Operating on (3.1) with

$$(-i/m)^{2s}T^{\mu_1\mu_2\cdots\mu_{2s}}\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{2s}}, \tag{3.8}$$

we get

$$\sum_{\sigma'}\left(\frac{-i}{m}\right)^{2s}T_{\sigma\sigma'}{}^{\mu_1\mu_2\cdots\mu_{2s}}\partial_{\mu_1\mu_2}\cdots\partial_{\mu_{2s}}\phi_{\sigma'}(x)$$

$$=(2\pi)^{-3/2}\int\frac{d^3p}{[2\omega(\mathbf{p})]^{1/2}}\sum_{\sigma'\sigma''}$$

$$\times\{[M_+(\mathbf{p})]_{\sigma\sigma'}D_{\sigma'\sigma''}{}^{(s,0)}((\mathbf{p}))a(\mathbf{p},\sigma'')e^{ip\cdot x}+[M_-(-\mathbf{p})]_{\sigma\sigma'}$$

$$\times[D^{(s,0)}(L(\mathbf{p}))C^{-1}]_{\sigma'\sigma''}a^*(\mathbf{p},\sigma'')e^{-ip\cdot x}\}, \tag{3.9}$$

where relation (2.9) has been used. Using (2.8a) and (2.8b), it simplifies to

$$\sum_{\sigma'}\left(\frac{-i}{m}\right)^{2s}T_{\sigma\sigma'}{}^{\mu_1\mu_2\cdots\mu_{2s}}\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{2s}}\phi_{\sigma'}(x)$$

$$=(2\pi)^{-3/2}\int\frac{d^3p}{[2\omega(\mathbf{p})]^{1/2}}\sum_{\sigma'}\{D_{\sigma\sigma'}{}^{(s,0)}(L(-\mathbf{p}))a(\mathbf{p},\sigma')e^{ip\cdot x}$$

$$+(-1)^{2s}[D^{(s,0)}(L(-\mathbf{p}))C^{-1}]_{\sigma\sigma'}a^*(\mathbf{p},\sigma')e^{-ip\cdot x}\}. \tag{3.10}$$

Since the right-hand side of (3.10) is identical with (3.7), then finally we get

$$(-i)^{2s}T^{\mu_1\mu_2\cdots\mu_{2s}}\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{2s}}\phi(x)=(m)^{2s}\phi, \tag{3.11}$$

where the spin indices σ,σ' have been suppressed. Thus (3.11) is the q-number version of (21.2a), which is what we set out to prove.

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Canonical Representation of Sugawara's Theory*

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We present canonical Lagrangian representations of Sugawara's theory of currents. Although there are some important differences, these Lagrangians resemble currently discussed phenomenological Lagrangians.

I. INTRODUCTION

Recently, Sugawara[1] put forth a nontrivial theory of currents, which has subsequently been extended[2] to include electromagnetism, PCAC (partially conserved axial-vector current), and $SU(3)$ breaking. As written, the theory has no manifest Lagrangian or canonical structure. Our purpose in this paper is to show that canonical Lagrangian representations of the theory do exist, although they are complicated. An important lesson one learns is that, although many field theories can presumably be rewritten as unesthetic current theories[3] (involving inverse densities, etc.), some very nonlinear field theories can be rewritten as beautiful current theories. This may be a reason to begin taking these particular field theories seriously.

The plan of the paper is as follows: In Sec. II, we recall Sugawara's theory and specialize to the case of $SU(2)$. Then we discuss the equal-time representation problem and solve the fixed-time constraint equation. The Appendix is devoted to a detailed discussion of this equation. In Sec. III, we write a Lagrangian representation of the $SU(2)$ theory. This Lagrangian is essentially a free Lagrangian for a Lorentz-scalar complex isospinor field u, but with the constraint $u^\dagger u=1$. The constraint can be removed, and the Lagrangian in terms of unconstrained fields is essentially the σ model[4] of Gell-Mann and Lévy, but with an important difference: The unconstrained fields do not have definite isospin. In this representation, this prevents us from being able to define an approximation procedure (such as perturbation theory) which conserves isospin at each order. Another curious feature of the representation is the presence of an additional $O(4)$ symmetry (and three new isoscalar conserved quantities), which is (as yet) unobserved in nature. Although we have not been able to prove that all representations of the $SU(2)$ theory must have this symmetry, we have not yet been able to find any that do not. The same turns out to be true at the level of $SU(2)\otimes SU(2)$, as discussed in Sec. IV. With the inclusion of PCAC, however, we have the opportunity to break the symmetry if we choose. At the end of Sec. IV, we note a representation with unconstrained fields of definite isospin. Unfortunately, such representations all appear to have singular interaction terms (inverse fields), and, again, no perturbation expansion is possible.

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II. HAMILTONIAN FORMULATION

Sugawara's theory is defined by the stress-energy-momentum tensor[5]

$$\theta_{\mu\nu}=(1/2C)\{[V_\mu{}^\alpha,V_\nu{}^\alpha]_+-g_{\mu\nu}(V_\lambda{}^\alpha V_\alpha{}^\lambda)+[A_\mu{}^\alpha,A_\nu{}^\alpha]_+-g_{\mu\nu}(A_\lambda{}^\alpha A_\alpha{}^\lambda)\} \quad (2.1)$$

together with the equal-time algebra among the (vector and axial-vector) currents

$$\begin{aligned}
[V_0{}^\alpha(\mathbf{x}),V_0{}^\beta(\mathbf{y})]&=[A_0{}^\alpha(\mathbf{x}),A_0{}^\beta(\mathbf{y})]\\
&=if_{\alpha\beta\gamma}V_0{}^\gamma(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\
[V_0{}^\alpha(\mathbf{x}),V_i{}^\beta(\mathbf{y})]&=[A_0{}^\alpha(\mathbf{x}),A_i{}^\beta(\mathbf{y})]\\
&=if_{\alpha\beta\gamma}V_i{}^\gamma(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})\\
&\quad+iC\delta_{\alpha\beta}\partial_i{}^{(x)}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\
[V_0{}^\alpha(\mathbf{x}),A_i{}^\beta(\mathbf{y})]&=[A_0{}^\alpha(\mathbf{x}),V_i{}^\beta(\mathbf{y})]\\
&=if_{\alpha\beta\gamma}A_i{}^\gamma(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\
[V_i{}^\alpha,V_j{}^\beta]&=[V_i{}^\alpha,V_j{}^\beta]=[A_i{}^\alpha,A_j{}^\beta]=0,
\end{aligned} \quad (2.2)$$

where α runs from 1 to 8 and $f_{\alpha\beta\gamma}$ are the structure constants of $SU(3)$. The generators of the Poincaré group are defined as

$$\begin{aligned}
M_{\mu\nu}&=\int(x_\mu\theta_{\nu0}-x_\nu\theta_{\mu0})d\mathbf{x},\\
P_\mu&=\int\theta_{0\mu}d\mathbf{x},\\
i[P_\mu,J_\nu{}^\alpha]&=\partial_\mu J_\nu{}^\alpha,
\end{aligned} \quad (2.3)$$

where $J_\nu{}^\alpha$ means either vector or axial vector. The resulting "equations of motion" are

$$\begin{aligned}
&\partial_\mu V_\alpha{}^\mu=\partial_\mu A_\alpha{}^\mu=0,\\
&\partial_\mu V_\nu{}^\alpha-\partial_\nu V_\mu{}^\alpha=(1/2C)f^{\alpha\beta\gamma}\times\{[V_\mu{}^\beta,V_\nu{}^\gamma]_++[A_\mu{}^\beta,A_\nu{}^\gamma]_+\},\\
&\partial_\mu A_\nu{}^\alpha-\partial_\nu A_\mu{}^\alpha=(1/2C)f^{\alpha\beta\gamma}\times\{[V_\mu{}^\beta,A_\nu{}^\gamma]_++[A_\mu{}^\beta,V_\nu{}^\gamma]_+\}.
\end{aligned} \quad (2.4)$$

