Majorana Equations for Composite Systems

G. BISIACCHI,* P. BUDINI, AND G. CALUCCI†
International Atomic Energy Agency, International Centre for Theoretical Physics, Trieste, Italy
(Received 27 December 1967)

A relativistic wave equation for the description of a composite system with nonrelativistic internal motion is deduced from the Bethe-Salpeter equation. It is shown that when the internal motion can be described by algebraic methods, as in the case of the hydrogen atom, the proposed equation for the motion of the system as a whole (motion of the c.m.) is equivalent to a Majorana-type equation, free from the well-known difficulties such as a spacelike solution. The hydrogen-atom example is discussed in some detail. It is shown how its $SO(4,1)$ spectrum-generating algebra contains the spin part of the generators of the Lorentz group. The solutions found to the Majorana-type equation, which is able to describe both discrete and continuum eigenstates, span a representation of an $SO(4,2)$ algebra.

1. INTRODUCTION

IN 1932 Majorana[1] proposed a linear relativistic wave equation based on a unitary representation of the Lorentz group. This representation is infinite dimensional and, as a consequence, the wave function of the equation has infinite components and represents a multimass system. Majorana pointed out that his equation allowed solutions not only for timelike total momentum P of the system, in which case the total mass $M^2=P^2$ possessed a discrete, angular-momentum-dependent spectrum, but also for spacelike P, which implied the existence of a continuum spectrum of imaginary masses.

This type of equation was later rediscovered by Gel'fand and Yaglom,[2] who, however, did not discuss the difficulty of the spacelike solutions. They showed that the simplest forms of these equations implied a not too encouraging feature of a mass spectrum going to zero with increasing angular-momentum quantum number j.

Lately, generalizations of the Majorana equations have been proposed by Nambu[3] and studied by Fronsdal,[4] who showed that the characteristic of spectra going to zero with increasing quantum numbers could be avoided by allowing the possibility of representing by these equations known physical systems such as the hydrogen atom. But the unwanted spacelike solutions are in general still present in these equations; other difficulties also appear as one attempts to take these equations as a basis for a "fundamental" field theory and proceeds to a second quantization.[5]

Nevertheless, the interest in the Majorana-type approach still persists because it represents an almost unique theoretical scheme to describe relativistically multimass systems with internal degrees of freedom having definite symmetry properties.

When these properties are expressed in algebraic form the Majorana approach represents an excellent example of realization of the dynamical-group program[6] and a possible way of obtaining insight into the so-called elementary-particle dynamics, especially on the role which they might have in connection with the Reggeization of scattering amplitudes[7] and with the current-algebra approach.[8]

In this paper we shall attempt to understand a possible role in physics of the Majorana-type equations and of a possible way of eliminating the known difficulties.

To begin with, there are, in nature, well-known "infinite-component" systems,[9] for example, the hydrogen atom and the harmonic oscillator. In our opinion, any acceptable Majorana-type theory should also be able to represent them. Another motivation for this kind of attempt is due to the fact that, in field theory, any bound system can be represented by a local free field,[10] so it is admissible that the free motion of composite systems with infinite states of mass may be represented by a Majorana-type equation from which Feynman rules are definable. It is, then, natural to

* Permanent address: Istituto di Fisica Teorica and Istituto di Meccanica dell'Università, Trieste.
† Permanent address: Istituto di Fisica Teorica dell'Università, Trieste, and I.N.F.N., Sottosezione di Trieste.

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[7] L. Van Hove, Phys. Letters **24B**, 183 (1967); R. Delbourgo, M. A. Rashid, A. Salam, and J. Strathdee, *ibid.* **25B**, 475 (1967); C. Fronsdal, Phys. Rev. **168**, 1845 (1968).

[8] M. Gell-Mann, D. Horn, and J. Weyers, in *Proceedings of the Heidelberg International Conference on Elementary Particles Heidelberg, 1967*, edited by H. Filthuth (North-Holland Publishing Co., Amsterdam, 1968), p. 479.

[9] It is clear that the "infinite" number of components is a mathematical idealization never realized in nature because a shield to the potential, however far away, always exists; therefore, one should adopt the theory to represent a large number of states instead of an infinite number. However, it is possible that the difference between the infinite-component idealization and the probably more difficult finite one is only marginal.

[10] W. Zimmermann, Nuovo Cimento **4**, 597 (1958).

try to formulate an infinite-component field theory from a composite-model scheme. If this attempt is successful, then the resulting Majorana equation will contain in principle no more difficulties than the original composite-model equation; in particular, spacelike solutions will be avoided and only in the interaction with external fields will the structure of the composite system play a role which might become decisive for the distinction between the "elementary" and "composite" systems.

Other attempts at interpreting linear relativistic equations in terms of internal degrees of freedom have been proposed.[11]

Our attempt is characterized by the fact that we start from a known, soluble composite system and try to construct a relativistic equation for its motion as a whole. In Sec. 2 we take as a basis for the study of the composite system the Bethe-Salpeter equation in the ladder approximation. For the c.m. motion we shall keep the strictly relativistic description, while for the relative motion we shall take the nonrelativistic one. The reduction of the relative motion from the relativistic to the nonrelativistic form will be done by using only "relativistic covariant operations" in such a way that, step by step, the theory keeps its over-all relativistic covariance and, at the end, we shall obtain the equation of a system with slow relative motion as seen from a rapidly moving observer.

In Secs. 3 and 4 we take as a composite system with definite internal symmetry the hydrogen atom and, through a relativistic generalization of the Fock stereographic projection,[12] transform the original equation to a Klein-Gordon type. We shall show then that the algebraization of the internal motion amounts to considering the Klein-Gordon equation to be derivable from a Majorana equation that is fully equivalent to the original and represents the relativistic motion of the system as a whole. These equations have no spacelike solutions and, with a redefinition of the metric, can represent both bound and scattering states.

