the dynamical equation

$$\psi^0(0)=v^0(0)G(0)\psi^0(0)\,, \tag{23}$$

where $v^0(t)$ is the $j=0$ component of the potential. What is happening is that Eq. (22) is satisfied as one proceeds to $j=0$ by transitions to nonsense states, while in Eq. (23) nonsense states can play no role.

In our model, as long as there is only one odd-parity even-signature Regge trajectory which goes through $j=0$ at $t=0$ the Bethe-Salpeter normalization condition will require any zero-mass pseudoscalar meson corresponding to the trajectory to couple to equal- (virtual-) mass external states. As we have argued above, there appears to be no dynamical equation for such a coupling, and on this basis we draw the conclusion that, in our model, a trajectory which chooses nonsense in all likelihood cannot correspond to a physical particle even in the case that the trajectory chooses a nonsense at zero energy. In particular, we feel that an $M=1$ massless pion is very unlikely.

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General Form of Representations of the Current Algebra in the Two-Quark Model*†

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The algebra of vector and axial-vector charge densities at infinite momentum is solved (disregarding relativistic invariance), giving the most general form of these densities in the two-quark model of the mesons. The possiblity of these solutions satisfying the angular condition for relativistic invariance is considered, and for $SU(3)$ currents it is found that if we cannot find a covariant solution in the simple case where the current is a sum of contributions from each quark and the mass of the two-quark system is $SU(3)$-independent, then we cannot find *any* covariant $SU(3)$-symmetric solution of the current algebra in this model, even with a singlet-octet mass splitting.

I. INTRODUCTION

RECENT attempts have been made to classify the known strongly interacting elementary particles and resonances according to representations of the algebra of vector and axial-vector charge densities at infinite momentum.[1] This algebra is believed to hold to all orders in the strong interaction and has been tested through the sum rules derived by the method of Fubini and Furlan.[2] It is hoped that one can find representations of the algebra in which the known hadronic states are treated ideally as discrete particles; such a representation would describe the masses and quantum numbers of the particles as well as their electromagnetic and weak form factors and [approximately, through the hypothesis of partially conserved axial-vector current (PCAC)] their pionic decay amplitudes.

In constructing representations of the current algebra one commonly uses the quark model, in which the baryons are made of three quarks, and the mesons (with which we shall be concerned in this paper) are made of a quark and an antiquark. The lowest meson states then form an $SU(3)$ octet and a singlet, and we identify these as the pseudoscalar mesons π, K, $\bar{K}$, η, and X^0. The next excited states form another octet and singlet, the vector mesons ρ, K^*, $\bar{K}^*$, ϕ, and ω. There are an infinite number of levels in the whole representation, the number of multiplets of a given spin and parity depending on how many degrees of freedom the internal quarks have.

The current algebra to be represented consists of a set of operators $F_a(\mathbf{k})$ and $F_a{}^5(\mathbf{k})$ obeying the commutation relations

$$[F_a(\mathbf{k}),F_b(\mathbf{k}')]=ic_{abc}F_c(\mathbf{k}+\mathbf{k}')\,, \tag{1.1a}$$

$$[F_a(\mathbf{k}),F_b{}^5(\mathbf{k}')]=ic_{abc}F_c{}^5(\mathbf{k}+\mathbf{k}')\,, \tag{1.1b}$$

$$[F_a{}^5(\mathbf{k}),F_b{}^5(\mathbf{k}')]=ic_{abc}F_c(\mathbf{k}+\mathbf{k}')\,, \tag{1.1c}$$

where $\mathbf{k}=(k_x,k_y,0)$ is a two-dimensional momentum transfer. In the case of $SU(3)$, the subscripts run from 1 to 8 and $c_{abc}=f_{abc}$. We shall also consider the simpler $SU(2)$ current algebra, in which case the subscripts run

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[1] See, for example, R. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy, 1966* (W. H. Freeman and Co., San Francisco, 1966).

[2] S. Fubini and G. Furlan, Physics 1, 229 (1964).

from 1 to 3 and $c_{abc}=\epsilon_{abc}$. For brevity, we call the $F_a(\mathbf{k})$ and $F_a{}^5(\mathbf{k})$ "currents," although they are really defined in terms of the true vector and axial-vector current densities $\mathfrak{F}_a{}^\mu(x)$ and $\mathfrak{F}_a{}^{5\mu}(x)$ by[3]

$$\langle n'|F_a(\mathbf{k})|n\rangle = \lim_{\substack{P_z'=P_z\to\infty \\ \mathbf{P}'-\mathbf{P}=\mathbf{k}}} \frac{\langle P',n'|\mathfrak{F}_a{}^0(0)|P,n\rangle}{2P_z}, \quad (1.2)$$

and similarly for $F_a{}^5(k)$. Here the particle states are labeled by the four-momentum P and an internal index n. One can show[3] that the right side of (1.2) exists and depends on P and P' only through the momentum transfer $\mathbf{k}$; $F_a(\mathbf{k})$ and $F_a{}^5(\mathbf{k})$ then operate only on the internal variables described by n. In particular, $F_a(\mathbf{0})$ is the a component of the isospin or unitary "charge" F_a, and $F_a{}^5(\mathbf{0})$ the corresponding "axial charge" $F_a{}^5$.

The purpose of this paper is to find the most general form of the currents $F_a(\mathbf{k})$ and $F_a{}^5(\mathbf{k})$ satisfying (1.1) in the quark-antiquark model of mesons. Aside from conveniently describing the $SU(3)$ properties of the currents, the results will be useful in considering the requirements of relativistic invariance.

The commutation relations at infinite momentum, (1.1), are not manifestly covariant, but they are compatible with relativistic invariance provided that $F_a(\mathbf{k})$ is derivable from some covariant density $\mathfrak{F}_a{}^\mu(x)$ by means of (1.2), and similarly for $F_a{}^5(\mathbf{k})$. A necessary condition that $F_a(\mathbf{k})$ and $F_a{}^5(\mathbf{k})$ be so derivable is the so-called angular condition,[3,4] which is a linear constraint to be imposed on both $F_a(\mathbf{k})$ and $F_a{}^5(\mathbf{k})$. (The angular condition is summarized in Appendix A.) Once a set of currents satisfying (1.1) has been found, the angular condition puts additional severe restrictions on them and on the mass operator (which enters explicitly into the angular condition), and may, therefore, almost uniquely determine these operators and the resulting form factors and mass spectra.