In this section, we shall focus our attention on the corresponding $SU(2)$ version of this theory, that is, the theory defined by

$$\theta_{\mu\nu}=(1/2C)\{[V_\mu{}^\alpha,V_\nu{}^\alpha]_+-g_{\mu\nu}(V_\lambda{}^\alpha V_\alpha{}^\lambda)\}, \quad (2.5a)$$

$$\partial_\mu V_\nu{}^\alpha-\partial_\nu V_\mu{}^\alpha=(1/2C)\epsilon^{\alpha\beta\gamma}[V_\mu{}^\beta,V_\nu{}^\gamma]_+, \quad (2.5b)$$

plus the relevant commutators from the list above

$$\begin{aligned}
[V_0{}^\alpha(\mathbf{x}),V_0{}^\beta(\mathbf{y})]&=i\epsilon_{\alpha\beta\gamma}V_0{}^\gamma(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\
[V_0{}^\alpha(\mathbf{x}),V_i{}^\beta(\mathbf{y})]&=i\epsilon_{\alpha\beta\gamma}V_i{}^\gamma(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})\\
&\quad+iC\delta_{\alpha\beta}\partial_i{}^x\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\
[V_i{}^\alpha(\mathbf{x}),V_j{}^\beta(\mathbf{y})]&=0,
\end{aligned} \quad (2.6)$$

where now α, β, and γ run from 1 to 3.

The problem of finding a fixed-time representation for this theory is more complicated than for ordinary theories because, not only must we represent the commutators Eq. (2.6), but also the fixed-time constraint equation

$$\partial_i V_j{}^\alpha-\partial_j V_i{}^\alpha=(1/C)\epsilon^{\alpha\beta\gamma}V_i{}^\beta V_j{}^\gamma, \quad (2.7)$$

where i and j range from 1 to 3, must be solved. Actually, the solution of this equation is not difficult. Note first that it is a C-number equation (although nonlinear) because the spatial currents commute. In terms of the 2×2 matrices $V_i=\tau^\alpha V_i{}^\alpha$ (τ^α are the Pauli matrices), the equation takes the form

$$\partial_i V_j-\partial_j V_i=(1/2iC)[V_i,V_j]. \quad (2.8)$$

Now it is easy to see that if V_i solves this equation then so does

$$V_i'=S^{-1}V_iS-2iCS^{-1}\partial_iS, \quad (2.9)$$

where S is any 2×2 nonsingular matrix—although for the current to be Hermitian, S must be unitary. This feature occurs because the equations of motion (2.5b) admit a gauge invariance of the second kind.[6] In particular,

$$V_i=-2iCS^{-1}\partial_iS \quad (2.10)$$

itself solves the equation. In the (massless) Yang-Mills theory[7] this "longitudinal part" is trivial, but as we shall see, such is not the case here. In the Appendix, we show in fact that Eq. (2.10) is the most general solution to (2.8), and we shall use it in what follows.

An approach to equal-time representation could now be phrased as follows: Having a space of solutions for V_i as functions of the unconstrained S, one can define variational derivatives with respect to S, say π, and try to find $V_0{}^\alpha$ as a function of S, π such that the additional commutators are satisfied. In fact this can be done [see, e.g., Eq. (3.27) below], but simply to state the result here would be unenlightening. We prefer to motivate the representation via the Lagrangian approach of Sec. III. One thing should be said here though. We have the most general representation for the spatial currents and particular representations for $V_0{}^\alpha$ (as given below), but what about more general $V_0{}^\alpha$, perhaps not corresponding to a Lagrangian at all? We have no general answer to this question, but one thing seems clear: Suppose one is given (say, from our Lagrangian) $V_i{}^\alpha$, $V_0{}^\alpha$ and one tries to modify $V_0{}^\alpha$ to, say, $V_0{}^\alpha+f^\alpha$, still satisfying the algebra. Under the assumption that anything which commutes with the spatial currents is a function of the spatial currents, one can show that $f^\alpha=0$.

III. LAGRANGIAN FORMULATION [$SU(2)$]

The appropriate Lagrangian is best motivated in terms of the limit procedure of Ref. 2. There the for-

[5] Our metric is $-1=-g_{00}=g_{11}=g_{22}=g_{33}$, $A_\mu B^\mu=A_0B_0-\mathbf{A}\cdot\mathbf{B}$. Four-dimensional indices are $\mu\nu$. Spatial indices are $i=1,2,3$. Internal symmetry labels are taken as α, β, γ.

[6] Despite this, the commutators do not exhibit a gauge invariance of the second kind; thus the theory as a whole does not have it.

[7] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

mal scale transformation

$$g_0 \to 0, \quad m_0 \to 0, \quad m_0^2/g_0^2 = C \tag{3.1}$$

on the massive Yang-Mills theory gave Sugawara's theory. Although not explicitly stated in Ref. 2, the same limit applied to the (massive) Yang-Mills Lagrangian results in

$$\mathcal{L} = (1/2C) V_\mu{}^\alpha V_\alpha{}^\mu , \tag{3.2}$$

that is, only the mass term of the original theory survives the limit. We propose to use this Lagrangian with the covariant generalization[8] of the form for the currents derived in Sec. II, namely,[9]

$$\mathcal{L} = (1/4C)\ \mathrm{Tr}(V_\mu V^\mu) , \tag{3.3a}$$

$$V_\mu = -2iC[S^{-1}\partial_\mu S - \tfrac{1}{2}\ \mathrm{Tr}(S^{-1}\partial_\mu S)] . \tag{3.3b}$$

We have explicitly made the currents traceless[10] to ensure the absence of an isoscalar current. It will also prove valuable to take S as a function of a Lorentz scalar complex isospinor

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u^\dagger = (u_1{}^* \quad u_2{}^*) \text{ (like the kaon system)},$$

$$S = \begin{pmatrix} u_1 & u_2 \\ -u_2{}^* & u_1{}^* \end{pmatrix}, \quad S^{-1} = \frac{1}{u^\dagger u} \begin{pmatrix} u_1{}^* & -u_2 \\ u_2{}^* & u_1 \end{pmatrix}, \tag{3.4}$$

$$u^\dagger u = u_1{}^* u_1 + u_2{}^* u_2 .$$

In terms of the isospinor, the Lagrangian is

$$\mathcal{L} = 2C \left\{ \partial_\mu \left(\frac{u^\dagger}{u^\dagger u} \right) \partial^\mu u \right\} + \tfrac{1}{2} C (\partial_\mu \ln u^\dagger u)(\partial^\mu \ln u^\dagger u) . \tag{3.5}$$

The currents can be calculated directly from (3.3b),

$$V_\mu{}^\alpha = \frac{-iC}{u^\dagger u} u^\dagger \tau^\alpha \overleftrightarrow{\partial}_\mu u \tag{3.6}$$

or, via Noether's theorem from the Lagrangian, noting the (isospin) invariance under the transformation

$$u \to u + i\epsilon \tau^\alpha \psi^\alpha u , \tag{3.7}$$

where ψ^α is a constant external field.