Finally, we discuss briefly possible further developments of this kind of approach.

2. COMPOSITE-SYSTEM EQUATION

We take as a basis for our discussion the Bethe-Salpeter equation for two spinors of masses m_1 and m_2:

$$(\hat{p}_1-m_1)(\hat{p}_2-m_2)\psi(p_1,p_2)=\frac{i}{2\pi}\int\int G(p_1,p_2;\,p_1'p_2')\psi(p_1',p_2')dp_1'dp_2', \quad (2.1)$$

which, introducing the total and relative momenta P and p, reads in ladder approximation

$$(c_1\hat{P}^{(1)}+\hat{p}^{(1)}-m_1)(c_2\hat{P}^{(2)}-\hat{p}^{(2)}-m_2)\psi_P(p)=\frac{i}{2\pi}\int G(q)\psi_P(p+q)dq, \quad (2.2)$$

where

$$P=p_1+p_2, \qquad p=c_2p_1-c_1p_2,$$
$$c_1=m_1/(m_1+m_2), \quad c_2=m_2/(m_1+m_2),$$

the metric is $g_{\mu\nu}=+1,\,-1,\,-1,\,-1$, and the superscripts on the $\hat{P}^{(i)}$ and $\hat{p}^{(i)}$ refer to the γ matrices. We define the projection operators

$$\Lambda_\pm{}^{(i)}=[\mathcal{E}_i(p)\pm\mathcal{K}_i]/2\mathcal{E}_i(p),$$

where

$$\mathcal{E}_i(p)=[P^2(m_i^2-p^2)+(p\cdot P)^2]^{1/2},$$
$$\mathcal{K}_1=[m_1\hat{P}^{(1)}-iP^\mu\sigma_{\mu\nu}{}^{(1)}p^\nu], \quad \mathcal{K}_2=[m_2\hat{P}^{(2)}+iP^\mu\sigma_{\mu\nu}{}^{(2)}p^\nu],$$

with

$$\sigma_{\mu\nu}{}^{(i)}=(1/2i)[\gamma_\mu{}^{(i)},\gamma_\nu{}^{(i)}],$$

and we obtain, multiplying on the left first by $\hat{P}^{(1)}\hat{P}^{(2)}$ and then by $\Lambda_+{}^{(1)}\Lambda_+{}^{(2)}$,

$$[c_1P^2-\mathcal{E}_1(p)+p\cdot P]\cdot[c_2P^2-\mathcal{E}_2(p)+p\cdot P]\times\Lambda_+{}^{(1)}\Lambda_+{}^{(2)}\psi_P(p)=\Lambda_+{}^{(1)}\Lambda_+{}^{(2)}\hat{P}^{(1)}\hat{P}^{(2)}\times\frac{i}{2\pi}\int G(p-q)\psi_P(q)dq \quad (2.3)$$

and analogous equations with the projectors

$$\Lambda_+{}^{(1)}\Lambda_-{}^{(2)}, \quad \Lambda_-{}^{(1)}\Lambda_+{}^{(2)}, \quad \Lambda_-{}^{(1)}\Lambda_-{}^{(2)}.$$

In order to simplify this equation without losing the formal relativistic covariance, we shall define the transverse relative momenta

$$p_\mu{}^T=p_\mu-p^Lu_\mu, \quad q_\mu{}^T=q_\mu-q^Lu_\mu,$$

where

$$p^L=p\cdot u, \quad q^L=q\cdot u, \quad \text{and} \quad u^\mu=P^\mu/|P|.$$

We then obtain for the integral on the right-hand side of Eq. (2.3)

$$\int G(p-q)\psi(q)dq=\int G(p^T-q^T,p^L-q^L)\psi(q^T,q^L)dq$$
$$=|P|\int G(p^T-l,p^L-q^L)\psi(l,q^L)\delta(l\cdot P)dldq^L. \quad (2.4)$$

We shall now make the hypothesis that the action between the two particles is instantaneous in the c.m. system[13] (but not in any other). This means that we

[11] P. A. M. Dirac, Proc. Roy. Soc. (London) **A183**, 284 (1944); H. Yukawa, Progr. Theoret. Phys. (Kyoto) Suppl. **37** and **38**, 512 (1966); T. Takabayasi, *ibid.* **36**, 185 (1966); **36**, 187 (1966).

[12] V. Fock, Z. Physik **98**, 145 (1935).

[13] This means that the relative motion is slow enough to make the retardation effects small. It would always be possible to insert them *a posteriori* as perturbations.

suppose G not to depend on p^L-q^L. In the above expression the integral can be performed on the variable q^L and we obtain

$$\int G(p-q)\psi(q)dq=|P|\int G(p^T-l)\varphi(l)\delta(l\cdot P)dl, \quad (2.5)$$

with

$$\varphi(l)=\int_{-\infty}^{+\infty}\psi(l,q^L)dq^L.$$

Therefore, inserting Eq. (2.5) in Eq. (2.3), we see that the dependence on the longitudinal variable appears only in the left-hand side (note that neither $\mathcal{E}_i$ nor $\mathcal{K}_i$ depends on p^L). We can divide the whole equation by $[c_1P^2-\mathcal{E}_1(p)+p\cdot P]\cdot[c_2P^2-\mathcal{E}_2(p)+p\cdot P]$ and integrate both sides in p^L with the prescription[14] $m_i\to m_i-i\epsilon$, arriving at

$$(P^2-\mathcal{K}_1-\mathcal{K}_2)\varphi(p^T)=-(\Lambda_+^{(1)}\Lambda_+^{(2)}-\Lambda_-^{(1)}\Lambda_-^{(2)})\hat{P}^{(1)}\hat{P}^{(2)}\times\int G(p^T-l)\varphi(l)\delta(l\cdot P)dl, \quad (2.6)$$

which is the relativistic generalization to an arbitrary reference system of Eq. (13) of Ref. 14.