Many models proposed to covariantly represent the current algebra have centered about the following simplifying assumptions[4]: (a) The mass operator for the two-quark system is completely independent[5] of $SU(3)$, and (b) the current is a sum of contributions from the individual quarks, so that for (1.1) to be satisfied,

$$F_a(\mathbf{k}) = \tfrac{1}{2}\lambda_a{}^{(1)}e^{i\mathbf{k}\cdot\mathbf{h}^{(1)}} + \tfrac{1}{2}\lambda_a{}^{(2)}e^{i\mathbf{k}\cdot\mathbf{h}^{(2)}},$$
$$F_a{}^5(\mathbf{k}) = \tfrac{1}{2}\lambda_a{}^{(1)}\omega^{(1)}e^{i\mathbf{k}\cdot\mathbf{h}^{(1)}} + \tfrac{1}{2}\lambda_a{}^{(2)}\omega^{(2)}e^{i\mathbf{k}\cdot\mathbf{h}^{(2)}}, \quad (1.3)$$

where $\lambda_a{}^{(1)}$ and $\lambda_a{}^{(2)}$ are the unitarity spin matrices for the quark and antiquark, $h_x{}^{(1)}$, $h_y{}^{(1)}$, $\omega^{(1)}$, $h_x{}^{(2)}$, $h_y{}^{(2)}$, and $\omega^{(2)}$ are commuting $SU(3)$-independent operators, and $\omega^{(1)2}=\omega^{(2)2}=1$. Since the angular condition is linear and $SU(3)$-independent, it must be satisfied by $e^{i\mathbf{k}\cdot\mathbf{h}^{(1)}}$, $\omega^{(1)}e^{i\mathbf{k}\cdot\mathbf{h}^{(1)}}$, $e^{i\mathbf{k}\cdot\mathbf{h}^{(2)}}$, and $\omega^{(2)}e^{i\mathbf{k}\cdot\mathbf{h}^{(2)}}$ separately. Also, we require $h^{(1)}\leftrightarrow h^{(2)}$ and $\omega^{(1)}\leftrightarrow -\omega^{(2)}$ under interchange of the two quarks in order for the currents to have the right charge-conjugation properties.

These criteria are so restrictive that no representation has been found that satisfies them all, except for a trivial case in which the "meson" consists of two free quarks.[4] For two quarks bound in a potential, one can find, for example, $\mathbf{h}^{(1)}$ and $\mathbf{h}^{(2)}$ using the angular condition, but then they do not commute with each other. The same difficulty (noncommutativity of $\mathbf{h}^{(1)}$ and $\mathbf{h}^{(2)}$) has arisen in other attempts to solve the current algebra using a Lorentz-group formalism.

One way out of this difficulty is to avoid it completely by considering a simplified problem in which the currents are isospin currents and one of the quarks is isoscalar (this representation would describe, for example, the K meson and its excited states). Only one quark then contributes to the current, so we write

$$F_a(\mathbf{k}) = \tfrac{1}{2}\tau^{(1)}e^{i\mathbf{k}\cdot\mathbf{h}^{(1)}} \quad (1.4)$$

and similarly for $F_a{}^5(k)$. With no $\mathbf{h}^{(2)}$ or $\omega^{(2)}$ to worry about, the problem of finding $\mathbf{h}^{(1)}$ and $\omega^{(1)}$ seems to be more readily soluble.[6]

If we cannot find any nontrivial representations of the form (1.3) that satisfy the angular condition, we can ask whether we have oversimplified the problem. For example, it might be essential that the mass operator contain a term with $\lambda_a{}^{(1)}\lambda_a{}^{(2)}$, which is still invariant under $SU(3)$ but separates the octet from the singlet in each level, in which case the angular condition mixes the $\lambda^{(1)}$ and $\lambda^{(2)}$ terms in the current and therefore cannot be imposed on $e^{i\mathbf{k}\cdot\mathbf{h}^{(1)}}$ and $e^{i\mathbf{k}\cdot\mathbf{h}^{(2)}}$ separately. Or we may have to replace (1.3) by more general solutions of (1.1). Assuming that we still keep the two-quark model of the mesons, the general solutions of (1.1) are found in Sec. III. But first, in Sec. II, we shall solve the simpler $SU(2)$ problem in which the current is the three-component isospin current and the mesons are made of two $I=\frac{1}{2}$ nonstrange quarks. We also shall see how the solution is modified when we include the extra isoscalar state made from two $I=0$ strange quarks. In Sec. IV we discuss the implications of our results when we impose the angular condition.

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[3] R. Dashen and M. Gell-Mann, Phys. Rev. Letters **17**, 340 (1966). The factor $1/2P_z$ is necessary when one uses a covariant normalization of states.

[4] M. Gell-Mann, in *Strong and Weak Interactions—Present Problems, 1966 International School of Physics "Ettore Majorana"* (Academic Press Inc., New York, 1966). In the notation of Gell-Mann, $h_x{}^{(1)}=U^{(1)-1}(x/2)U^{(1)}$, $h_x{}^{(2)}=U^{(2)-1}(-x/2)U^{(2)}$, $\omega^{(1)}=U^{(1)-1}\sigma_z{}^{(1)}U^{(1)}$, and $\omega^{(2)}=U^{(2)-1}(-\sigma_z{}^{(2)})U^{(2)}$, with $U^{(1)}=e^{-iS(1)}$ and $U^{(2)}=e^{-iS(2)}$.

[5] Note that by "$SU(3)$-independent" we mean that the mass operator contains no λ matrices for either quark (so that the octet of a given level has the same mass as the singlet), while in an "$SU(3)$-symmetric" mass we allow the possibility of a $\lambda_a{}^{(1)}\lambda_a{}^{(2)}$ term splitting the singlet from the octet.

[6] The "simplified problem" is treated in detail by M. Gell-Mann D. Horn, and J. Weyers, in *Proceedings of the Heidelberg International Conference on Elementary Particles, Heidelberg, 1967*, eidted by H. Filthuth (John Wiley & Sons, Inc., New York, 1968). See also S. Fubini, in *Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energy, 1967* (W. H. Freeman and Co., San Francisco, 1967); H. Leutwyler, Phys. Rev. Letters **20**, 561 (1968); H. Bebié and H. Leutwyler, *ibid.* **19**, 618 (1967).

II. $SU(2)$ CURRENTS

For $SU(2)$, a runs from 1 to 3, and if $\tau_a^{(1)}$ and $\tau_a^{(2)}$ are the isospin matrices for the two quarks, we can write as the most general form for the currents,

$$F_a(\mathbf{k})=F^{(1)}(\mathbf{k})\tau_a^{(1)}/2+F^{(2)}(\mathbf{k})\tau_a^{(2)}/2+F^{(3)}(\mathbf{k})\epsilon_{abc}(\tau_b^{(1)}/2)(\tau_c^{(2)}/2),$$
$$F_a^5(\mathbf{k})=F^{5(1)}(\mathbf{k})\tau_a^{(1)}/2+F^{5(2)}(\mathbf{k})\tau_a^{(2)}/2+F^{5(3)}(\mathbf{k})\epsilon_{abc}(\tau_b^{(1)}/2)(\tau_c^{(2)}/2). \quad (2.1)$$