In order to quantize this system, we must discover the number of independent degrees of freedom in the Lagrangian. That there are in fact only three is most easily seen[11] with the change of variables from u to

$$\{\hat{u} \equiv u/(u^\dagger u)^{1/2},\ u^\dagger u\} . \tag{3.8}$$

In terms of these variables, the Lagrangian can be reexpressed

$$\mathcal{L} = 2C(\partial_\mu \hat{u}^\dagger \partial^\mu \hat{u}), \quad \hat{u}^\dagger \hat{u} = 1 . \tag{3.9}$$

Because all reference to the variable $u^\dagger u$ disappears, we must take $u^\dagger u$ as a constant, say,

$$u^\dagger u = 1 . \tag{3.10}$$

The Lagrangian is then a "free" Lagrangian for a constrained isospinor,

$$\begin{gathered} \mathcal{L} = 2C(\partial_\mu u^\dagger \partial^\mu u) , \\ V_\mu{}^\alpha = -iC(u^\dagger \tau^\alpha \overleftrightarrow{\partial}_\mu u) , \quad u^\dagger u = 1 . \end{gathered} \tag{3.11}$$

Because of the constraint, the theory is by no means a free theory, as will be evident below.

Again, because of the constraint, not all 4-momenta

$$\begin{gathered} \pi_i = \delta\mathcal{L}/\delta\partial_0 u_i = 2C\partial_0 u_i{}^* , \\ \pi_i{}^* = \delta\mathcal{L}/\delta\partial_0 u_i{}^* = 2C\partial_0 u_i \end{gathered} \tag{3.12}$$

can be taken independent, that is, π and u cannot be taken canonically conjugate in the usual way. Instead, a consistent quantization is

$$[u_i(\mathbf{x}), \pi_j(\mathbf{y})] = i(\delta_{ij} - u_i u_j{}^*) \delta^{(3)}(\mathbf{x} - \mathbf{y}) , \tag{3.13}$$

etc. With Eqs. (3.11), (3.12), and (3.13), one can show that the currents satisfy Sugawara's algebra and that $\theta_{\mu\nu}$ (constructed from $\mathcal{L}$ in the usual way) is precisely Sugawara's. However, rather more structure is seen by rewriting the theory in terms of three independent degrees of freedom.

Lagrangian with Three Independent Degrees of Freedom

To project out three independent degrees of freedom, we return to the Lagrangian in the form of Eq. (3.3) and introduce new variables:

$$\begin{gathered} S = S_0 - i\tau^\alpha S^\alpha, \quad S^{-1} = (S_0 + i\tau^\alpha S^\alpha)/S^2 , \\ S^2 = S_0{}^2 + S^\alpha S^\alpha . \end{gathered} \tag{3.14}$$

That is,

$$\begin{gathered} \mathrm{Re}u_1 = S_0, \quad \mathrm{Im}u_1 = -S_3 , \\ \mathrm{Re}u_2 = -S_2, \quad \mathrm{Im}u_2 = -S_1 . \end{gathered} \tag{3.15}$$

These new variables evidently have no definite isospin. In terms of them, the Lagrangian and currents become

$$\begin{aligned} \mathcal{L} = 2C\{ \partial_\mu (S^\alpha/S^2) \partial^\mu S^\alpha + \partial_\mu (S_0/S^2) \partial^\mu S_0 \} \\ + \tfrac{1}{2} C (\partial_\mu \ln S^2)(\partial^\mu \ln S^2) , \\ V_\mu{}^\alpha = (2C/S^2) \epsilon^{\alpha\beta\gamma} S^\beta \partial_\mu S^\gamma + (2C/S^2) \\ \times (S^\alpha \partial_\mu S_0 - S_0 \partial_\mu S^\alpha) . \end{aligned} \tag{3.16}$$

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[8] In fact even the $(0,i)$ form of the "equations of motion" can be put into the form of Eq. (2.8); noting that $\partial_i{}^x \delta(\mathbf{x}-\mathbf{y})|_{x=y} = 0$, one can write the generalization of Eq. (2.8) as $\partial_\mu V_\nu - \partial_\nu V_\mu = (1/2iC)[V_\mu, V_\nu]$. The immediate solution is $V_\mu = -2iCS^{-1}\partial_\mu S$.

[9] Throughout the paper, normal ordering is neglected. Because all structures will (as usual) be linear in canonical momenta, this does not affect the algebraic structure. The usual prescription $AB \to \frac{1}{2}[A,B]_+$ can always be appended when needed.

[10] One can always add or subtract $t_\mu \equiv \mathrm{Tr}(S^{-1}\partial_\mu S)$ and still have a solution of the equations of motion. In fact, t_μ can be expressed as a total derivative and $\partial_\mu t_\nu - \partial_\nu t_\mu = 0$. Of course t_μ is an isoscalar current, but it must in fact be subtracted out or there would be a zero-mass scalar particle in the theory.

[11] This can also be seen by noting the invariance of the Lagrangian under

$$u \to u + \epsilon\sigma u,$$

where σ is an external constant scalar field.

Again, the change of variables from $\{S_0,S^\alpha\}$ to

$$\{\hat{S}^\alpha\equiv S^\alpha/(S^2)^{1/2},\quad S^2\} \tag{3.17}$$

eliminates all reference to S^2. Thus we must take S^2 a constant, e.g.,

$$S_0{}^2+S^\alpha S^\alpha=1\,, \tag{3.18}$$

which is equivalent to Eq. (3.10). (Another way of seeing this is to calculate the momenta

$$\pi^\alpha\equiv\delta\mathcal{L}/\delta\partial_0S^\alpha,\quad \pi_0\equiv\delta\mathcal{L}/\delta\partial_0S_0\,, \tag{3.19}$$

and note that they are not all independent; in fact, $\pi_0S_0+\pi^\alpha S^\alpha=0$.) Thus the theory can be rewritten

$$\begin{aligned}\mathcal{L}&=2C\{\partial_\mu S^\alpha\partial^\mu S^\alpha+\partial_\mu S_0\partial^\mu S_0\}\,,\\ V_\mu{}^\alpha&=2C\{\epsilon^{\alpha\beta\gamma}S^\beta\partial_\mu S^\gamma+S^\alpha\partial_\mu S_0-S_0\partial_\mu S^\alpha\}\,,\\ &S_0{}^2+S^\alpha S^\alpha=1.\end{aligned} \tag{3.20}$$

With a trivial rescaling of the fields, this Lagrangian looks like the σ model of Gell-Mann and Lévy[4] $[S_0\leftrightarrow(C)^{-1/2}\sigma, S^\alpha\leftrightarrow(C)^{-1/2}\phi^\alpha]$, but with the important difference that S_0, S_α do not have definite isospin. (S_0, S^α also have the same parity.) Connected with this is the fact that the isospin currents $V_\mu{}^\alpha$ are the sums of the vector and axial-vector currents of the σ model. We shall return to this below, after verifying the algebra of $V_\mu{}^\alpha$.

Toward verifying the algebra, we eliminate S_0 in favor of the S^α:

$$S_0=(1-S^\alpha S^\alpha)^{1/2}\,, \tag{3.21}$$

which results in the Lagrangian and currents[12]

$$\mathcal{L}=2C\left\{\partial_\mu S^\alpha\partial^\mu S^\alpha+\frac{S^\alpha\partial_\mu S^\alpha S^\beta\partial^\mu S^\beta}{1-S^2}\right\}\,,\qquad S^2\equiv S^\alpha S^\alpha \tag{3.22a}$$

$$V_\mu{}^\alpha=2C\left\{\epsilon^{\alpha\beta\gamma}S^\beta\partial_\mu S^\gamma-\frac{S^\alpha S^\beta\partial_\mu S^\beta}{(1-S^2)^{1/2}}-(1-S^2)^{1/2}\partial_\mu S^\alpha\right\}. \tag{3.22b}$$

This makes it evident that the theory is not a free theory. Because all S^α are now independent degrees of freedom, the 3-momenta

$$\pi^\alpha=\frac{\delta\mathcal{L}}{\delta\partial_0S^\alpha}=4C\left(\delta^{\alpha\beta}+\frac{S^\alpha S^\beta}{1-S^2}\right)\partial_0S^\beta \tag{3.23}$$

can be taken canonically conjugate to the S^α in the usual way,

$$[S^\alpha(\mathbf{x}),\pi^\beta(\mathbf{y})]=i\delta^{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,. \tag{3.24}$$

To express $V_0{}^\alpha$ as a function of π^α, S^α, we need the inverse of Eq. (3.23),

$$\partial_0S^\alpha=(1/4C)(\delta^{\alpha\beta}-S^\alpha S^\beta)\pi^\beta. \tag{3.25}$$

[12] In general, now that $S^2=S_0{}^2+S_\alpha S_\alpha$ has been eliminated, we refer below to $S_\alpha S_\alpha$ as S^2.