Since as a first example, we wish to apply our considerations to completely soluble examples of nonrelativistic bound systems without spin, we adopt in (2.6) the approximation[15]

$$\Lambda_+^{(1)}\Lambda_+^{(2)}-\Lambda_-^{(1)}\Lambda_-^{(2)}=+1, \quad (2.7)$$

which means that we take only positive-energy states for the constituents. But since (2.7) is covariant, the relativistic covariance of (2.6) is maintained.

At this point we neglect the effects of the spins and, since $\Lambda_+^{(i)}=1$ is equivalent to

$$\mathcal{K}_i=\mathcal{E}_i=(m_i^2-(p^T)^2)^{1/2}|P|$$

and in the following we shall be concerned with the Coulomb interaction, we take, in the same way as in Ref.14, $\hat{P}^{(1)}\hat{P}^{(2)}=P^2$. On the left-hand side of (2.6), in the hypothesis of slow relative motion, we take only the first term of the expansion of $\mathcal{E}_i$ in terms of $(p^T)^2/m_i^2$ (p^T four-vector). We then obtain the equation

$$[P^2-|P|(m_1+m_2-(p^T)^2/2\mu)]\varphi(p^T)=P^2\int G(p^T-l)\varphi(l)\delta(l\cdot P)dl, \quad (2.8)$$

$$\mu=m_1m_2/(m_2+m_2),$$

where the argument p^T of φ satisfies the condition $p^T\cdot P=0$. This is the starting point for the algebraization of the c.m. motion of a composite system with nonrelativistic internal motion. [In the c.m. system, (2.8) reduces to the Schrödinger equation.] Therefore this equation, in general, will have solution only when P^2 is timelike, provided that the coupling is small enough. Obviously, if one wants to consider one or two of the constituent particles with spin or some effect of relativity in the relative motion, one has to start some steps sooner.

We now try to transform Eq. (2.8) into a relativistic equation for the motion of the c.m. of the system. The relative motion will be expressed only by the transformation properties of the wave function of the system. The actual physical system we shall deal with will be the hydrogen atom, often considered in this kind of attempt.

3. HYDROGEN ATOM

In this case the kernel is $(e^2/2\pi)(p^T-l)^{-2}$ and Eq. (2.8) becomes

$$[|P|-(m_1+m_2-(p^T)^2/2\mu)]\varphi(p^T)=|P|\frac{e^2}{2\pi}\int\frac{1}{(p^T-l)^2}\delta(l\cdot P)\varphi(l)dl. \quad (3.1)$$

Generalizing the well-known Fock method, we shall now project stereographically the four-dimensional p space on a five-dimensional hyperboloid:

$$\xi_\mu=2ap_\mu/(a^2-p^2),\ \xi_4=(a^2+p^2)/(a^2-p^2),\ \mu=0,\cdots,3,$$

where $p^2=p_\mu p^\mu$ and a is an arbitrary constant. We then have $\xi_b\xi^b=\xi_\mu\xi^\mu-\xi_4\xi_4=-1$, where $b=0,\cdots,4$.

Using the ξ variables, Eq. (3.1) becomes

$$\left[|P|-m_1-m_2-\frac{a^2}{2\mu}\frac{1-\xi_4}{1+\xi_4}\right](1+\xi_4)^2\Psi_P(\xi)=|P|a(1+\xi_4)\frac{e^2}{2\pi}\int\frac{1}{(\xi-\eta)^2}\Psi_P(\eta)\delta(\eta_\mu P^\mu)\frac{d^4\eta}{\eta_4}, \quad (3.2)$$

$$\eta_4=(1+\eta_\mu\eta^\mu)^{1/2},\quad \varphi(p^T)=(1+\xi_4)^2\Psi_P(\xi),$$

and the variables ξ^μ must satisfy the condition $\xi_\mu P^\mu=0$. If we now set

$$a=[2\mu(m_1+m_2-|P|)]^{1/2}, \quad (3.2')$$

Eq. (3.2) reduces to the form

$$(a/\mu)\Psi_P(\xi)=-|P|\frac{e^2}{2\pi}\int\frac{1}{(\xi-\eta)^2}\Psi_P(\eta)\delta(\eta\cdot P)\frac{d^4\eta}{\eta_4}. \quad (3.3)$$

This equation with $\eta\cdot P=\eta_\mu P^\mu$ is the relativistic generalization of the well-known Fock integral equation on the hypersphere $\xi^2=1$. It has a symmetry $SO(3,1)$

[14] E. E. Salpeter, Phys. Rev. 87, 328 (1952).

[15] The choice $\Lambda_+\Lambda_+-\Lambda_-\Lambda_-=-1$ would have meant taking only negative-energy states for the system and would correspond to charge conjugation for the c.m. motion.

because of its relativistic covariance and, moreover, a symmetry $SO(4)$ represented by the stability group of P in the five-dimensional ξ space. We can put this equation into an $SO(4,1)$ covariant form if we add a new spacelike dimension to the total momentum space. By allowing rotations in the space so extended, P_μ will gain another component P_4, and in Eq. (3.3) we now have

$$\eta\cdot P=\eta_b P^b, \tag{3.4a}$$

$$|P|=(P_\mu P^\mu-P_4P_4)^{1/2}, \tag{3.4b}$$

while ξ_b satisfies

$$\xi_b P^b=0. \tag{3.4c}$$

For the physical interpretation we have to rotate back into the reference system where $P_4=0$.[16]

The Lie algebra of this $SO(4,1)$ group is clearly

$$M_{ab}=i\left(P_a\frac{\partial}{\partial P^b}-P_b\frac{\partial}{\partial P^a}\right)+i\left(\xi_a\frac{\partial}{\partial\xi^b}-\xi_b\frac{\partial}{\partial\xi^a}\right). \tag{3.5}$$

In Eq. (3.5) we call the first part, constructed with the P_b, "orbital" and the second part, built up with the ξ_b, the "spin part."