We could plug these expressions directly into (1.1), collect the coefficients of the independent isotopic matrices, and get equations involving $F^{(i)}$ and $F^{5(i)}$ evaluated at $\mathbf{k}$, $\mathbf{k}'$, and $\mathbf{k}+\mathbf{k}'$. It turns out to be simpler, however, not to use (2.1) directly but to work with eigenstates of the total isospin, $I_a=\frac{1}{2}\tau_a^{(1)}+\frac{1}{2}\tau_a^{(2)}$. Let the isotriplet states be $|3,a\rangle$, $(a=1, 2, 3)$ and the isosinglet state $|1\rangle$ (suppressing the other internal variables). The state with $I=1$ and $I_3=1$, for example, is $-(1/\sqrt{2})(|3,1\rangle+i|3,2\rangle)$. The matrix elements of I_a are then

$$\langle 3,c|I_a|3,b\rangle=i\epsilon_{cab}$$

with all other elements zero; I_a is the only isovector operator connecting the triplet with the triplet. We also define $A_a^{(+)}$ to connect the singlet and triplet as follows:

$$\langle 3,b|A_a^{(+)}|1\rangle=\delta_{ba}$$

with all other matrix elements zero, and define $A_a^{(-)}=A_a^{(+)\dagger}$. In terms of the original τ matrices,

$$A_a^{(\pm)}=\tfrac{1}{4}(\tau_a^{(1)}-\tau_a^{(2)}\pm i\epsilon_{abc}\tau_b^{(1)}\tau_c^{(2)}).$$

Any isovector operator must be a linear combination of I_a, $A_a^{(+)}$, and $A_a^{(-)}$, so we can replace (2.1) by an alternative expression

$$F_a(\mathbf{k})=G^{(I)}(\mathbf{k})I_a+G^{(-)}(\mathbf{k})A_a^{(+)}+G^{(+)}(\mathbf{k})A_a^{(-)},$$
$$F_a^5(\mathbf{k})=G^{5(I)}(\mathbf{k})I_a+G^{5(-)}(\mathbf{k})A_a^{(+)}+G^{5(+)}(\mathbf{k})A_a^{(-)}. \quad (2.2)$$

Since the original currents $\mathfrak{F}_a^\mu(x)$ are Hermitian, $F_a(\mathbf{k})^\dagger=F_a(-\mathbf{k})$, so that $G^{(I)}(\mathbf{k})^\dagger=G^{(I)}(-\mathbf{k})$ and $G^{(\pm)}(\mathbf{k})^\dagger=G^{(\mp)}(-\mathbf{k})$. Similar relations hold for the axial operators. When $\mathbf{k}=0$, F_a is just the isospin I_a, so that $G^{(I)}(\mathbf{0})=1$ and $G^{(\pm)}(\mathbf{0})=0$. We cannot make such a definite statement about the "axial charge," so we just let $G^{5(I)}(\mathbf{0})=\omega^{(I)}$ and $G^{5(\pm)}(\mathbf{0})=\omega^{(\pm)}$.

We now impose the commutation relations using (2.2) for the F's, and collect the coefficients of the isotopic operators to get relations among the G's. Multiplication involving γ_a, $A_a^{(+)}$, $A_a^{(-)}$ is fairly simple, and among all of the products I_aI_b, $I_aA_b^{(+)}$, $\cdots$, $A_b^{(-)}A_a^{(-)}$ one finds six independent operators. From (1.1a) we correspondingly obtain six relations involving the G's:

$$[G^{(I)}(\mathbf{k}),G^{(I)}(\mathbf{k}')]=0, \quad (2.3a)$$
$$G^{(-)}(\mathbf{k})G^{(+)}(\mathbf{k}')-G^{(-)}(\mathbf{k}')G^{(+)}(\mathbf{k})=0, \quad (2.3b)$$
$$G^{(+)}(\mathbf{k})G^{(-)}(\mathbf{k}')-G^{(+)}(\mathbf{k}')G^{(-)}(\mathbf{k})=0, \quad (2.3c)$$
$$\tfrac{1}{2}\{G^{(I)}(\mathbf{k}),G^{(I)}(\mathbf{k}')\}+\tfrac{1}{2}G^{(-)}(\mathbf{k})G^{(+)}(\mathbf{k}')+\tfrac{1}{2}G^{(-)}(\mathbf{k}')G^{(+)}(\mathbf{k})=G^{(I)}(\mathbf{k}+\mathbf{k}'), \quad (2.3d)$$
$$G^{(+)}(\mathbf{k})G^{(I)}(\mathbf{k}')+G^{(+)}(\mathbf{k}')G^{(I)}(\mathbf{k})=G^{(+)}(\mathbf{k}+\mathbf{k}'), \quad (2.3e)$$

and the Hermitian conjugate of (2.3e). Using (2.3a) and (2.3b), we can immediately simplify (2.3d) to

$$G^{(I)}(\mathbf{k})G^{(I)}(\mathbf{k}')+G^{(-)}(\mathbf{k})G^{(+)}(\mathbf{k}')=G^{(I)}(\mathbf{k}+\mathbf{k}'). \quad (2.3d')$$

From (1.1b) we get exactly the same set of equations as (2.3) but with $G^{(j)}(\mathbf{k}')$ replaced by $G^{5(j)}(\mathbf{k}')$ on the left and $G^{(j)}(\mathbf{k}+\mathbf{k}')$ by $G^{5(j)}(\mathbf{k}+\mathbf{k}')$ on the right ($j=I$, $+$, $-$). Similarly, the consequences of (1.1c) may be obtained from (2.3) by replacing $G^{(j)}(\mathbf{k})$ by $G^{5(j)}(\mathbf{k})$ and $G^{(j)}(\mathbf{k}')$ by $G^{5(j)}(\mathbf{k}')$ on the left.

To solve these equations, we define[7]

$$\mathbf{h}^{(I)}=(h_x^{(I)},h_y^{(I)},0) \quad \text{and} \quad \mathbf{h}^{(\pm)}=(h_x^{(\pm)},h_y^{(\pm)},0)$$

by

$$G^{(I)}(\mathbf{k})=1+i\mathbf{k}\cdot\mathbf{h}^{(I)}+O(k^2),$$
$$G^{(\pm)}(\mathbf{k})=i\mathbf{k}\cdot\mathbf{h}^{(\pm)}+O(k^2). \quad (2.4)$$

If we know $\mathbf{h}^{(I)}$ and $\mathbf{h}^{(+)}$ ($\mathbf{h}^{(-)}=\mathbf{h}^{(+)\dagger}$), then $G^{(I)}$ and $G^{(+)}$ are determined for all $\mathbf{k}$, because (2.3d) and (2.3e) will determine all the terms in power-series expansions of $G^{(I)}$ and $G^{(+)}$. If in addition we know $\omega^{(I)}$ and $\omega^{(+)}$, the "initial values" of $G^{5(I)}$ and $G^{5(+)}$, then $G^{5(I)}$ and $G^{5(+)}$ are determined for all $\mathbf{k}$ by letting $\mathbf{k}'=0$ in the $[F_a(k),F_b^5(k')]$ analog of (2.3d) and (2.3e). So to find the most general solution for $G^{(I)}$, $G^{(+)}$, $G^{5(I)}$, and $G^{5(+)}$, it suffices to find the constraints on $\mathbf{h}^{(I)}$, $\mathbf{h}^{(+)}$, $\omega^{(I)}$, and $\omega^{(+)}$, and then guess (or otherwise find) a solution for the G's satisfying all of the equations in (2.3) and their axial analogs.