Thus

$$V_0{}^\alpha=\tfrac{1}{2}[\epsilon^{\alpha\beta\gamma}S^\beta\pi^\gamma-(1-S^2)^{1/2}\pi^\alpha]. \tag{3.26}$$

Now we are in a position to calculate the algebra of $V_\mu{}^\alpha$. It is instructive to do this in analogy with the σ model, namely, to break up $V_\mu{}^\alpha$ into what would be the vector and axial-vector currents of the σ model:

$$\begin{aligned}&V_\mu{}^\alpha\equiv2C(v_\mu{}^\alpha-a_\mu{}^\alpha)\,,\\ &v_\mu{}^\alpha\equiv\epsilon^{\alpha\beta\gamma}S^\beta\partial_\mu S^\gamma,\quad a_\mu{}^\alpha\equiv\frac{S^\alpha S^\beta\partial_\mu S^\beta}{(1-S^2)^{1/2}}+(1-S^2)^{1/2}\partial_\mu S^\alpha,\\ &v_0{}^\alpha=\epsilon^{\alpha\beta\gamma}S^\beta\pi^\gamma,\quad a_0{}^\alpha=(1-S^2)^{1/2}\pi^\alpha,\\ &v_i{}^\alpha=\epsilon^{\alpha\beta\gamma}S^\beta\partial_iS^\gamma,\quad a_i{}^\alpha=\frac{S^\alpha S^\beta\partial_iS^\beta}{(1-S^2)^{1/2}}+(1-S^2)^{1/2}\partial_iS^\alpha.\end{aligned} \tag{3.27}$$

These have the commutation relations (of the σ model)

$$\begin{aligned}&[v_0{}^\alpha(\mathbf{x}),v_0{}^\beta(\mathbf{y})]=[a_0{}^\alpha(\mathbf{x}),a_0{}^\beta(\mathbf{y})]=i\epsilon^{\alpha\beta\gamma}v_0{}^\gamma(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\ &[v_0{}^\alpha(\mathbf{x}),a_0{}^\beta(\mathbf{y})]=i\epsilon^{\alpha\beta\gamma}a_0{}^\gamma(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\ &[v_0{}^\alpha(\mathbf{x}),v_i{}^\beta(\mathbf{y})]=i\epsilon^{\alpha\beta\gamma}v_i{}^\gamma(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})-i[\delta^{\alpha\beta}S^2(\mathbf{y})-S^\alpha(\mathbf{y})S^\beta(\mathbf{y})]\partial_i{}^y\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\ &[a_0{}^\alpha(\mathbf{x}),a_1{}^\beta(\mathbf{y})]=i\epsilon^{\alpha\beta\gamma}v_i{}^\gamma(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})-iS^\alpha(\mathbf{y})S^\beta(\mathbf{y})\partial_i{}^y\delta^{(3)}(\mathbf{x}-\mathbf{y})-i\delta^{\alpha\beta}[1-S^2(\mathbf{y})]\partial_i{}^y\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\ &[a_0{}^\alpha(\mathbf{x}),v_i{}^\beta(\mathbf{y})]=i\epsilon^{\alpha\beta\gamma}a_i{}^\gamma(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})-i\epsilon^{\alpha\beta\gamma}S^\gamma(\mathbf{y})[1-S^2(\mathbf{y})]^{1/2}\partial_i{}^y\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\ &[v_0{}^\alpha(\mathbf{x}),a_i{}^\beta(\mathbf{y})]=i\epsilon^{\alpha\beta\gamma}a_i{}^\gamma(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})+i\epsilon^{\alpha\beta\gamma}S^\gamma(\mathbf{y})[1-S^2(\mathbf{y})]^{1/2}\partial_i{}^y\delta^{(3)}(\mathbf{x}-\mathbf{y})\,.\end{aligned} \tag{3.28}$$

Schwinger terms abound in this algebra, but for the particular combinations of interest [Eq. (3.27)], we find exactly Sugawara's algebra,

$$\begin{aligned}&[V_0{}^\alpha(\mathbf{x}),V_0{}^\beta(\mathbf{y})]=i\epsilon^{\alpha\beta\gamma}V_0{}^\gamma(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\ &[V_0{}^\alpha(\mathbf{y}),V_i{}^\beta(\mathbf{y})]=i\epsilon^{\alpha\beta\gamma}V_i{}^\gamma(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})+iC\delta^{\alpha\beta}\partial_i{}^x\delta^{(3)}(\mathbf{x}-\mathbf{y})\,.\end{aligned} \tag{3.29}$$

To complete the identification with Sugawara's theory, we note that $\theta_{\mu\nu}$, as constructed in the usual way from Eq. (3.22a), namely,

$$\begin{aligned}\theta_{\mu\nu}&=\tfrac{1}{2}[\delta\mathcal{L}/\delta\partial^\mu S^\alpha,\partial_\nu S^\alpha]_+-g_{\mu\nu}\mathcal{L}\\ &=(1/2C)\{[V_\mu{}^\alpha,V_\nu{}^\alpha]_+-g_{\mu\nu}(V_\lambda{}^\alpha V_\alpha{}^\lambda)\}\end{aligned} \tag{3.30}$$

is exactly Sugawara's $\theta_{\mu\nu}$.

For completeness we also give the equations of motion of the fields S^α, as calculated from Lagrange's equations of motion

$$\Box^2S^\alpha+\frac{S^\alpha\partial_\mu S^\beta\partial^\mu S^\beta}{1-S^2}+\frac{S^\alpha S^\gamma\partial_\mu S^\gamma S^\beta\partial^\mu S^\beta}{(1-S^2)^2}=0. \tag{3.31}$$

It is curious to note that because $V_\mu{}^\alpha$ is built to satisfy all of Sugawara's "equations of motion" [Eq. (2.5b)], these Lagrangian equations of motion are essentially the requirement at the Hamiltonian level that $\partial_\mu V_\alpha{}^\mu = 0$.

One should also mention that theories with such complicated Lagrangians have been criticized on the grounds that, in certain approximations, certain axioms of field theory break down,[13] e.g., the scattering amplitudes may become nonunitary. In particular, the factor $(1-S^2)^{-1}$ in the Lagrangian provokes suspicion for obvious reasons. That this pole is in fact spurious can be seen by the following change of variable:

$$S = e^{i\tau\cdot\phi}; \quad S^\alpha = \frac{-\phi^\alpha}{\phi}\sin\phi, \qquad (1-S^2)^{1/2} = \cos\phi, \quad \phi = (\phi^2)^{1/2}. \tag{3.32}$$

In terms of these variables,

$$\mathfrak{L} = 2C\left[\partial_\mu\phi^\alpha\partial^\mu\phi^\alpha \frac{\sin^2\phi}{\phi^2} + \frac{\phi^\alpha\partial_\mu\phi^\alpha\phi^\beta\partial^\mu\phi^\beta}{\phi^2} \times\left(1-\frac{\sin^2\phi}{\phi^2}\right)\right], \tag{3.33}$$

which has no manifest singularities at all. Alternately, of course, rewriting the Lagrangian as Sugawara's theory certainly removes any obvious operator singularities.