The Lorentz subalgebra $SO(3,1)$ of $SO(4,1)$ is defined by the generators

$$M_{\mu\nu}=L_{\mu\nu}+S_{\mu\nu}, \tag{3.6}$$

where

$$L_{\mu\nu}=i(P_\mu\partial/\partial P^\nu-P_\nu\partial/\partial P^\mu)$$

and

$$S_{\mu\nu}=i\left[\xi_\mu\frac{\partial}{\partial\xi^\nu}-\xi_\nu\frac{\partial}{\partial\xi^\mu}\right]=i\left[p_\mu\frac{\partial}{\partial p^\nu}-p_\nu\frac{\partial}{\partial p^\mu}\right],$$

i.e., the orbital part of the relative motion becomes the spin part of the c.m. motion, as it should. Furthermore, the generators of the Poincaré group are (3.6) and P_μ. Note that because of the subsidiary condition (3.4c) reflected also in the δ function in the integrand of (3.3), this equation is covariant only with respect to the simultaneous action of the orbital and spin part of $M_{\mu\nu}$.

Since Eq. (3.3) with (3.4c) is $SO(4,1)$ covariant, its eigensolutions corresponding to the invariant eigenvalues

$$|P|=m_1+m_2-\mu e^4/2n^2 \tag{3.7}$$

span a representation of the $SO(4,1)$ algebra with generators (3.5), with respect to which P^2 is invariant. In the c.m. system these solutions coincide with the well-known ones of the Schrödinger equation, obtained by Fock with the same substitution.

It was established some time ago[17,18] that these solutions in the c.m. system build up a basis for a particular representation of $SO(4,1)$, which is characterized by its decomposition with respect to the $SO(4)$ subgroup in a direct sum of representations having the second quadratic Casimir operator equal to zero. The "noncompact" generators of $SO(4,1)$ connect eigensolutions with different eigenvalues of the principal quantum number and the relative algebra was obtained[4,18] by mapping the four-dimensional sphere on a five-dimensional hypercone and considering the group of rotations on this space. In our case we could proceed analogously and take as the $SO(4)$ group of internal invariance the one leaving the direction parallel to P_b, in the ξ space, unchanged. We should then enlarge this group to an $SO(4,1)$ which keeps the condition $\xi\cdot P=0$ and whose generators connect different solutions of the integral equation (3.3). This would be the closest extension of the procedure given in Refs. 17 and 18 to our case.

Instead of constructing explicitly these algebras, we proceed by transforming Eq. (3.3) into a differential form in the ξ_a space, and the result is

$$(D^T+1)\Psi_P(\xi)=(\mu^2e^4/a^2)\Psi_P(\xi), \tag{3.8}$$

where D^T is the angular part of the four-dimensional Laplace operator in the hyperplane $\xi\cdot P=0$, which can be written as (see Appendix)

$$D^T=\tfrac{1}{2}S_{ab}S^{ab}-u_au^cS^{ab}S_{cb}, \tag{3.8'}$$

with $u_a=P_a/|P|$.

The spectrum of D^T is n^2-1, with $n=1, 2, \cdots$, independent of the direction of u^a, provided that $u^2=1$ (timelike P). In fact, D^T is nothing else than the ratio between the "Pauli-Lubansky" invariant W of $IO(4,1)$ and P^2, and because of this both D^T and its eigenvalues are Poincaré invariant for the system. In the c.m. system $[P\equiv(P_0,0)]$ Eq. (3.8) reduces to the equation of the four-dimensional spherical harmonics $\Psi_{P_0}(\xi)$. In an arbitrary system $\Psi_P(\xi)$ is obtained by a Lorentz transformation

$$\Psi_{P'}(\xi')e^{iP_\mu'X^\mu}=\exp[\tfrac{1}{2}i\vartheta_{\mu\nu}(L^{\mu\nu}+S^{\mu\nu})]\Psi_{P_0}(\xi)e^{iP_0t}. \tag{3.9}$$

The eigenvalues of $|P|$ obtained from (3.8), which coincide with the Balmer energy levels are obviously given by Eq. (3.7). Substituting the value of a^2 given by Eq. (3.2') into (3.8) and squaring the resulting equation, we obtain

$$(P^2-\mathcal{K}^2)\Psi_P=0, \tag{3.10}$$

with

$$\mathcal{K}=m_1+m_2-\mu e^4/2N$$

and N^2 is the operator D^T+1. Equation (3.10) has the formal aspect of a Klein-Gordon equation, and the operator $\mathcal{K}$ determines the spectrum of P^2.

We noted above that $\Psi_{P_0}(\xi_\alpha)$, $\alpha=1, \cdots, 4$, form a basis for a representation of $SO(4,1)$. With the definition (3.8'), the condition $\xi\cdot P=0$ is a restriction on the

[16] Alternatively, one can obtain the same final results if one considers P_4 as an arbitrary auxiliary parameter which is identically zero for the physical interpretation.

[17] A. O. Barut, P. Budini, and C. Fronsdal, Proc. Roy. Soc. (London) **A291**, 106 (1966); P. Budini, Nuovo Cimento 44, 363 (1966).