Looking at low orders in $\mathbf{k}$ and $\mathbf{k}'$, one finds that the operators $h_i^{(I)}$, $\omega^{(I)}$, $h_i^{(-)}h_j^{(+)}$, $h_i^{(-)}\omega^{(+)}$, $\omega^{(-)}h_i^{(+)}$, and $\omega^{(-)}\omega^{(+)}$ form a commuting set, and furthermore,

$$h_i^{(\mp)}h_j^{(\pm)}=h_j^{(\mp)}h_i^{(\pm)}, \quad h_i^{(\mp)}\omega^{(\pm)}=\omega^{(\mp)}h_i^{(\pm)}, \quad (2.5)$$

and

$$\omega^{(I)2}+\omega^{(-)}\omega^{(+)}=1, \quad \omega^{(+)}\omega^{(I)}=\omega^{(I)}\omega^{(-)}=0. \quad (2.6)$$

Using (2.5) and appealing to theorems 1 and 2 of Appendix B, we can write $h_i^{(+)}$ and $\omega^{(+)}$ in the form $h_i^{(+)}=gh_i^{(J)}$ and $\omega^{(+)}=g\omega^{(J)}$, where $h_x^{(J)}$, $h_y^{(J)}$, and $\omega^{(J)}$ are Hermitian operators commuting with each other and with $h_i^{(I)}$ and $\omega^{(I)}$, and $g^\dagger g=1$ except possibly on states with $h_i^{(J)}=\omega^{(J)}=0$. From (2.6) we obtain the further conditions $\omega^{(I)2}+\omega^{(J)2}=1$ and $\omega^{(I)}\omega^{(J)}=0$.

Given the h's and ω's, we can now find the (unique) functions $G^{(I)}(\mathbf{k})$, $G^{(\pm)}(\mathbf{k})$, $G^{5(I)}(\mathbf{k})$, and $G^{5(\pm)}(\mathbf{k})$ satisfying (2.3) by expanding in powers of k, by solving a differential equation, or by simply guessing the

[7] Physically, the $\mathbf{h}$'s are the coefficients of I_a and $A_a^{(\pm)}$ in the dipole moment of the vector charge density at infinite momentum.

solution. The G's in the following paragraph indeed satisfy (2.3) with no further constraints on the h's and ω's. We give the solutions and summarize the constraints.

The most general two-isospinor-quark solution of (1.1) with $SU(2)$ currents is given by (2.2) with

$$\begin{aligned} G^{(I)}(\mathbf{k}) &= e^{i\mathbf{k}\cdot\mathbf{h}^{(1)}} \cos\mathbf{k}\cdot\mathbf{h}^{(J)},\\ G^{(+)}(\mathbf{k}) &= gG^{(J)}(\mathbf{k}),\\ G^{(J)}(\mathbf{k}) &= ie^{i\mathbf{k}\cdot\mathbf{h}^{(I)}} \sin\mathbf{k}\cdot\mathbf{h}^{(J)},\\ G^{(-)}(\mathbf{k}) &= G^{(J)}(\mathbf{k})g^\dagger,\\ G^{5(I)}(\mathbf{k}) &= \omega^{(I)}G^{(I)}(\mathbf{k})+\omega^{(J)}G^{(J)}(\mathbf{k}),\\ G^{5(+)}(\mathbf{k}) &= g[\omega^{(I)}G^{(J)}(\mathbf{k})+\omega^{(J)}G^{(I)}(\mathbf{k})],\\ G^{5(-)}(\mathbf{k}) &= [\omega^{(I)}G^{(J)}(\mathbf{k})+\omega^{(J)}G^{(I)}(\mathbf{k})]g^\dagger, \end{aligned} \tag{2.7}$$

where $h_x^{(I)}, h_y^{(I)}, h_x^{(J)}, h_y^{(J)}, \omega^{(I)}$, and $\omega^{(J)}$ all commute, and

$$\omega^{(I)2}+\omega^{(J)2}=1, \quad \omega^{(I)}\omega^{(J)}=0, \tag{2.8}$$

$g^\dagger g=1$ except possibly on states with $h_x^{(J)}=h_y^{(J)}=\omega^{(J)}=0$. (See the comment at the end of Appendix B.) Note that g need not commute with the h's or ω's.

This solution may, of course, also be described as the most general representation of the $SU(2)$ current algebra containing one isovector particle family and one isoscalar family. Now in the quark model we can form *two* isoscalar families: one from two $I=\frac{1}{2}$ quarks as described above, and another from two $I=0$ quarks. The $I=0$ pseudoscalar mesons, for example, are η and X^0, and the corresponding vector mesons are ϕ and ω. It is therefore more realistic to represent the isospin current algebra on a space of states containing an isotriplet $\{|\mathbf{3},a\rangle\}$ and two isosinglets $|\mathbf{1}\rangle$ and $|\mathbf{1}'\rangle$. The most general isospin current is then of the form

$$F_a(\mathbf{k})=G^{(I)}(\mathbf{k})I_a+G^{(-)}(\mathbf{k})A_a^{(+)}+G^{(+)}(\mathbf{k})A_a^{(-)} +G'^{(-)}(\mathbf{k})A_a'^{(+)}+G'^{(+)}(\mathbf{k})A_a'^{(-)}, \tag{2.9}$$

and similarly for $F_a^5(\mathbf{k})$, where I_a and $A_a^{(\pm)}$ are as before and $A_a'^{(+)}=A_a'^{(-)\dagger}$ has its as only nonzero element

$$\langle\mathbf{3},b|A_a'^{(+)}|\mathbf{1}'\rangle=\delta_{ba}.$$

Imposing the current algebra, we obtain equations similar to (2.3) and can solve them by slightly generalizing the theorems in Appendix B. The result is that $G^{(I)}$, $G^{(\pm)}$, $G^{5(I)}$, and $G^{5(\pm)}$ are still given by (2.7), and the expressions for $G'^{(\pm)}$ and $G^{5'(\pm)}$ are the same as for $G^{(\pm)}$ and $G^{5(\pm)}$ except that g is replaced by another operator g'. Conditions (2.8) remain the same except that $g^\dagger g=1$ is replaced by $g^\dagger g+g'^\dagger g'=1$; there are no further conditions on g and g'.

