals of (2) and (3) can be expressed by a linear combination of

$$\mathrm{Tr}\lambda_\alpha\lambda_\beta,\ \mathrm{Tr}\lambda_\alpha\ \mathrm{Tr}\lambda_\beta,\ \mathrm{Tr}\{\lambda_\alpha,\lambda_\beta\}\lambda_8,$$

and

$$\mathrm{Tr}\lambda_\alpha\ \mathrm{Tr}\lambda_\beta\lambda_8+\mathrm{Tr}\lambda_\beta\ \mathrm{Tr}\lambda_\alpha\lambda_8,$$

with the coefficients independent of $\alpha$ and $\beta$. Then the Okubo ansatz corresponds to excluding the terms proportional to $\mathrm{Tr}\lambda_\alpha$ and/or $\mathrm{Tr}\lambda_\beta$, so that we can get

$$J_{\alpha\beta}=a\delta_{\alpha\beta}+bd_{\alpha\beta8},\quad(4)$$

$$K_{\alpha\beta}=a'\delta_{\alpha\beta}+b'd_{\alpha\beta8},\quad(5)$$

where $d_{\alpha\beta\gamma}$ is the familair symmetric symbol.[7] If the vacuum expectation value of the Schwinger terms is not affected by the symmetry breaking, which is true in the case of field algebra of Lee, Weinberg, and

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[1] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).
[2] See, e.g., S. Okubo, in *Proceedings of 1967 International Conference on Particles and Fields* (Interscience Publishers, Inc., New York, 1967) for further references in addition to those we cite below.
[3] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **19**, 470 (1967).
[4] J. J. Sakurai, Phys. Rev. Letters **19**, 803 (1967).
[5] S. Okubo, Phys. Letters **5**, 165 (1963); also G. L. Glashow, in *High-Energy Physics, Tokyo Summer Lectures in Theoretical Physics, 1965* (W. A. Benjamin, Inc., New York, 1966).
[6] S. Coleman and H. J. Schnitzer, Phys. Rev. **134**, B863 (1964).
[7] Generally, $a$, $b$, $a'$, and $b'$ may be different for vector and axial-vector currents. If we assume that the symmetry is broken by a term $\{\lambda_\alpha,\lambda_\beta\}_{33}$, we can also arrive at an equivalent form of (4) and (5), i.e., $J_{\alpha\beta}=a\delta_{\alpha\beta}+c(\delta_{\alpha\beta}-\sqrt{3}d_{\alpha\beta8})$ and a similar relation for $K_{\alpha\beta}$. Note that $a$ and $a'$ are universal for vector and axial-vector currents of the $SU(2)\times SU(2)$. This may be the reason for the remarkable agreement of Weinberg's mass relation $m_{A_1}{}^2=2m_\rho{}^2$. This alternative form of $J_{\alpha\beta}$ and $K_{\alpha\beta}$ comes from a phenomenological Lagrangian model in which the symmetry is broken by the (3,3) component of a scalar nonet spurion (vacuon), but we shall present this elsewhere.

Zumino,[8] then we may set $b=0$. It follows from (4) and (5) that

$$J_I=J_{\Delta S}=J_{88}=J_{00}, \quad J_{08}=0 \tag{6}$$

$$K_I-K_{\Delta S}=(\tfrac{3}{8}\sqrt{8})K_{08}, \quad K_I+2K_{\Delta S}=3K_{00}, \quad 4K_{\Delta S}-K_I=3K_{88}, \tag{7}$$

where the subscripts $I$ and $\Delta S$ denote $\alpha=\beta=1, 2, 3$ and $\alpha=\beta=4, 5, 6, 7$, respectively. We mention that the original form of Weinberg's second sum rule is replaced by the three sum rules of (7) due to the symmetry breaking.