[18] P. Budini, Acta Phys. Austriaca Suppl. 4, 118 (1967).

solutions of Eq. (3.8) which, in itself, is defined even for $\xi\cdot P\neq 0$. We can now use the freedom we have obtained in introducing a function[19] $\Phi_n(\xi_\alpha,\xi_0)$, defined by the condition that $\Phi_n(\xi_\alpha,0)=\Psi_{P_0}(\xi_\alpha)$ (n being the eigenvalue of N corresponding to P_0) and we can span with these functions a representation of the spectrum-generating algebra $SO(4,1)$. Note that the $SO(3,1)$ subalgebra with generators $S_{\mu\nu}$ is just the spin part of the physical Lorentz group. The representations suitable for the description of the hydrogen levels are known to have[17] the quadratic invariant negative and the biquadratic invariant zero. This in fact ensures the correct decomposition with respect to $SO(4)$. We can realize this representation on the space of the homogeneous harmonic function of ξ_a with degree of homogeneity ν; in this case the invariant operator $F=\frac{1}{2}S_{ab}S^{ab}$ is equal to $\nu(\nu+3)$, so that we can have either $\nu=-\frac{3}{2}+i\rho$, ρ real, or $0>\nu>-3$.

In conclusion, we can write

$$[P^2-(m_1+m_2-\mu e^4/2N)^2]\Phi=0\,, \qquad (3.10')$$

with $N=(W/P^2+1)^{1/2}$, and this new equation is fully equivalent to Eq. (3.10) when $P^2>0$ (we will show that there are no solutions for $P^2\leqslant 0$).

4. DETERMINATION OF A MAJORANA-TYPE EQUATION

We may ask whether Eq. (3.10) is the "quadratic" form of a Majorana-type equation. We write, then, a general, $SO(4,1)$-covariant equation such as

$$(\Gamma_a P^a-\Omega)\Phi=0\,, \qquad (4.1)$$

where Ω is an $IO(4,1)$-invariant operator.

The $SO(4,1)$ algebra is already given by (3.5) and the condition for the $SO(4,1)$ covariance of the above equation is that the Γ matrices behave like components of a five-vector in $SO(4,1)$, i.e.,

$$[S_{ab},\Gamma_c]=i(g_{bc}\Gamma_a-g_{ac}\Gamma_b)\,.$$

The commutator $[\Gamma_a,\Gamma_b]$ is then an antisymmetric tensor in $SO(4,1)$; analogously with what happens in the original Majorana equation, we shall now postulate, for the Minkowsky tensor, $[\Gamma_\mu,\Gamma_\nu]=-iS_{\mu\nu}$. Then by $SO(4,1)$ covariance we are led to

$$[\Gamma_a,\Gamma_b]=-iS_{ab} \qquad (4.2)$$

in such a way that the S_{ab}, together with the Γ_c, build up an $SO(4,2)$ algebra. In this formulation the introduction of the $SO(4,2)$ group does not seem to be essential. It is only useful in defining the properties of the Γ-matrices.[20] To give a realization of the $SO(4,2)$ algebra,[21] we shall introduce the Beltrami coordinates $\xi_a=\pi_a/\pi_5$, with metric tensor $g_{AA}=+1,\ -1,\ -1,\ -1,\ -1,\ +1$, $A=0,\cdots 5$. In this way, we go from the hyperboloid $\xi^2+1=0$ to the cone $g_{AB}\pi^A\pi^B=0$.

In terms of these coordinates the algebra of $SO(4,2)$ is

$$S_{AB}=i(\pi_A\partial/\partial\pi^B-\pi_B\partial/\partial\pi^A) \qquad (4.3)$$

and its $SO(4,1)$ subalgebra coincides with the spin part of the algebra (3.5), while the new generators are

$$\Gamma_a=S_{5a}=i(\pi_5\partial/\partial\pi^a-\pi_a\partial/\partial\pi^5)\,.$$

A basis for these operators can be found in the homogeneous functions of π_A, with degree of homogeneity λ. In this way we can express the Γ_a in terms of the ξ as

$$\Gamma_a=-i[\xi_a(\lambda-\xi\cdot\partial/\partial\xi)-\partial/\partial\xi^a]\,. \qquad (4.4)$$

In order to show the equivalence of the "Majorana" equation (4.1) with the original equation (3.10′) we must obtain it by "squaring" (4.1) with a procedure analogous to that by which one gets the Klein-Gordon equation from the Dirac equation. To this end we fix the representation of $SO(4,2)$ by postulating the anticommutator[18]

$$\{\Gamma_a,\Gamma_b\}=\alpha g_{ab}+\beta g^{cd}\{S_{ac},S_{bd}\}\,. \qquad (4.5)$$

We can explicitly verify that this choice is compatible with the realization (4.4), provided that $\alpha=2\lambda$, $\beta=-1$, $\lambda=-1$, and, moreover, $\nu(\nu+3)=-2$. In fact the determination of λ defines the quadratic invariant of $SO(4,2)$ whose eigenvalue is $\lambda(\lambda+4)$, while the determination of ν fixes the representation of $SO(4,1)$; we see that this is one of the acceptable representations, since $F=-2$ (more precisely it belongs to the supplementary series).

We can now multiply Eq. (4.1) on the left by $\Gamma_a P^a+\Omega$, obtaining

$$(P^2+g^{ab}P^cP^dS_{ac}S_{bd}-\Omega^2)\Phi=0\,,$$

provided that

$$[\Gamma\cdot P,\Omega]=0\,.$$

Inserting $W=P^2F-g^{ab}P^cP^dS_{ac}S_{bd}$, which is the biquadratic invariant operator of $IO(4,1)$ (and remembering $F=-2$), we get

$$[P^2+W-\Omega^2]\Phi=0\,. \qquad (4.6)$$

[19] In what follows we shall indicate by Ψ the solution of Eq. (3.8) with constraint (3.4c) and by Φ the solution of the same equation without any restriction.