III. $SU(3)$ CURRENTS

For a two-quark system, the most general $SU(3)$ current is of the form

$$F_a(\mathbf{k})=F^{(1)}(\mathbf{k})\lambda_a^{(1)}/2+F^{(2)}(\mathbf{k})\lambda_a^{(2)}/2 +F^{(f)}(\mathbf{k})f_{abc}(\lambda_b^{(1)}/2)(\lambda_c^{(2)}/2) +F^{(d)}(\mathbf{k})d_{abc}(\lambda_b^{(1)}/2)(\lambda_c^{(2)}/2), \tag{3.1}$$

and similarly for $F_a^5(k)$, where a now runs from 1 to 8 and $\lambda_a^{(1)}$ and $\lambda_a^{(2)}$ are the $SU(3)$ matrices for the two quarks. The symmetric and antisymmetric "couplings" d_{abc} and f_{abc} are defined by

$$\lambda_a\lambda_b=\tfrac{2}{3}\delta_{ab}+(d_{abc}+if_{abc})\lambda_c.$$

Here it is important that one quark (say, No. 2) be considered an antiquark. If $\lambda_a^{(1)}$ is the usual λ_a matrix, then $\lambda_a^{(2)}$ is, with suitable conventions, $-\lambda_a^*$. As in the isospin case, we prefer to work with eigenstates of the total F spin, $F_a=\frac{1}{2}\lambda_a^{(1)}+\frac{1}{2}\lambda_a^{(2)}$. We label the octet states $|\mathbf{8},a\rangle$ $(a=1, \cdots, 8)$, and the singlet state $|\mathbf{1}\rangle$. Then

$$\langle\mathbf{8},c|F_a|\mathbf{8},b\rangle=if_{cab}$$

with all other matrix elements of F_a zero. Also define D_a by

$$\langle\mathbf{8},c|D_a|\mathbf{8},b\rangle=d_{cab}$$

with all other matrix elements zero, and define $A_a^{(+)}$ and $A_a^{(-)}=A_a^{(+)\dagger}$ by

$$\langle\mathbf{8},b|A_a^{(+)}|\mathbf{1}\rangle=\langle\mathbf{1}|A_a^{(-)}|\mathbf{8},b\rangle=(\surd\tfrac{2}{3})\delta_{ab}$$

with all other elements zero. Using F_a, D_a, $A_a^{(+)}$, and $A_a^{(-)}$ as a basis in terms of which any operator transforming like $\mathbf{8}$ can be expressed, we write

$$\begin{aligned} F_a(\mathbf{k}) &= G^{(F)}(\mathbf{k})F_a+G^{(D)}(\mathbf{k})D_a +G^{(-)}(\mathbf{k})A_a^{(+)}+G^{(+)}(\mathbf{k})A_a^{(-)},\\ F_a^5(\mathbf{k}) &= G^{5(F)}(\mathbf{k})F_a+G^{5(D)}(\mathbf{k})D_a +G^{5(-)}(\mathbf{k})A_a^{(+)}+G^{5(+)}(\mathbf{k})A_a^{(-)}. \end{aligned} \tag{3.2}$$

The "new basis" is related to the "old basis" as follows:

$$\begin{aligned} F_a &= \tfrac{1}{2}\lambda_a^{(1)}+\tfrac{1}{2}\lambda_a^{(2)},\\ D_a &= (5/18)(\lambda_a^{(1)}-\lambda_a^{(2)})+\tfrac{1}{3}d_{abc}\lambda_b^{(1)}\lambda_c^{(2)},\\ A_a^{(\pm)} &= \tfrac{1}{9}(\lambda_a^{(1)}-\lambda_a^{(2)})+\tfrac{1}{6}(-d_{abc}\pm if_{abc})\lambda_b^{(1)}\lambda_c^{(2)}. \end{aligned} \tag{3.3}$$

As with $SU(2)$, we shall use (1.1) to find all of the G's in terms of the h's and ω's defined by

$$\begin{aligned} G^{(F)}(\mathbf{k}) &= 1+i\mathbf{k}\cdot\mathbf{h}^{(F)}+O(\mathbf{k}^2),\\ G^{(D)}(\mathbf{k}) &= 0+i\mathbf{k}\cdot\mathbf{h}^{(D)}+O(k^2),\\ G^{(\pm)}(\mathbf{k}) &= 0+i\mathbf{k}\cdot\mathbf{h}^{(\pm)}+O(k^2),\\ G^{5(F)}(\mathbf{k}) &= \omega^{(F)}+O(k),\\ G^{5(D)}(\mathbf{k}) &= \omega^{(D)}+O(k),\\ G^{5(\pm)}(\mathbf{k}) &= \omega^{(\pm)}+O(k). \end{aligned} \tag{3.4}$$

Using (3.2) in (1.1) turns out to be somewhat more complicated than for $SU(2)$, because there are more independent operators among the products F_aF_b, F_aD_b, $\cdots$. It helps to separate out the parts of $[F_a(\mathbf{k}),F_b(\mathbf{k}')]$ having definite $SU(3)$ transformation properties. The representations that occur in this commutator are those found in $\mathbf{8}\otimes\mathbf{8}$, namely, $\mathbf{1}$, $\mathbf{8}$, $\mathbf{27}$ (symmetric in a, b) and $\mathbf{8}$, $\mathbf{10}$, $\mathbf{10^*}$ (antisymmetric in a, b). There are two possible operators transforming like $\mathbf{1}$: one connecting only the octet states $|\mathbf{8},a\rangle$ to themselves and one connecting $|\mathbf{1}\rangle$ to itself. There are four kinds of $\mathbf{8}$ operators $(F_a,D_a,A_a^{(+)},A_a^{(-)})$ giving eight in all[8] since $\mathbf{8}$ appears twice. Finally, there is one operator a piece transforming like $\mathbf{10}$, $\mathbf{10^*}$, and $\mathbf{27}$. Hence there are 13 independent operators, giving us 13 equations from (1.1a):

$$[G^{(F)}(\mathbf{k}),G^{(F)}(\mathbf{k}')]=0\,,$$
$$[G^{(D)}(\mathbf{k}),G^{(D)}(\mathbf{k}')]=0\,,$$
$$[G^{(F)}(\mathbf{k}),G^{(D)}(\mathbf{k}')]=0 \text{ and the (equivalent) one obtained by interchanging } k \text{ and } k',$$
$$\begin{aligned}&G^{(-)}(\mathbf{k})G^{(+)}(\mathbf{k}')-G^{(-)}(\mathbf{k}')G^{(+)}(\mathbf{k})=0\,,\\&G^{(+)}(\mathbf{k})G^{(-)}(\mathbf{k}')-G^{(+)}(\mathbf{k}')G^{(-)}(\mathbf{k})=0\,,\\&G^{(D)}(\mathbf{k})G^{(D)}(\mathbf{k}')-G^{(-)}(\mathbf{k})G^{(+)}(\mathbf{k}')=0\,,\\&G^{(+)}(\mathbf{k})G^{(D)}(\mathbf{k}')-G^{(+)}(\mathbf{k}')G^{(D)}(\mathbf{k})=0 \text{ and its Hermitian conjugate,}\\&G^{(F)}(\mathbf{k})G^{(F)}(\mathbf{k}')+G^{(D)}(\mathbf{k})G^{(D)}(\mathbf{k}')=G^{(F)}(\mathbf{k}+\mathbf{k}')\,,\\&G^{(F)}(\mathbf{k})G^{(D)}(\mathbf{k}')+G^{(F)}(\mathbf{k}')G^{(D)}(\mathbf{k})=G^{(D)}(\mathbf{k}+\mathbf{k}')\,,\\&G^{(+)}(\mathbf{k})G^{(F)}(\mathbf{k}')+G^{(+)}(\mathbf{k}')G^{(F)}(\mathbf{k})=G^{(+)}(\mathbf{k}+\mathbf{k}') \text{ and its Hermitian conjugate.}\end{aligned}\tag{3.5}$$