To see the symmetry-breaking effect to the masses and coupling constants, let us first consider (6) and (7) applied to the nine vector currents, i.e., the $SU(3)$ currents plus baryon current. It is customary to introduce the hypercharge and baryon currents by $Y_\mu=\frac{2}{3}\sqrt{3}\times J_\mu{}^{V,8}$ and $B_\mu=\frac{2}{3}{}^{1/2}J_\mu{}^{V,0}$, respectively. Then by saturating (6) and (7) by the known vector mesons and by inserting the results of (6) into (7), one obtains

$$\tfrac{1}{3}(m_\rho{}^2-m_{K^*}{}^2)=(\tfrac{1}{8}\sqrt{8})(m_\phi{}^2-m_\omega{}^2)\sin\theta_V\cos\theta_V, \tag{8a}$$

$$\tfrac{1}{3}(m_\rho{}^2+2m_{K^*}{}^2)=m_\phi{}^2\sin^2\theta_V+m_\omega{}^2\cos^2\theta_V, \tag{8b}$$

$$\tfrac{1}{3}(4m_{K^*}{}^2-m_\rho{}^2)=m_\phi{}^2\cos^2\theta_V+m_\omega{}^2\sin^2\theta_V, \tag{8c}$$

where $\theta_V$ is related to the two $\omega$-$\phi$ mixing angles $\theta_Y$ and $\theta_B=\theta_N$ of Kroll, Lee, and Zumino[9] by

$$\tan\theta_V\equiv(m_\omega/m_\phi)\tan\theta_Y=(m_\phi/m_\omega)\tan\theta_B \tag{9}$$

and the covariant matrix elements of the vector currents are defined as those of Oakes and Sakurai.[10,11] Excluding a solution that corresponds to degenerate masses, i.e., $b'=0$ in (5), we get from (8) that

$$m_\rho{}^2=m_\omega{}^2, \quad m_\omega{}^2+m_\phi{}^2=2m_{K^*}{}^2, \tag{10a}$$

$$\tan\theta_V=(m_\phi/m_\omega)\tan\theta_B=(m_\omega/m_\phi)\tan\theta_Y=-\tfrac{1}{2}\sqrt{2}, \tag{10b}$$

and therefore it follows from (6) that

$$\frac{m_\rho{}^2}{f_\rho{}^2}=\frac{m_{K^*}{}^2}{f_{K^*}{}^2}=\frac{9m_\phi{}^2}{8f_Y{}^2}\left(1+\frac{1}{2}\frac{m_\phi{}^2}{m_\omega{}^2}\right)^{-1}=\frac{9m_\omega{}^2}{4f_B{}^2}\left(1+\frac{1}{2}\frac{m_\omega{}^2}{m_\phi{}^2}\right)^{-1}. \tag{11}$$

Note that (10a) is the well-known mass formula of $SU(6)$ theory,[12] and that (11) combined with the relation[13] $f_\rho{}^2=m_\rho{}^2/2F_\pi{}^2$ reproduces the $VPP$ coupling constant of current algebra.[14] The mixing angles of (10b)

$$\theta_V=-35.3°, \quad \theta_Y=-42.6°, \quad \theta_B=-28.5° \tag{12}$$

are consistent with those of Das, Mathur, and Okubo,[3] who used the sum rules (6) and the third relation of (7). We note that also (10b) and (11) give the relation

$$m_\omega\Gamma(\omega\to l^+l^-)=\tfrac{1}{2}m_\phi\Gamma(\phi\to l^+l^-)=\tfrac{1}{9}m_\rho\Gamma(\rho\to l^+l^-) \tag{13}$$

for the lepton pair decays of neutral vector mesons. This relation was derived by Bose and Torgerson[15] before by assuming $\langle 0|\frac{1}{2}Y_\mu+B_\mu|\phi\rangle=\langle 0|-Y_\mu+B_\mu|\omega\rangle=0$, which can be shown to be equivalent to (6) with (10b).

Next, let us turn to the nine axial-vector currents. Although the vector and pseudoscalar nonets are fairly well established, the situation of the corresponding nonet of axial-vector mesons is not very clear. In our discussion, we assume that[16] $A_1$(1070 MeV), $K_A$(1320 MeV), $E$(1420 MeV), and $D$(1285 MeV) form a nonet analogous to the vector nonet $\rho$, $K^*$, $\phi$, and $\omega$. While we introduce again two mixing angles $\theta_8$ and $\theta_0$ for axial-vector mesons, we use only one mixing angle $\theta_P$ for the $\eta$-$X$ mixing of the pseudoscalar nonet, for simplicity. This mixing effect is very small from experiments. By saturating (6) and (7) again by the axial-vector and pseudoscalar mesons and by inserting the results of (6) into (7), we get[17]