[20] If one would construct first the spectrum-generating algebra of the system at rest, one would find the group $SO(4,1)$ built up by $S_{\alpha\beta}$, and Γ_α given by the corresponding part of Eq. (4.4) (see Refs. 17 and 18). One could then generalize the problem and in this case the compact subgroup $SO(4)$ would go to the $SO(4,1)$ and the spectrum-generating algebra would become $SO(4,2)$. From this point of view it seems essential that Ψ span a representation of $SO(4,2)$ also.

[21] Another possibility would be to express the algebra with the method of Nambu (Ref. 3) in terms of creation and annihilation operators.

It is remarkable that the representation obtained is the extension to the (4+1) space of the Majorana one and it is identical with the representation on which the Nambu and Fronsdal equations are based. The operator Ω can now be explicitly calculated;

$$\Omega=(m_1+m_2)N-e^4\mu/2N. \tag{4.7}$$

Therefore, it is a function of W and P^2, but its P^2 dependence is only apparent, because N can always be expressed as a differential operator in the ξ subspace orthogonal to P and its eigenvalues are independent of the direction of P. We must still show that Ω commutes with $\Gamma\cdot P$, and in order to achieve this result it is sufficient to show that $[\Gamma\cdot P,W]=0$. In fact, both $\Gamma\cdot P$ and W are invariants of the complete $SO(4,1)$ group as given by Eq. (3.5). We can calculate the commutator in the rest frame, i.e., where $P\equiv(P_0,0)$; in this case $\Gamma\cdot P$ contains only Γ_0 while W contains only $S_{\alpha\beta}$, $\alpha,\beta=1,\cdots,4$, and so it is clear that the two operators commute. For Ω given by (4.7), Eq. (4.6) becomes identical with Eq. (3.10′). So, finally, the equation looked for is

$$[\Gamma\cdot P-(m_1+m_2)N+e^4\mu/2N]\Phi_P=0. \tag{4.8}$$

It has the mass spectrum (3.7) of Eq. (4.1) and both coincide with that of the hydrogen atom for $P_4=0$. The eigenfunctions Φ have infinite components. For a Lorentz transformation, Φ will be transformed like (3.9), where $S_{\mu\nu}=i[\Gamma_\mu,\Gamma_\nu]$. This will, in general, mix the indices of Φ, but the mass eigenvalue defined by the Pauli-Lubansky operator will remain invariant. There are no problems with the normalization condition $\Phi^*\Phi=1$, since Φ spans a unitary representation.

The negative-energy eigenstates obey an equation, formally identical with (4.8), that can be obtained by taking the projection operator (2.7) equal to -1.

We shall now show that, as expected, Eq. (4.8) does not admit spacelike solutions. In fact, for spacelike P_μ, the stability group of P_a is $SO(3,1)$ and, correspondingly, the eigenvalues of W/P^2 are $(\lambda+1)^2-1$, with λ complex and ranging from $-1-i\infty$ to $-1+i\infty$, since the $SO(4,1)$ representation considered, when decomposed with respect to $SO(3,1)$, contains only representations belonging to the principal series.[22]

In this case the mass spectrum is given by

$$P^2=[m_1+m_2-\mu e^4/2(\lambda+1)^2]^2, \tag{4.9}$$

with $(\lambda+1)^2$ continuous from 0 to $-\infty$; clearly this has no solution[23] for $P^2<0$.

If we wish to take into account the possibility $P^2=0$, we see very easily that it is impossible to have solutions for $W\neq0$. When $W=0$ we can very naturally consider the ratio W/P^2 as the "helicity" of the particle in the (4+1) space. So this quantity turns out to be quantized as $j(j+1)$, and the equation $\Omega^2\Phi=0$, to which Eq. (4.6) reduces, can have solutions only for particular values of the masses and of the charge; for the physical values of these parameters in our case (H atom) these solutions are certainly excluded.

Equation (4.9) for timelike P_μ represents the ionization spectrum for $(\lambda+1)^2$ going from $-\infty$ to 0. In fact, for $|P|>m_1+m_2$ the previously defined parameter a becomes imaginary and the same happens with $\xi_\mu\to i\xi_\mu$ while ξ_4 remains real. Then the spin group $SO(4,1)$ becomes $SO(3,2)$ and the stability group of timelike P_μ becomes $SO(3,1)$. The eigenvalue of N^2 in Eq. (4.6) becomes negative, with a range from 0 to $-\infty$, and Φ builds up a representation of $SO(3,2)$.

These eigenvalues and eigenfunctions can still be obtained from Eq. (4.8), where one has only to change the metric from $(+1,-1,-1,-1,-1)$ to $(+1,-1,-1,-1,+1)$.

5. CONCLUSION AND OUTLOOK

It has been shown that it is possible to give a relativistic description of the c.m. motion of a composite system with nonrelativistic internal motion. This possibility corresponds to the fact that if a system has a simple, nonrelativistic behavior for an observer at rest, the same must be true for a moving observer.

In particular, if the relative motion is soluble and the eigenstates of the internal motion at rest build up a representation of a noncompact group, as in the case of the hydrogen atom, then the relative motion can be completely algebraized and the c.m. motion can be described by a relativistic equation of the Majorana type. Therefore, it is not surprising that for the hydrogen atom, where the symmetry algebra is $SO(4)$ and the spectrum-generating algebra is $SO(4,1)$, the Majorana equation describing the relativistic behavior of the system as a whole acts on a space spanned by a representation of the algebra $SO(4,2)$.