Extra Conserved Quantities

A very curious feature of this representation is the presence of additional conserved quantities (besides $V_\mu{}^\alpha$). This is evident from the "σ model" form of the Lagrangian Eq. (3.20). This Lagrangian has a "chiral" invariance (in the space of S_α, S_0) under the transformations

$$\begin{aligned} \text{I.}\quad & S^\alpha \to S^\alpha + \epsilon\epsilon^{\alpha\beta\gamma}\lambda^\beta S^\gamma, \\ \text{II.}\quad & S^\alpha \to S^\alpha + \epsilon\phi^\alpha S_0, \\ & S_0 \to S_0 - \epsilon\phi^\alpha S^\alpha, \end{aligned} \tag{3.34}$$

where λ^β, ϕ^β are constant external fields. In the σ model itself, these invariances are of course responsible for the conservation of vector (I) and axial-vector (II) currents. In our case, however, because S_α, S_0 do not have definite isospin[14] [as generated by $Q_\alpha = \int d\mathbf{x}\, V_0{}^\alpha(\mathbf{x})$], these invariances are not connected with isospin. Nevertheless, the invariances lead to the separate conservation of $v_\mu{}^\alpha$, $a_\mu{}^\alpha$ (as defined above, that is, conservation of what would be the vector and axial-vector currents in the σ model).

For definiteness, call Sugawara's currents ${}^-V_\mu{}^\alpha = 2C(v_\mu{}^\alpha - a_\mu{}^\alpha)$; then

$${}^+V_\mu{}^\alpha = 2C(v_\mu{}^\alpha + a_\mu{}^\alpha) \tag{3.35}$$

is also conserved. The new charges

$$\Sigma^\alpha \equiv \int d\mathbf{x}\, {}^+V_0{}^\alpha(\mathbf{x}) \tag{3.36}$$

commute with Q^α and are hence new isoscalar conserved quantities. The ${}^+V_\mu{}^\alpha$ also satisfy Sugawara's algebra among themselves. Note, however, that $[{}^+V_\mu{}^\alpha, {}^-V_\nu{}^\beta] \neq 0$ because of the Schwinger terms. In the next paragraph we discuss more generally why we find the Σ's and why, together with Q^α, they generate the algebra of $O(4)$.

$O(4)$ Symmetry

That one should have expected an additional $O(4)$ symmetry is easily seen from the Lagrangian Eq. (3.5). This Lagrangian is constructed entirely out of isospinor combinations like $u^\dagger u$, which is the invariant length in a 4-dimensional Euclidean space. That is,

$$u^\dagger u = (\mathrm{Re}u_1)^2 + (\mathrm{Im}u_1)^2 + (\mathrm{Re}u_2{}^2) + (\mathrm{Im}u_2)^2 \tag{3.37}$$

is invariant under a rotation in the Euclidean space $(\mathrm{Re}u_1, \mathrm{Re}u_2, \mathrm{Im}u_1, \mathrm{Im}u_2)$. We can best discuss the symmetry in terms of a new quantity. Define

$$U = \begin{pmatrix} u_2{}^* & u_1 \\ -u_1{}^* & u_2 \end{pmatrix}, \quad \det U = u^\dagger u. \tag{3.38}$$

The columns of U are the spinors $\dot{u}^\dagger$, and u where $\dot{u}$, the dotted spinor, is defined as

$$\dot{u}_\alpha = \epsilon_{\alpha\beta} u_\beta \tag{3.39}$$

and $\epsilon_{\alpha\beta}$ is the 2-dimensional antisymmetric symbol with $\epsilon_{12} = 1$. In terms of U, the Lagrangian is invariant under the (unitary) rotation

$$U \to U\Theta, \quad \Theta = \begin{pmatrix} \alpha^* & \beta \\ -\beta^* & \alpha \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1. \tag{3.40}$$

Note that the $O(4)$ transformation rotates U from the right while the isospin transformation rotates U from the left

$$U \to \mathcal{g}U = (1 + i\epsilon\tau\cdot\psi)U. \tag{3.41}$$

In this notation, the currents

$$V_\mu{}^\alpha = -iCu^\dagger\tau^\alpha \overleftrightarrow{\partial_\mu} u = -\tfrac{1}{2}iC\,\mathrm{Tr}\{U^\dagger\tau^\alpha \overleftrightarrow{\partial_\mu} U\} \tag{3.42}$$

rotate like vectors under isospin, but are $O(4)$ invariants. One can take the $O(4)$ rotation Θ in infinitesimal form and vary the Lagrangian with respect to the relevant parameters to obtain the conserved currents associated with the $O(4)$ symmetry

$${}^+V_\mu{}^\alpha = -\tfrac{1}{2}iC\,\mathrm{Tr}\{\tau^\alpha U^\dagger \partial_\mu U\}, \tag{3.43}$$

[13] See, e.g., H. M. Fried (unpublished), R. Jackiw (unpublished). If this should turn out to be true independent of approximation, it might be interesting to entertain the following possibility: Nonunitarity of the S matrix indicates incompleteness of the asymptotic states. Could this be an indication that fermionic states must be included in the complete set?

[14] Because $V_0{}^\alpha$ contains both $v_0{}^\alpha$ and $a_0{}^\alpha$ of the σ model, the commutator of $V_0{}^\alpha$ with, say, S^β, has terms in $\epsilon^{\alpha\beta\gamma}S^\gamma$ and $\delta^{\alpha\beta}S_0$. Similarly, S_0 mixes with S_α under isospin rotation.

which are equal to the forms displayed in Eq. (3.35). These are also $O(4)$ invariant but now rotate like scalars under isospin. Writing out each component, we find

$$\begin{aligned}
{}^{+}V_\mu{}^1&=iC(u_2{}^*\overleftrightarrow{\partial_\mu}u_1{}^*-u_2\overleftrightarrow{\partial_\mu}u_1)\,,\\
{}^{+}V_\mu{}^2&=C(u_2{}^*\overleftrightarrow{\partial_\mu}u_1{}^*+u_2\overleftrightarrow{\partial_\mu}u_1)\,,\\
{}^{+}V_\mu{}^3&=iCu^\dagger\overleftrightarrow{\partial_\mu}u.
\end{aligned}\tag{3.44}$$

The third component can be interpreted as a particle-number density, but the other two are less transparent.

One can check that $\theta_{\mu\nu}$ and the algebra are also invariant under the $O(4)$ transformation. One learns, however, that the symmetry is representation-dependent, as anticommutators are needed copiously in the check. Thus, although we cannot prove that all representations [of the $SU(2)$ theory] must have this extra symmetry, we have not yet been able to find a representation that does not. The same will be true at the level of $SU(2)\otimes SU(2)$ as seen in Sec. IV. On the other hand, with the inclusion of PCAC we will in fact have a choice of whether or not to break the symmetry.