Here the first few equations have already been used to simplify some of the remaining ones. From (1.1b) and (1.1c) we get the same equations with certain G's replaced by G^5's as in Section II.

The method of solving these equations is similar to that of Sec. II, and the solution looks almost the same: The superscript (I) is replaced by (F) and (J) by (D). It is interesting, however, that although $\mathbf{h}^{(J)}$ and $\omega^{(J)}$ had to be defined by theorem 1 in the $SU(2)$ case, $\mathbf{h}^{(D)}$ and $\omega^{(D)}$ are already defined by (3.4) in the $SU(3)$ case. For $SU(3)$ we therefore do not need theorem 1 but only use theorem 2 to define g. The result is that the most general two-quark solution of (1.1) with $SU(3)$ currents is given by (3.2) with

$$\begin{aligned}&G^{(F)}(\mathbf{k})=e^{i\mathbf{k}\cdot\mathbf{h}^{(F)}}\cos\mathbf{k}\cdot\mathbf{h}^{(D)}\,,\\&G^{(D)}(\mathbf{k})=ie^{i\mathbf{k}\cdot\mathbf{h}^{(F)}}\sin\mathbf{k}\cdot\mathbf{h}^{(D)}\,,\\&G^{(+)}(\mathbf{k})=gG^{(D)}(\mathbf{k})\,,\\&G^{(-)}(\mathbf{k})=G^{(D)}(\mathbf{k})g^\dagger\,,\\&G^{5(F)}(\mathbf{k})=\omega^{(F)}G^{(F)}(\mathbf{k})+\omega^{(D)}G^{(D)}(\mathbf{k})\,,\\&G^{5(D)}(\mathbf{k})=\omega^{(D)}G^{(F)}(\mathbf{k})+\omega^{(F)}G^{(D)}(\mathbf{k})\,,\\&G^{5(+)}(\mathbf{k})=gG^{5(D)}(\mathbf{k})\,,\\&G^{5(-)}(\mathbf{k})=G^{5(D)}(\mathbf{k})g^\dagger\,,\end{aligned}\tag{3.6}$$

where $h_x^{(F)}$, $h_y^{(F)}$, $h_x^{(D)}$, $h_y^{(D)}$, $\omega^{(F)}$, and $\omega^{(D)}$ all commute, and

$$\omega^{(F)2}+\omega^{(D)2}=1\,,\quad \omega^{(F)}\omega^{(D)}=0\,,\tag{3.7}$$

and $g^\dagger g=1$ except possibly on states with $h_x^{(D)}=h_y^{(D)}=\omega^{(D)}=0$.

[8] I.e., in $[F_a(\mathbf{k}),F_b(\mathbf{k}')]$ we can have $f_{abc}F_c$, $f_{abc}D_c$, $f_{abc}A_c^{(+)}$, $f_{abc}A_c^{(-)}$, $d_{abc}F_c$, $d_{abc}D_c$, $d_{abc}A_c^{(+)}$, and $d_{abc}A_c^{(-)}$.

Note that this $SU(3)$ solution has exactly the same form as the $SU(2)$ solution in Sec. II, except that in the $SU(2)$ case $G^{(J)}(\mathbf{k})$ does not appear by itself as a "form factor" in (2.2), while in the $SU(3)$ case the corresponding operator $G^{(D)}(\mathbf{k})$ multiplies D_a in (3.2).

As a check on our $SU(3)$ solution, we may observe that if we find $F_a(\mathbf{k})$ and $F_a^5(\mathbf{k})$ for $SU(3)$ and restrict a to 1, 2, 3 only, then we have a reducible representation of the $SU(2)$ current algebra. The "nonstrange" states $|\mathbf{8},1\rangle$, $|\mathbf{8},2\rangle$, $|\mathbf{8},3\rangle$, $|\mathbf{1}\rangle$, and $|\mathbf{8},8\rangle$ are taken into each other and we may identify them[9] with $|\mathbf{3},1\rangle$ $|\mathbf{3},2\rangle$ $|\mathbf{3},3\rangle$, $|\mathbf{1}\rangle$, and $|\mathbf{1}'\rangle$ of Sec. II. Restricting F_a, D_a, and $A_a^{(\pm)}$ ($a=1$, 2, 3) to this five-dimensional subspace we find $F_a\to I_a$, $D_a\to(\sqrt{\tfrac{1}{3}})(A_a'^{(+)}+A_a'^{(-)})$, and $A_a^{(\pm)}\to(\sqrt{\tfrac{2}{3}})A_a^{(\pm)}$. Comparing (3.2) with (2.9) and (2.7), we then find $\mathbf{h}^{(I)}=\mathbf{h}^{(F)}$, $\mathbf{h}^{(J)}=\mathbf{h}^{(D)}$, $\omega^{(I)}=\omega^{(F)}$, $\omega^{(J)}=\omega^{(D)}$, $g_{SU(2)}=(\sqrt{\tfrac{2}{3}})g_{SU(3)}$, and $g_{SU(2)}'=(\sqrt{\tfrac{1}{3}})$. Therefore we have obtained a special case of the representation with one isovector and two isoscalars considered at the end of Sec. II.

Our $SU(3)$ representation also contains two $SU(2)$ representations with $I=\tfrac{1}{2}$ particles. For example, let $|+\rangle=(1/\sqrt{2})(|\mathbf{8},4\rangle+i|\mathbf{8},5\rangle)$ and $|-\rangle=(1/\sqrt{2})(|\mathbf{8},6\rangle+i|\mathbf{8},7\rangle)$ (corresponding to K^+ and K^0). Then with respect to these states $F_a\to\tfrac{1}{2}\tau_a$, $D_a\to\tfrac{1}{2}\tau_a$, and $A_a^{(\pm)}\to 0$ ($a=1$, 2, 3), so that

$$\begin{aligned}&F_a(\mathbf{k})\to[G^{(F)}(\mathbf{k})+G^{(D)}(\mathbf{k})]\tfrac{1}{2}\tau_a=e^{i\mathbf{k}\cdot(\mathbf{h}^{(F)}+\mathbf{h}^{(D)})}\tfrac{1}{2}\tau_a\,,\\&F_a^5(\mathbf{k})\to(\omega^{(F)}+\omega^{(D)})e^{i\mathbf{k}\cdot(\mathbf{h}^{(F)}+\mathbf{h}^{(D)})}\tfrac{1}{2}\tau_a\,,\end{aligned}\tag{3.8}$$

which is just equivalent to (1.4), the solution to the problem of only one isospin-carrying quark.