This equation, then, represents a good example of how a radial internal quantum number of nonrelativistic origin can become a good index for a Poincaré transformation applied to the system by taking into account its covariance with respect to the $IO(4,1)$. This happens because that quantum number is connected with the Pauli-Lubansky operator of a group having the Poincaré group as a subgroup.

At this point one could think of introducing, into the found equations, perturbative effects which will break the symmetry. In particular, we could take into account the effect of considering the internal motion as relativistic. One should then, at the cost of simplicity, start some steps sooner in the derivation of the equation of motion and, for instance, include the spin of one or both of the constituents.

[22] A. Böhm, in Proceedings of the Summer Institute, University of Colorado, Boulder, 1967 (to be published).

[23] This can also be seen directly from Eq. (3.18). For spacelike P_μ it can be reduced to the form

$$[-\Gamma_1P_1-(m_1+m_2)N+e^4\mu/2N]\Phi=0,$$

which has no solutions, because Γ_1 has a real spectrum while N has an imaginary one.

In the example considered this will certainly break the $SO(4)$ degeneracy but, in general, insofar as the corrections come from covariant operators, they will always act inside the $SO(4,1)$ space. The final Majorana equation, if obtainable at all, will certainly be more complicated but the essence of the method will not be destroyed.

The equation that we obtained is obviously free from many of the known defects of the infinite-component equations postulated as a basis of a local-field theory. This is because the original equation from which one starts is free from these defects. Thus there are no solutions for spacelike or lightlike P_μ because they were not admitted by the original equation. For this reason, from such an equation it will be possible to derive a Green's function approach and obtain Regge poles for the scattering amplitude. The electromagnetic properties of the system will also be correct as far as they were correct in the original equation. Obviously, since the interaction of the composite system with an external electromagnetic field is essentially nonlocal, the equivalent Majorana equation will also interact nonlocally and contain other terms besides those dictated by the minimal electromagnetic interaction. We intend to examine this problem further, but we can anticipate that the electromagnetic vertex function for elastic scattering for such a system will contain form factors of kinematical origin (because the scattered wave Φ_{P+Q} is Lorentz-transformed with respect to the incoming Φ_P) and form factors of dynamical nature (due to the structure of the composite system).

From the example considered, it appears that for the hydrogen atom the Majorana-type equation is a good approximation for the motion of the system as a whole. One is then led to consider the hypothesis that the same applies also to other systems, such as the so-called elementary particles, where the Majorana equation seems to play a role.[7,8] It is interesting to think that a check of this hypothesis could come from the investigation of the electromagnetic form factor where both the kinematical and structural effects should, in this hypothesis, make a contribution. If this is the case and if there is some truth in the hypothesis that so-called elementary particles are composite systems, with nearly nonrelativistic internal motion, then the proposed approach could be of some help in their description.

[*Note added in manuscript.* We have obtained a relativistic infinite-dimensional equation with neither spacelike nor lightlike solutions. This seems to be in contradiction with what is stated in the "no-go theorem" by Streater and Grodsky.[24] In fact the contradiction is only apparent since our equation is and remains a wave equation and second quantization would not bring about local commutators of the type used in the no-go theorem. It is our opinion that this should be a general feature for composite systems whose fields obey complicated commutation relations as shown by Zimmermann.[10] Further, the P dependence of Ω, which brings our model outside the domain of validity of the no-go theorem, gives no trouble for the solution of our equation because in the rest system Ω is P-independent and its eigenvalues are Poincaré-invariant.

Leaving aside the question of the possibility of a second quantization, the use of equations like ours is to give Green's functions from which to proceed to an S-matrix theory by usual methods.]

[*Note added in proof.* Using Eq. (4.5) we can write $W=(\Gamma\cdot P)^2-P^2$. Introducing this relation in the expression for N and Ω, we can reduce Eq. (4.8) in the simpler form

$$[(\Gamma\cdot P)^2(|P|-m_1-m_2)+\tfrac{1}{2}e^4\mu P^2]\Phi_P=0,$$

But the cost of this simplification is that we have thus introduced solutions with $P^0<0$ when $|P|>m_1+m_2$.]

ACKNOWLEDGMENT

This paper was initiated during the 1967 International Heidelberg Conference. Two of the authors (P. B. and G. C.) take pleasure in expressing their gratitude to Professor J. H. D. Jensen for the hospitality offered to them at the Institut für theoretische Physik der Universität Heidelberg.

APPENDIX

In order to transform Eq. (3.3) into a differential equation we introduce the longitudinal component $\eta^L=\eta\cdot P/|P|$ and the corresponding four transversal components η^T and write (keeping in mind that, from the beginning, $\xi^L=0$)

$$\overset{a}{\underset{\mu}{-}}\Psi(\xi^T,0)=\frac{e^2}{2\pi}|P|\int\frac{1}{(\xi^T-\eta^T)^2}\Psi(\eta^T,\eta^L)\delta(\eta^L|P|)$$

$$\times\delta((\eta^L)^2-(\eta^T)^2+1)d\eta^L d^4\eta^T=\frac{e^2}{2\pi}\int\frac{1}{(\xi^T-\eta^T)^2}$$

$$\times\Psi(\eta^T,0)\delta(1-(\eta^T)^2)d^4\eta^T. \quad \text{(A1)}$$

It is now possible to express $(\xi^T-\eta^T)^{-2}$ in terms of the hyperspherical harmonics[25]:

$$(\xi^T-\eta^T)^{-2}=\sum_{nlm}\frac{1}{2n}\mathcal{Y}_{nlm}(\xi^T)\mathcal{Y}_{nlm}{}^*(\eta^T), \quad \text{(A2)}$$

where $(\xi^T)^2=(\eta^T)^2=1$. Thus, by iterating Eq. (A1)

[24] I. T. Grodsky and R. F. Streater, Phys. Rev. Letters **20**, 695 (1968).