IV. $SU(2)\otimes SU(2)$ AND PCAC

To represent the $SU(2)\otimes SU(2)$ theory [Eqs. (2.1) and (2.2) with $f_{\alpha\beta\gamma}\to\epsilon_{\alpha\beta\gamma}$, indices running from 1 to 3], we first need solve the "equations of motion" including axial-vector currents. We can write these as one matrix equation by defining

$$J_\mu=V_\mu+\gamma_5A_\mu\,,\quad V_\mu=\tau^\alpha V_\mu{}^\alpha\,,\quad A_\mu=\tau^\alpha A_\mu{}^\alpha\,,\qquad \gamma_5=\begin{pmatrix}0&1\\1&0\end{pmatrix},\tag{4.1}$$

where γ_5 doubles the space. V_μ and A_μ can be obtained from J_μ by obvious traces in γ_5 space. In terms of J, all equations of motion become

$$\partial_\mu J_\nu-\partial_\nu J_\mu=\frac{1}{2iC}[J_\mu,J_\nu]\tag{4.2}$$

with the immediate solution

$$J_\mu=-2iC[S^{-1}\partial_\mu S-\tfrac14\operatorname{Tr}(S^{-1}\partial_\mu S)-\tfrac14\gamma_5\operatorname{Tr}(\gamma_5S^{-1}\partial_\mu S)],\tag{4.3}$$

where Tr means trace in both γ_5 and τ spaces. As in the $SU(2)$ case, we have explicitly taken the currents traceless to avoid neutral currents. We now take S as a function of two isospinors u and v, respectively, pseudoscalar and scalar:

$$S=\gamma_5\begin{pmatrix}u_1&u_2\\-u_2{}^*&u_1{}^*\end{pmatrix}+\begin{pmatrix}v_1&v_2\\-v_2{}^*&v_1{}^*\end{pmatrix}.\tag{4.4}$$

The inverse of S is easily calculated:

$$S^{-1}=\left\{\gamma_5\begin{pmatrix}u_1{}^*&-u_2\\u_2{}^*&u_1\end{pmatrix}+\begin{pmatrix}v_1{}^*&-v_2\\v_2{}^*&v_1\end{pmatrix}\right\}\left(\frac{s-\gamma_5p}{s^2-p^2}\right),\qquad s=u^\dagger u+v^\dagger v,\quad p=u^\dagger v+v^\dagger u.\tag{4.5}$$

As a Lagrangian, we take again the form suggested by the limit procedure on the massive Yang-Mills theory,

$$\mathcal{L}=(1/2C)(V_\mu{}^\alpha V_\alpha{}^\mu+A_\mu{}^\alpha A_\alpha{}^\mu)=(1/8C)\operatorname{Tr}(J_\mu J^\mu).\tag{4.6}$$

In terms of the spinors, the currents are

$$\begin{aligned}
V_\mu{}^\alpha&=[-iC/(s^2-p^2)][s(v^\dagger\tau^\alpha\overleftrightarrow{\partial_\mu}v+u^\dagger\tau^\alpha\overleftrightarrow{\partial_\mu}u)-p(u^\dagger\tau^\alpha\overleftrightarrow{\partial_\mu}v+v^\dagger\tau^\alpha\overleftrightarrow{\partial_\mu}u)],\\
A_\mu{}^\alpha&=[-iC/(s^2-p^2)][s(v^\dagger\tau^\alpha\overleftrightarrow{\partial_\mu}u+u^\dagger\tau^\alpha\overleftrightarrow{\partial_\mu}v)-p(u^\dagger\tau^\alpha\overleftrightarrow{\partial_\mu}u+v^\dagger\tau^\alpha\overleftrightarrow{\partial_\mu}v)].
\end{aligned}\tag{4.7}$$

The Lagrangian in terms of spinors is rather long, and we shall not need its explicit form at this stage, so we omit it; rather, we find the constraints directly by invariance arguments. Because the currents are invariant under

$$\begin{aligned}
&\text{I.}\quad u\to u+\epsilon\sigma u,\quad v\to v+\epsilon\sigma v\,,\\
&\text{II.}\quad u\to u+\epsilon\eta v,\quad v\to v+\epsilon\eta u\,,
\end{aligned}\tag{4.8}$$

(where σ,η are scalar and pseudoscalar external fields), so is the Lagrangian. Then one learns that invariance I implies $s^2-p^2=\text{const}$, while invariance II implies $p=0$. With these two constraints in mind, the Lagrangian and the currents become

$$\begin{aligned}
\mathcal{L}&=2C(\partial_\mu u^\dagger\partial^\mu u+\partial_\mu v^\dagger\partial^\mu v)\,,\\
V_\mu{}^\alpha&=-iC(v^\dagger\tau^\alpha\overleftrightarrow{\partial_\mu}v+u^\dagger\tau^\alpha\overleftrightarrow{\partial_\mu}u)\,,\\
A_\mu{}^\alpha&=-iC(v^\dagger\tau^\alpha\overleftrightarrow{\partial_\mu}u+u^\dagger\tau^\alpha\overleftrightarrow{\partial_\mu}v)\,,\\
&u^\dagger u+v^\dagger v=1,\quad u^\dagger v+v^\dagger u=0.
\end{aligned}\tag{4.9}$$

Alternatively one could derive the currents directly from the Lagrangian (via Noether's theorem) because of the chiral symmetry under

$$\begin{aligned}
&\text{I.}\quad u\to u+i\tfrac12\epsilon\tau\cdot\psi u,\quad v\to v+i\tfrac12\epsilon\tau\cdot\psi v\,,\\
&\text{II.}\quad u\to u+i\tfrac12\epsilon\tau\cdot\chi v,\quad v\to v+i\tfrac12\epsilon\tau\cdot\chi u\,,
\end{aligned}\tag{4.10}$$

where ψ^α, χ^α are scalar and pseudoscalar isovector external fields.

To verify the algebra, it is most convenient to project out the six independent degrees of freedom in the following manner: First define spinors without definite parity

$$u=\tfrac12(\chi_+-\chi_-),\quad v=\tfrac12(\chi_++\chi_-)\tag{4.11}$$

in terms of which the constraints become simply

$$\chi_\pm{}^\dagger\chi_\pm=1\,.\tag{4.12}$$

Then introduce two sets of variables $\{S_\pm{}^0,S_\pm{}^\alpha\}$, one for each χ, analogous to the $\{S^0,S^\alpha\}$ of the $SU(2)$ case. Using the constraints in the form $S_\pm{}^0=(1-S_\pm{}^2)^{1/2}$ to eliminate $S_\pm{}^0$, one arrives at a Lagrangian and currents which are essentially the parity-doubling of the forms encountered in the $SU(2)$ theory,

$$\mathcal{L}=C\sum_{i=\pm}\left(\partial_\mu S_i{}^\alpha\partial^\mu S_i{}^\alpha+\frac{S_i{}^\alpha\partial_\mu S_i{}^\alpha S_i{}^\beta\partial^\mu S_i{}^\beta}{1-S_i{}^2}\right),$$

$$V_\mu{}^\alpha=C\sum_{i=\pm}\left(\epsilon^{\alpha\beta\gamma}S_i{}^\beta\partial_\mu S_i{}^\gamma+S_i{}^\alpha\partial_\mu S_i{}^0-S_i{}^0\partial_\mu S_i{}^\alpha\right),\quad(4.13)$$

$$A_\mu{}^\alpha=C[\epsilon^{\alpha\beta\gamma}S_+{}^\beta\partial_\mu S_-{}^\gamma+S_-{}^\alpha\partial_\mu S_+{}^0-S_+{}^0\partial_\mu S^\alpha+(-)\leftrightarrow(+)].$$

Thus, checking the algebra essentially reduces to the previous case [$SU(2)$]. Although we omit any further details, everything, including $\theta_{\mu\nu}$, works out as it should.

Note also that, because the Lagrangian is still expressed in terms of $u^\dagger u$, $v^\dagger u$, $v^\dagger v$, etc., the $O(4)$ symmetry persists here. Indeed, one can show again that there exist three isoscalar conserved quantities in the theory. We shall return to this below, after discussing the inclusion of PCAC.