In the special case $g=1$, the $SU(3)$ solution becomes

$$\begin{aligned}F_a(\mathbf{k})&=G^{(F)}(\mathbf{k})F_a+G^{(D)}\mathbf{k}(D_a+A_a^{(+)}+A_a^{(-)})\,,\\F_a^5(\mathbf{k})&=G^{5(F)}(\mathbf{k})F_a+G^{5(D)}(\mathbf{k})(D_a+A_a^{(+)}+A_a^{(-)})\,.\end{aligned}$$

Using (3.3) and (3.6), we find that the currents are given by the simple form (1.3), where

$$\begin{aligned}&\mathbf{h}^{(1)}=\mathbf{h}^{(F)}+\mathbf{h}^{(D)},\quad \omega^{(1)}=\omega^{(F)}+\omega^{(D)}\,,\\&\mathbf{h}^{(2)}=\mathbf{h}^{(F)}-\mathbf{h}^{(D)},\quad \omega^{(2)}=\omega^{(F)}-\omega^{(D)}\,.\end{aligned}\tag{3.9}$$

In other words, $g=1$ gives the form of the currents usually assumed in the quark model.

IV. SIGNIFICANCE OF THE RESULTS

Suppose as an approximation that the mass operator for the two-quark system is $SU(3)$-independent. Then if the angular condition is to be satisfied by $F_a(\mathbf{k})$ and $F_a^5(\mathbf{k})$, it must be satisfied by all of the G's, namely [in the $SU(3)$ case] by $G^{(F)}$, $G^{(D)}$, $gG^{(D)}$, $G^{5(F)}$, $G^{5(D)}$, and $gG^{5(D)}$. Now if all of these operators satisfy the condition, then they satisfy it *a fortiori* with $g=1$, which means that there exists a simpler solution of the form

[9] This identification is partly arbitrary, since the $SU(2)$ states $|\mathbf{1}\rangle$ and $|\mathbf{1}'\rangle$ could correspond to any two orthogonal linear combinations of the $SU(3)$ states $|\mathbf{1}\rangle$ and $|\mathbf{8},8\rangle$.

(1.3) which obeys the angular condition. In other words, if we cannot make (1.3) work (and the only success so far has been in the free-quark model), then we cannot make the general solution (3.6) work either.

If we are content to deal with only $SU(2)$ currents, then the operators that have to satisfy the angular condition are $G^{(I)}$, $gG^{(J)}$, and the corresponding axial operators, but $G^{(J)}$ itself need not satisfy it. It is conceivable, therefore, that there might be a solution for nontrivial g that does not continue to satisfy the angular condition when g is replaced by 1; this possibility has not been further investigated.

Our results also have profound implications in tackling the more general case of an $SU(3)$-dependent mass. Suppose that the mass operator M contains a term proportional to $\lambda_a^{(1)}\lambda_a^{(2)}$ (summation over a understood). Then M is still invariant under $SU(3)$ but splits the octets from the singlets, so that we can write

$$M=M_1P_1+M_8P_8,$$

where P_1 and P_8 are the projection operators into the singlet and octet states, respectively, and M_1 and M_8 are $SU(3)$-independent operators. Now let the angular condition be imposed on $F_a(\mathbf{k})$ as given by (3.2). A small amount of inspection will reveal that the operators F_a, D_a, $A_a^{(+)}$, and $A_a^{(-)}$ are not mixed by the angular condition, and that if we examine the coefficients of F_a and D_a we find that $G^{(F)}(\mathbf{k})$ and $G^{(D)}(\mathbf{k})$ must each satisfy the angular condition with mass M_8, and so must $G^{5(F)}(\mathbf{k})$ and $G^{5(D)}(\mathbf{k})$. But these conditions are the same as those which would result in trying the simple solution (1.3) with an $SU(3)$-independent mass operator M_8. In other words, if we cannot make (1.3) work for an $SU(3)$-*independent* mass, then we cannot make (3.1) or (3.2) work for *any* mass which is invariant under $SU(3)$.

It appears, then, that in looking for a relativistic $SU(3)$-symmetric two-quark[10] representation of the current algebra, it is not an oversimplification to assume the simple form (1.3) for the currents[11] or to assume the mass operator $SU(3)$-independent. Although more complicated currents and masses may approximate nature more closely, it is sufficient to use the simple ones to find out whether we can get any representation at all.

Note added in proof. We can arrive at the same conclusions even if the mass is not $SU(3)$-invariant at all [but still $SU(2)$-invariant] by considering the two $I=\frac{1}{2}$ representations of the $SU(2)$ current algebra contained in the $SU(3)$ representation, given by (3.8) with a similar expression for $\bar{K}^0$ and $\bar{K}^-$. If the angular condition is satisfied here, then by making the identifications of (3.9) we can make (1.3) also satisfy it.

[10] The results can also be extended to systems with, e.g., one octet and several singlets in each level, the form of the current being analogous to (2.9) for the isospin current.

[11] Note further that if it *is* possible to find a covariant current of the form (1.3) with $SU(3)$-independent mass, then it is of course possible to find one of the simpler form (1.4). If we cannot even do the latter, i.e., if we cannot find any operator of the form $\exp(i\mathbf{k}\cdot\mathbf{h})$ satisfying the angular condition (for a given set of internal quark variables and mass operator), then we cannot find any relativistic $SU(3)$-symmetric two-quark current.

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APPENDIX A

For reference we give the angular condition for relativistic invariance as applied to the currents at infinite momentum. Derivations can be found in Refs. 3 and 4.

In the following, all operators act on internal variables of the particle system, the total momentum having disappeared from the picture in the passage to infinite momentum [Eq. (1.2)]. Let M be the operator for the mass of the system, and let $\mathbf{J}$ be the part of the total angular momentum that acts on the internal variables. For an operator X it is convenient to define $\mathcal{A}_X$ and $\mathcal{B}_X$ to be the operations of commuting and anticommuting with X, respectively. That is, if Y is another operator, then

$$\mathcal{A}_X Y=[X,Y],\quad \mathcal{B}_X Y=\{X,Y\}.$$

The currents $F_a(\mathbf{k})$ and $F_a{}^5(\mathbf{k})$ are assumed to be invariant under rotations about the z axis (remember $\mathbf{k}$ is two-dimensional), so we take $\mathbf{k}$ in the x direction. The angular condition can then be stated concisely as follows:

$$\{\exp i[\mathcal{A}_{J_y}\tan^{-1}(\mathcal{A}_M/k)-\mathcal{B}_{J_y}\tan^{-1}(k/\mathcal{B}_M)]\}F_a(k\mathbf{e}_x)$$

must have $|\Delta J_x|=0$ or 1, i.e., it must transform as a scalar or a vector under rotations about the x axis.