[25] J. Schwinger, J. Math. Phys. **5**, 1606 (1964).

and using (A2), we obtain

$$\Psi(\xi^T,0)=\left(\frac{e^2\mu}{2\pi a}\right)^2\int\int\frac{1}{(\xi^T-\eta^T)^2}\cdot\frac{1}{(\eta^T-\zeta^T)^2}\delta(1-(\eta^T)^2)\delta(1-(\zeta^T)^2)\Psi(\zeta^T)d^4\eta^T d^4\zeta^T$$

$$=\left(\frac{e^2\mu}{2\pi a}\right)^2\int\int\tfrac{1}{2}(\eta^T)^2d(\eta^T)^2d\Omega_\eta\tfrac{1}{2}(\zeta^T)^2d(\zeta^T)^2d\Omega_\zeta\delta(1-(\eta^T)^2)\delta(1-(\zeta^T)^2)$$
$$\times\sum_{nlm}\frac{1}{2n}\mathcal{Y}_{nlm}(\xi^T)\mathcal{Y}_{nlm}{}^*(\eta^T)\sum_{\nu\lambda\mu}\frac{1}{2\nu}\mathcal{Y}_{\nu\lambda\mu}(\eta^T)\mathcal{Y}_{\nu\lambda\mu}{}^*(\zeta^T)\Psi(\zeta^T,0)$$

$$=\left(\frac{e^2\mu}{2\pi a}\right)^2\int\sum_{nlm}\frac{1}{4n^2}\mathcal{Y}_{nlm}(\xi^T)\mathcal{Y}_{nlm}{}^*(\zeta^T)\Psi(\zeta^T,0)d\Omega_\zeta. \tag{A3}$$

Now, applying the operator D^T+1 and using the completeness of the $\mathcal{Y}$ functions,

$$\sum_{nlm}\mathcal{Y}_{nlm}(\Omega)\mathcal{Y}_{nlm}{}^*(\Omega')=\delta(\Omega-\Omega'),$$

we obtain, finally,[26]

$$(D^T+1)\Psi(\xi^T,0)=\tfrac{1}{4}(e^2\mu/2\pi a)^2\Psi(\xi^T,0). \tag{A4}$$

Concerning the operator D^T, it can be clearly obtained from the transverse Laplace operator

$$\Delta^T=\sum_\alpha\frac{\partial}{\partial\xi_\alpha{}^T}\frac{\partial}{\partial\xi_\alpha{}^T}.$$

We can now express D^T by using the generators of the internal $SO(4,1)$. The detailed calculations can be performed through a tetrad of spacelike unit vectors $e_a{}^{(\alpha)}$ orthogonal to u_a, such that

$$\sum_1^4{}_\alpha e_a{}^{(\alpha)}e_b{}^{(\alpha)}=g_{ab}-u_au_b. \tag{A5}$$

We now introduce

$$S_{\alpha\beta}{}^T=-i(\xi_\alpha{}^T\partial/\partial\xi_\beta{}^T-\xi_\beta{}^T\partial/\partial\xi_\alpha{}^T)$$

in terms of which D^T reads $S_{\alpha\beta}{}^TS_{\alpha\beta}{}^T$, and by using the unit vectors $e_a{}^{(\alpha)}$, we express $S_{\alpha\beta}{}^T$ as $S_{\alpha\beta}{}^T=e_a{}^{(\alpha)}S^{ab}e_b{}^{(\beta)}$, which can be easily obtained by noting that $\xi_\alpha{}^T=\xi^ae_a{}^{(\alpha)}$ and $\partial/\partial\xi_\beta{}^T=e_b{}^{(\beta)}\partial/\partial\xi_b$. By the completeness relation (A5) we get

$$D^T=\tfrac{1}{2}S_{ab}S^{ab}-u_au^cS^{ab}S_{cb}. \tag{A6}$$

Once D^T has been expressed in this way, we can extend the definition of Eq. (A4) to every ξ, simply writing[27]

$$(D^T+1)\Phi(\xi)=(e^2\mu/4\pi a)^2\Phi(\xi). \tag{A4'}$$

It is easily seen that the operator D^T does not commute with the S_{ab}, which therefore connects states corresponding to different eigenvalues, but it commutes with the invariant operators of $SO(4,1)$, so that it can be diagonalized together with them. So, making the correspondence $\Psi_{P_0}(\xi_\alpha,0)\rightarrow\Phi_n(\xi_\alpha,\xi_0)$, we can use the Φ_n as a basis for the $SO(4,1)$ of spin, which will transform the $\Phi_n(\xi)$ into a linear combination $\Phi_n'(\xi)=\sum_r c_{nr}\Phi_r(\xi)$; this transformation corresponds to the transformation $\Psi_n\rightarrow\Sigma_r c_{nr}\Psi_r$. Now it is sufficient to choose a correct representation of $SO(4,1)$ [i.e., having the desired decomposition with respect to $SO(4)$] in order to obtain the final result that the spin group is the noncompact group connecting all the different solutions of Eq. (A4').

[26] Note that the usual formulas of nonrelativistic quantum mechanics are obtained after the substitution $e^2/4\pi\rightarrow e^2$, since here we have the Lorentz-Heaviside system of electrical units instead of the Gaussian system; this has been done in Eq. (3.8).

[27] Even here we denote by Φ the wave function without restriction on the ξ.