$$\tfrac{1}{3}(m_{A_1}{}^2p_\pi-m_{K_A}{}^2p_K)=(\tfrac{1}{8}p_0p_8)^{1/2}(m_E{}^2-m_D{}^2)\times\cos\theta_A\sin\theta_A, \tag{14a}$$

$$\tfrac{1}{3}(m_{A_1}{}^2p_\pi+2m_{K_A}{}^2p_K)=p_0(m_E{}^2\sin^2\theta_A+m_D{}^2\cos^2\theta_A), \tag{14b}$$

$$\tfrac{1}{3}(4m_{K_A}{}^2p_K-m_{A_1}{}^2p_\pi)=p_8(m_E{}^2\cos^2\theta_A+m_D{}^2\sin^2\theta_A), \tag{14c}$$

where

$$\tan\theta_A=(m_E/m_D)\tan\theta_0=(m_D/m_E)\tan\theta_8, \tag{15}$$

$$p_i=m_\rho{}^2/f_\rho{}^2-F_i{}^2, \quad i=\pi, K, 8, 0. \tag{16}$$

We assume that the masses of the axial-vector mesons as well as the relation $f_\rho{}^2=m_\rho{}^2/2F_\pi{}^2$ are given, say, from experiments and try to calculate the ratios of the decay constants of the pseudoscalar mesons from (14). For this purpose, it is convenient to introduce a small parameter $\epsilon$ and an angle $\psi$ which is very close to $\frac{1}{4}\pi$ by

$$\epsilon\equiv(m_E{}^2-m_D{}^2)/(m_E{}^2+m_D{}^2) \text{ and } \tan\psi\equiv(p_0/p_8)^{1/2}.$$

[8] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).

[9] N. M. Kroll, T. D. Lee, and B. Zumino, Phys. Rev. **157**, 1376 (1967).

[10] R. J. Oakes and J. J. Sakurai, Phys. Rev. Letters **19**, 1266 (1967).

[11] Notice that (8) is independent of the coupling constants defined in the covariant matrix elements of currents.

[12] B. Sakita, Phys. Rev. **136**, B1756 (1964); also J. Schwinger, in *Lectures on Particles and Field Theory, Summer School Proceedings, Brandeis University, 1964* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1965), Vol. II. Had we included the $\kappa$ meson, (10a) would have changed to $m_\omega{}^2=m_\phi{}^2+2m_{K^*}{}^2(1-F_\kappa{}^2/2F_\pi{}^2)$. $F_\kappa{}^2$ is the measure of the deviation from the Gell-Mann–Okubo mass formula $4m_{K^*}{}^2-m_\rho{}^2-3m_{V8}{}^2=0$, where $m_{V8}{}^2=\frac{1}{3}(2m_\phi{}^2+m_\omega{}^2)$.

[13] This is known as the KSRF relation: K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966).

[14] T. Akiba and K. Kang, Phys. Rev. **160**, 1283 (1967); T. Akiba, Nuovo Cimento **55A**, 170 (1968).

[15] S. K. Bose and R. Torgerson, Phys. Rev. Letters **19**, 1151 (1967).

[16] The masses of the axial-vector mesons are taken from A. H. Ronsenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968).

[17] In deriving (14), we have used $J_I=m_\rho{}^2/f_\rho{}^2$ and $\int_0^\infty dm^2\times\rho_I{}^{(0)}(m^2)=F_\pi{}^2$, etc.

Then, after some elementary manipulation, one can easily show from (14) that

$$p_0/p_8=(1+\epsilon x)/(1-\epsilon x)\,, \tag{17a}$$

$$(F_K/F_\pi)^2=2-\frac{m_{A_1}{}^2}{m_{K_A}{}^2}\times\frac{1-\epsilon^2 x\cos2\theta_A}{1+3\epsilon(x-\cos2\theta_A-\frac{1}{3}\epsilon x\cos2\theta_A)}\,, \tag{17b}$$

$$\left(\frac{F_8}{F_\pi}\right)^2+\left(\frac{F_0}{F_\pi}\right)^2=4-\frac{4m_{A_1}{}^2}{m_E{}^2+m_D{}^2}\times[1+3\epsilon(x-\cos2\theta_A-\tfrac{1}{3}\epsilon x\cos2\theta_A)]^{-1}\,, \tag{17c}$$

where

$$x=\cos2\theta_A+(\tfrac{1}{8}\sqrt{8})\sin2\theta_A\sin2\psi=-(\cos2\psi)/\epsilon\,. \tag{18}$$