PCAC

There are a number of ways to break the axial-vector conservation, of which we discuss two representative forms. The first is analogous to the σ model[4] of Gell-Mann and Lévy, and forms a representation of the PCAC form of Sugawara's theory written down by Bardakci, Frishman, and Halpern[2]: We can define objects in the theory which transform like pions and a σ field

$$\phi^\alpha\equiv i(u^\dagger\tau^\alpha v-v^\dagger\tau^\alpha u),\qquad \sigma\equiv u^\dagger u-v^\dagger v.\quad(4.14)$$

Moreover, these have the commutation relations (among themselves and with the currents) specified in Ref. 2. That is, the ϕ's and σ commute among themselves, rotate like an isovector and an isoscalar under isospin, etc. That they have the right (σ-model) commutation relations with the axial charges is most easily seen by noting that under the axial transformation (II) of Eq. (4.10),

$$\sigma\to\sigma+\epsilon X^\alpha\phi^\alpha,\qquad \phi^\alpha\to\phi^\alpha-\epsilon X^\alpha\sigma.\quad(4.15)$$

Thus, by simply adding to the Lagrangian a term proportional to σ, one has a representation of the PCAC theory of Ref. 2.

Another (independent) way of breaking axial-vector conservation is the following: One can define three scalar and three pseudoscalar densities

$$\phi^\alpha=\dot{u}^T\tau^\alpha v+\dot{v}^T\tau^\alpha u+\text{H.c.},\qquad S^\alpha=\dot{u}^T\tau^\alpha u+\dot{v}^T\tau^\alpha v+\text{H.c.},\quad(4.16)$$

where $\dot{u}^T$ is the transpose of the dotted spinor introduced in Sec. III, and H.c. means Hermitian conjugate. ϕ^α and S^α transform like isovectors and commute among themselves. Their commutators with the axial charges are again seen most easily through the transformation (II) of Eq. (4.10):

$$S^\alpha\to S^\alpha-\epsilon\epsilon^{\alpha\beta\gamma}X^\beta\phi^\alpha,\qquad \phi^\alpha\to\phi^\alpha-\epsilon\epsilon^{\alpha\beta\gamma}X^\beta S^\gamma.\quad(4.17)$$

Thus one can break the axial current conservation by adding a term to the Lagrangian proportional to, say, $\phi^\alpha\phi^\alpha$. This results in a PCAC statement of the form

$$\partial_\mu A_\alpha{}^\mu\sim\epsilon_{\alpha\beta\gamma}S^\beta\phi^\gamma.\quad(4.18)$$

One would then identify this particular combination of ϕ^α, S^α as the physical "pion" field. This then is a representation of a PCAC Sugawara theory not given in Ref. 2.

Breaking the $O(4)$ Symmetry

Although the $SU(2)$ and $SU(2)\otimes SU(2)$ representations have the $O(4)$ symmetry, we find that, at the level of PCAC, we have a choice of whether or not to break it. Evidently, the first PCAC model retains the symmetry, but the second, using dotted spinors, does not. In fact, it is evident that bilinears like $\dot{u}v$, etc., are not invariant under the rotations in question. A little formalism may be of help in pinning this down: In terms of U as defined above, and

$$V\equiv\begin{pmatrix}v_2{}^* & v_1\\ -v_1{}^* & v_2\end{pmatrix},\quad \dot{U}\equiv\begin{pmatrix}u_1{}^* & u_2\\ u_2{}^* & -u_1\end{pmatrix},\quad(4.19)$$

the ϕ^α of Eq. (4.16) can be written as

$$\phi^\alpha=\operatorname{Tr}\{\dot{U}^T\tau^\alpha V\}.\quad(4.20)$$

Under the $O(4)$ transformation

$$V\to V\Theta,\quad \dot{U}^T\to\Theta\dot{U}^T.\quad(4.21)$$

Thus ϕ^α is not $O(4)$ invariant. Moreover,

$$\phi^\alpha\phi^\alpha=4[(\dot{u}v+\dot{u}^\dagger v^\dagger)^2]+16(u^\dagger v)(v^\dagger u)\quad(4.22)$$

is not $O(4)$ invariant. Thus adding $\phi^\alpha\phi^\alpha$ to the Lagrangian breaks the symmetry.

Unconstrained Fields with Definite Isospin

Thus far all our representations are written in terms either (a) of constrained fields with definite isospin (isospinors) or (b) of unconstrained fields without definite isospin. As a consequence of this, one can easily be satisfied that ordinary perturbative approaches lead to a set of Feynman graphs which, though relativistically invariant, violate isospin conservation at each order. For example, breaking the Lagrangian Eq. (3.22a) into obvious free and interaction parts and expanding the denominator $(1-S_\alpha S_\alpha)^{-1}\sim 1+S_\alpha S_\alpha+\cdots$ in the usual way is equivalent to approximating the constraint

$$\operatorname{Re}u_1=[1-(\operatorname{Im}u_1)^2-(\operatorname{Re}u_2)^2-(\operatorname{Im}u_2)^2]^{1/2}\cong 1-\tfrac{1}{2}[(\operatorname{Im}u_1)^2+(\operatorname{Re}u_2)^2+(\operatorname{Im}u_2)^2]+\cdots,\quad(4.23)$$

which obviously breaks isospin at each order. One would need to sum the series (or perhaps certain infinite subsets) to regain isospin conservation. The situation is unpleasantly reminiscent of the similar problem in the Feynman-graph approach to the Dirac monopole.[15]

The only hope of doing an ordinary perturbation expansion would appear to necessitate writing a representation in terms of unconstrained fields of definite isospin. This is in fact possible at the $SU(2)\otimes SU(2)$ level. One needs, e.g., to write down a set of independent scalar and pseudoscalar isovector fields as functions of u and v. One possible choice (among many) is

$$\begin{aligned}\phi^\alpha&=i(u^\dagger\tau^\alpha v-v^\dagger\tau^\alpha u)\,,\\ S^\alpha&=(\dot u\tau^\alpha v+\dot v\tau^\alpha u+\text{H.c.})(\dot u v+\text{H.c.})\,.\end{aligned}\tag{4.24}$$

By independent we mean that no relation exists between ϕ^α and S^α (e.g., no relation between ϕ^2, S^2, and $S\cdot\phi$). This gives us six independent degrees of freedom, so that the equations can be inverted, expressing u, v (six degrees of freedom) as functions of ϕ^α, S^α. Note that, at the $SU(2)$ level (with one spinor u, $u^\dagger u=1$) such an inversion is not possible—the isovector field squared turns out always to be a function of $u^\dagger u=1$, and the inversion is impossible. Note also that the relations (4.24) are quartic in the spinors. We have not been able to find any bilinear S^α, ϕ^α which are independent.

In fact this can be done and leads to curious expressions for isospinors constructed out of isovectors. This will be presented elsewhere. Here we give a simpler method (of obtaining unconstrained fields with definite isospin) that procedes via a higher dimensional representation for the matrix S. Suppose, for example, we define a quantity

$$J_i\equiv V_i{}^\alpha L^\alpha+A_i{}^\alpha K^\alpha\,,\tag{4.25}$$

where L^α,K^α are a 4×4 representation of $SU(2)\otimes SU(2)$,

$$\begin{aligned}[L^\alpha,L^\beta]&=[K^\alpha,K^\beta]=i\epsilon^{\alpha\beta\gamma}L^\gamma\,,\\ [L^\alpha,K^\beta]&=[K^\alpha,L^\beta]=i\epsilon^{\alpha\beta\gamma}K^\gamma\,.\end{aligned}\tag{4.26}$$

Such a representation is

$$L^\alpha=\begin{bmatrix}0&0&0&0\\0&&&\\0&&t^\alpha&\\0&&&\end{bmatrix}\quad(t^\alpha)_{\beta\gamma}=-i\epsilon^{\alpha\beta\gamma}\,,$$

$$K^1=i\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad K^2=i\begin{bmatrix}0&0&-1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{bmatrix},\tag{4.27}$$

$$K^3=i\begin{bmatrix}0&0&0&-1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix}.$$