The same condition holds with F_a replaced by $F_a{}^5$. The quantity in curly brackets is an "operator on operators" and may be expanded in powers of, e.g., the mass splitting, giving polynomials in the $\mathcal{A}$'s and $\mathcal{B}$'s; each $\mathcal{A}$ and $\mathcal{B}$ then operates on F_a or $F_a{}^5$.

In this paper the detailed form of the angular condition is not important; we use the fact that it is linear in the current and depends on $SU(3)$ variables only through commutations and anticommutations with M.

APPENDIX B

We prove here two theorems used in Secs. II and III.

Theorem 1: Suppose we have a set of operators $A_n{}^{(+)}=A_n{}^{(-)\dagger}$ such that the operators $A_m{}^{(-)}A_n{}^{(+)}$ commute with each other for all m and n, and

$$A_m{}^{(\mp)}A_n{}^{(\pm)}=A_n{}^{(\mp)}A_m{}^{(\pm)}. \tag{B1}$$

Then there exists a set of commuting Hermitian operators H_n such that

$$A_m{}^{(-)}A_n{}^{(+)}=H_mH_n. \tag{B2}$$

Proof: Let $H_{mn}=A_m^{(-)}A_n^{(+)}$. Then $H_{mn}=H_{mn}^\dagger=H_{nm}$, and $H_{kl}H_{mn}=H_{kn}H_{ml}$, because of (B1). Now since the H_{mn} all commute, they can be simultaneously diagonalized, so assume that this has been done and let $|\alpha\rangle$ be any eigenstate of H_{mn} with eigenvalue $H_{mn}^{(\alpha)}$. The properties of H_{mn} are reflected in the eigenvalues:

$$H_{mn}^{(\alpha)}=H_{mn}^{(\alpha)*}=H_{nm}^{(\alpha)}$$

and

$$H_{kl}^{(\alpha)}H_{mn}^{(\alpha)}=H_{kn}^{(\alpha)}H_{ml}^{(\alpha)},$$

or

$$\begin{vmatrix} H_{kl}^{(\alpha)} & H_{ml}^{(\alpha)} \\ H_{kn}^{(\alpha)} & H_{mn}^{(\alpha)} \end{vmatrix}=0.$$

Therefore $H_{mn}^{(\alpha)}$ can be factored: $H_{mn}^{(\alpha)}=H_m^{(\alpha)}H_n^{(\alpha)}$, and we define H_n to be that operator with only the diagonal elements $H_n^{(\alpha)}$. The H_n clearly commute and satisfy (B2).

Note, by the way, that if there are other operators C_p commuting with each other as well as with $A_m^{(-)}A_n^{(+)}$, then we can diagonalize them along with H_{mn}, so that the operators H_n will commute with C_p also.

This theorem was applied in Sec. II with $\{A_n^{(+)}\}=\{h_x^{(+)},h_y^{(+)},\omega^{(+)}\}$ and $\{C_p\}=\{h^{(I)},\omega^{(I)}\}$, to define $\{H_n\}=\{h_x^{(J)},h_y^{(J)},\omega^{(J)}\}$. We then appealed to the following theorem:

Theorem 2: Suppose (B2) holds for a set of commuting H_n. Then there exists an operator g such that

$$A_n^{(+)}=gH_n \text{ (and therefore } A_n^{(-)}=H_ng^\dagger),$$
$$g^\dagger g=1, \text{ except possibly on states where all}$$
$$H_n=0. \tag{B3}$$

Proof: Diagonalize all H_n so that $\langle\alpha|H_n|\beta\rangle=H_n^{(\alpha)}\delta_{\alpha\beta}$. We want to define g by $\langle\alpha|g|\beta\rangle=\langle\alpha|A_n^{(+)}|\beta\rangle/H_n^{(\beta)}$ but we have to show that the right side is independent of n, and also worry about $H_n^{(\beta)}$ being zero. If $H_m^{(\beta)}$ and $H_n^{(\beta)}$ are both nonzero, then putting (B2) between $\langle\beta|$ and $|\beta\rangle$, we find

$$\sum_\alpha\left(\frac{\langle\alpha|A_m^{(+)}|\beta\rangle}{H_m^{(\beta)}}\right)^*\left(\frac{\langle\alpha|A_n^{(+)}|\beta\rangle}{H_n^{(\beta)}}\right)=1.$$

The same holds, of course, if m is replaced by n or vice versa. Then

$$\sum_\alpha\left|\frac{\langle\alpha|A_m^{(+)}|\beta\rangle}{H_m^{(\beta)}}-\frac{\langle\alpha|A_n^{(+)}|\beta\rangle}{H_n^{(\beta)}}\right|^2=1+1-2\,\mathrm{Re}1=0,$$

so that

$$\frac{\langle\alpha|A_m^{(+)}|\beta\rangle}{H_m^{(\beta)}}=\frac{\langle\alpha|A_n^{(+)}|\beta\rangle}{H_n^{(\beta)}} \quad\text{whenever } H_m^{(\beta)},\, H_n^{(\beta)}\neq 0.$$

Define g by

$$\langle\alpha|g|\beta\rangle=\frac{\langle\alpha|A_n^{(+)}|\beta\rangle}{H_n^{(\beta)}} \quad\text{if}\quad H_n^{(\beta)}\neq 0 \quad\text{for some } n,$$
$$=0 \quad\text{if}\quad H_n^{(\beta)}=0 \quad\text{for all } n.$$

Now if $H_n^{(\beta)}=0$ for some n and β, then

$$\textstyle\sum_\alpha|\langle\alpha|A_n^{(+)}|\beta\rangle|^2=0$$

from (B2), so that $\langle\alpha|A_n^{(+)}|\beta\rangle=0$. Therefore

$$\langle\alpha|gH_n|\beta\rangle=\langle\alpha|g|\beta\rangle H_n^{(\beta)}=\langle\alpha|A_n^{(+)}|\beta\rangle,$$

whether or not $H_n^{(\beta)}=0$, so the first part of (B3) holds. Using (B2) and the definition of g, one also finds

$$\langle\gamma|g^\dagger g|\beta\rangle=\delta_{\gamma\beta} \quad\text{if}\quad H_n^{(\beta)}\neq 0 \text{ for some } n,$$
$$=0 \quad\text{if}\quad H_n^{(\beta)}=0 \text{ for all } n,$$

or $g^\dagger g=1-P_0$, where P_0 is the projection operator onto the set of states on which all $H_n=0$. On this set of states, g can be arbitrarily redefined, so it might be (but is not always) possible to make $g^\dagger g=1$ on all states.