From (17), we note that the case $\epsilon=0$ gives $F_K/F_\pi=1.16$, which was derived by Nieh[18] from Weinberg's sum rules for the chiral $SU(3)\times SU(3)$ without, however, taking into account the symmetry breaking. Although one cannot determine $\theta_A$ uniquely from our discussion, the $E$-$D$ mixing is presumably very similar to the $\phi$-$\omega$ mixing so that $x$ is small. In particular, $x=0$ gives $\theta_A=\theta_V$ and $p_0=p_8$. Note that if $p_0\approx p_8$, then the deviation from the $SU(6)$ mass relation for the axial-vector nonet can be shown to be due to the deviation of $p_K/p_8$ and $p_\pi/p_8$ from unity.

The fact that $\epsilon$ is very small, i.e., $\epsilon\simeq0.1$ from experiment, allows us to determine the decay constants fairly rigorously, without assuming any particular value for $x$ or $\theta_A$. We find from (17) in general that

$$F_K/F_\pi=1.12\text{–}1.20\,,\quad (F_8/F_\pi)^2+(F_0/F_\pi)^2=2.60\text{–}2.88\,,\quad p_0/p_8=0.79\text{–}1.24\,. \tag{19}$$

We remark that the decay constants are determined in (19) to within 10% accuracy and therefore are not sensitive to $x$. In particular, $F_K/F_\pi$ is affected only less than 5% in the case of $\epsilon=0$ by the symmetry breaking, in good agreement with experiment $F_K/F_\pi=1.20$.[19] If $\tan2\theta_A=-\sqrt{8}$ is used in (17), we get $F_K/F_\pi=1.13$ and $F_8/F_\pi=F_0/F_\pi=1.15$.[20] Finally, we note from (14) that the deviation from the Gell-Mann–Okubo mass formula for the axial-vector mesons is given by[21]

$$4m_{K_A}{}^2-m_{A_1}{}^2-3m_8{}^2=(4m_{K_A}{}^2F_K{}^2-m_{A_1}{}^2F_\pi{}^2-3m_8{}^2F_8{}^2)/2F_\pi{}^2\,. \tag{20}$$

To conclude, our second sum rules (7) modified by the Okubo ansatz of the symmetry breaking, together with the first sum rule (6) give the mass relations and mixing angles of the $SU(6)$ theory for the vector mesons, while they yield fairly reliable estimates of the decay constants for the pseudoscalar mesons.

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[18] H. T. Nieh, Phys. Rev. Letters 19, 43 (1967); also, G. L. Glashow, H. J. Schnitzer, and S. Weinberg, *ibid.* 19, 139 (1967).

[19] L. B. Auerbach *et al.*, Phys. Rev. 155, 1505 (1967). This number is based on the universality of the bare Cabibbo angle for vector and axial-vector currents and on neglecting the second-order $SU(3)$ breaking effect in the $K_{e3}$ form factor $f^+(0)$.

[20] Had we used $m_{K_A}=1230$ MeV instead of 1320 MeV, we would have obtained $F_K/F_\pi=1.08$ and $F_8/F_\pi=F_0/F_\pi=1.15$.

[21] Note that to first order in the $SU(3)$-symmetry breaking, one obtains from (20) the relation $4F_K{}^2-F_\pi{}^2-3F_8{}^2=0$, which has been obtained before by K. Kawarabayashi and W. W. Wada, Phys. Rev. Letters 19, 1193 (1967). If instead of $E$ and $D$ mesons a pure eighth component of octet and unitarity singlet axial-vector mesons were used, we would obtain from (14) $m_{A_1}{}^2p_\pi=m_{K_A}{}^2p_K=m_8{}^2p_8=m_0{}^2p_0$, which implies that if the masses are degenerate, then $F_\pi{}^2=F_K{}^2=F_8{}^2=F_0{}^2$.