$V_i{}^\alpha$ and $A_i{}^\alpha$ can be obtained from J_i by obvious traces. Then the constraint equations for $SU(2)\otimes SU(2)$ are equivalent to the equation for J_i:

$$\partial_iJ_j-\partial_jJ_i=(1/iC)[J_i,J_j]\,,\tag{4.28}$$

with the proviso that J_i can be written in the form (4.25); that is, J_i must be a 4×4 antisymmetric matrix. The solution to (4.28) is, as usual,

$$J_i=-iCS^{-1}\partial_iS\,,\tag{4.29}$$

where the antisymmetry condition means that S must be orthogonal. We take S,S^T (transpose) as $S_{ij}\equiv e_{(i)}{}^j$, $(S^T)_{ij}=e_{(j)}{}^i$, with i and j running from 0 to 3. Then the constraints become (summation convention)

$$e_{(i)}{}^je_{(i)}{}^k=\delta_{jk},\quad e_{(i)}{}^je_{(l)}{}^j=\delta_{il}\,.\tag{4.30}$$

The currents are

$$\begin{aligned}V_i{}^\alpha&=\tfrac12C\epsilon^{\alpha\beta\gamma}e_{(j)}{}^\beta\partial_ie_{(j)}{}^\gamma,\quad\alpha=1,2,3\\ A_i{}^\alpha&=\tfrac12C[\partial_ie_{(j)}{}^\alpha e_{(j)}{}^0-\partial_ie_{(j)}{}^0e_{(j)}{}^\alpha]\,,\end{aligned}\tag{4.31}$$

from which we learn that $e_{(i)}{}^0$ are pseudoscalar isoscalars, whereas $e_{(i)}{}^\alpha$ are pseudoscalar isovectors (very much like the σ model again). The Lagrangian is again (4.6) and is proportional to $\partial_\mu e_{(j)}{}^i\partial^\mu e_{(j)}{}^i$ (i and j summed from 0 to 3). S has six degrees of freedom which we may take as $e_{(1)}{}^\alpha$, $e_{(2)}{}^\alpha$, $e_{(3)}{}^\alpha$ with the constraints (i, $j=1,2,3$, not summed),

$$(1-e_{(i)}{}^\alpha e_{(i)}{}^\alpha)^{1/2}(1-e_{(j)}{}^\beta e_{(j)}{}^\beta)^{1/2}+e_{(i)}{}^\alpha e_{(j)}{}^\alpha=0\,.\tag{4.32}$$

Now we can easily express these in terms of two unconstrained pseudoscalar isovectors ϕ^α, χ^α. Write

$$e_{(1)}{}^\alpha=\phi^\alpha,\quad e_{(2)}{}^\alpha=\chi^\alpha+f_1\phi^\alpha,\quad e_{(3)}{}^\alpha=f_2\phi^\alpha+f_3\chi^\alpha\tag{4.33}$$

and use (4.32) to solve for the f's. This results in a Lagrangian $\mathfrak{L}=\mathfrak{L}_0(\varphi,\chi)+\mathfrak{L}_I(\varphi,\chi)$, where $\mathfrak{L}_0$ consists of ordinary kinetic energy terms for ϕ,χ. Unfortunately, $\mathfrak{L}_I$ has the property that, because it contains inverse fields, it is singular as both ϕ,χ go to zero. This is an interaction that cannot be "turned off," and again one cannot do perturbation theory around $\mathfrak{L}_0$. Why this happened is obvious: A little thought about (4.30) convinces one that not all 6 independent variables can go to zero simultaneously (because of the constraint $S^TS=1$). We have tried a number of other models with higher-dimensional representations (higher than two), and always find similar problems. On the strength of this, we would conjecture that although many representations can be found in terms of unconstrained fields with definite isospin, when written in this form, the interactions are always singular. This seems to indicate that the basic fields in these Lagrangians do not correspond to observable particles.

Similar considerations can be made for $SU(3)$, $SU(3)\otimes SU(3)$, and various breakings of these symmetries.

[15] J. Schwinger, Phys. Rev. 144, 1087 (1966).

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APPENDIX: SPATIAL CONSTRAINT EQUATION

Our purpose here is to show that in fact Eq. (2.10) is the most general solution to the spatial constraint equation (2.8).

We first prove the following lemma: If V_i is any solution to Eq. (2.8) (with i,j running from 1 to 3, and the time variable fixed), then a 2×2 matrix S (a function only of space coordinates) can be found such that V_i' [defined by (2.9)] has one of its spatial components identically zero. For definiteness, take this component to be V_1'. Then we have to show that for some S, $V_1'=0$ or

$$S^{-1}V_1S-2iCS^{-1}\partial_1S=0. \qquad \text{(A1)}$$

Defining $S=S^\alpha\tau^\alpha+S_0$ and going back to the component notation, a sufficient condition for (A1) is that S^α, S_0 satisfy the differential equations

$$\partial_1S^\alpha-(1/2C)\epsilon^{\alpha\beta\gamma}V_1{}^\beta S^\gamma+(i/2C)V_1{}^0S^\alpha+(i/2C)S^0V_1{}^\alpha=0, \qquad \text{(A2)}$$
$$\partial_1S^0+(i/2C)V_1{}^0S^0+(i/2C)V_1{}^\gamma S^\gamma=0,$$

where, for convenience, we have allowed a neutral current $V_i{}^0$; that is, $V_i=\tau^\alpha V_i{}^\alpha+V_i{}^0$, but this could be omitted if we used traceless currents (see Ref. 10). Now consider $V_1{}^\alpha$ and $V_1{}^0$ as given functions of x_1 and treat the variables x_2, x_3 as fixed parameters. Then (A2) is a system of linear differential equations in one variable (x_1)—which is known always to have a solution (assuming nonsingular $V_i{}^\alpha$). This proves the lemma.

Setting $V_1'=0$, one finds that, for either i or j equal to 1, Eq. (2.8) reduces to

$$\partial_1V_k'=0, \quad k=2,3. \qquad \text{(A3)}$$

That is, with this particular S, all V_i' are independent of x_1 (and $V_1'=0$). The constraint equations of V_i' therefore are just the two-dimensional analog of (2.8):

$$\partial_iV_j'(x_2,x_3)-\partial_jV_i'(x_2,x_3)=(1/2iC)[V_i',V_j'] \qquad \text{(A4)}$$

(with $i,j=2,3$).

Using the lemma again for another spatial direction, the system (A4) can be reduced to a one-dimensional system. For example, one finds a transformation function S' which eliminates V_2'', where

$$V_i''=(S')^{-1}V_i'S'-2iC(S')^{-1}\partial_iS', \quad i=2,3. \qquad \text{(A5)}$$

This results in

$$\partial_2V_3''=0; \qquad \text{(A6)}$$

that is, the solution is only $V_i''=\delta_{i3}V_3(x_3)$. Finally, a further transformation S'' can be used to set $V_3'''=0$. Undoing the chain of transformations (S,S',S'') which mapped V_i into $V_i'''\equiv 0$, we obtain

$$V_i=2iC[\partial_i(SS'S'')](SS'S'')^{-1}. \qquad \text{(A7)}$$

Redefining

$$SS'S''\equiv\tilde{S}^{-1}, \qquad \text{(A8)}$$

this is the same as

$$V_i=-2iC(\tilde{S})^{-1}\partial_i\tilde{S}, \qquad \text{(A9)}$$

which proves that Eq. (2.10) is the most general solution to the spatial constraint equation. From Lorentz invariance, it now easily follows that the covariant generalization of (A9), namely, Eq. (3.3b), is the most general solution to the four-dimensional equations (2.5b).