PHYSICAL REVIEW VOLUME 172, NUMBER 5 25 AUGUST 1968

# Paraquarks, Structural Saturation, and Hard Core

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(Received 5 February 1968; revised manuscript received 8 April 1968)

This investigation, which is concerned with the effect of parastatistics for more than one baryon, has a twofold objective, viz., to show that (i) saturation of baryon energy levels and (ii) short-range repulsion between baryons can *both* be thought of as consequences of parastatistics symmetry in a multiquark system. An explicit method of construction of the multiquark wave functions of appropriate symmetry is outlined. The saturation of baryon energy levels is also found to depend crucially on a relation of the form $V_s=-2V_a$, where $V_s$ and $V_a$ are, respectively, the even and odd parts of the quark-quark potential. The problem of short-range repulsion is investigated with the help of a two-quark model of baryons, via a four-particle nonrelativistic Schrödinger equation for a [2, 2] symmetric wave function characteristic of parastatistics symmetry. An explicit solution for the baryon-baryon amplitude is obtained with the help of a factorable, *s*-wave interaction between the quarks and brings out clearly the effect of structural repulsion.

## 1. INTRODUCTION

The usefulness, if any, of the concept of parastatistics[1] in the quark model[2] has been intimately linked with the success of the **56** representation for the familiar octet and decuplet of baryon states, which gives Fermi statistics a very unattractive look because no reasonable force mechanism seems to produce a ground-state antisymmetric ($A$) quark wave function that goes with it. Symmetrical ($S$) quark wave functions for individual baryons provide *a priori* a much better basis for even a limited dynamical understanding of the **56** baryons as states of lowest energy.[3] This statement can be substantiated considerably by more explicit dynamical calculations[4,5] as well. Symmetrical wave functions are also favored by other, less dynamical, considerations like the behavior of baryon form factors.[6–8] Parastatistics represents one of the simplest possibilities for realizing symmetrical wave functions for individual baryons as $3Q$ composites, without having to extend the Gell-Mann–Zweig version of the quark model either by adding more quarks or by introducing additional quantum numbers.[9]

[1] H. S. Green, Phys. Rev. **90**, 270 (1953).
[2] O. W. Greenberg, Phys. Rev. Letters **13**, 598 (1964).
[3] See, e.g., R. H. Dalitz, in *Proceedings of the Oxford International Conference on Elementary Particles, 1965* (Rutherford High-Energy Laboratory, Chilton, Berkshire, England, 1966).
[4] A. N. Mitra, Phys. Rev. **151**, 1168 (1966).
[5] A. N. Mitra, Ann. Phys. (N. Y.) **43**, 126 (1967).
[6] A. N. Mitra and R. Majumdar, Phys. Rev. **150**, 1194 (1966).
[7] R. H. Dalitz, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967), p. 215.
[8] R. Kreps and J. deSwart, Phys. Rev. **162**, 1729 (1967) have reached a different conclusion by considering extremely complicated wave functions with a very singular behavior at the origin. However, we believe that the essential physics is considerably obscured by such artificial-looking wave functions, whose physical interpretation does not seem at all clear.
[9] For a list of references on extended quark models, see Ref. 5.

The next question that arises after the symmetries of individual baryon states is: What kind of symmetry, or symmetries, does parastatistics imply for a system consisting of *more* than one baryon? While for a single baryon the effect of parastatistics is almost indistinguishable from that of Bose statistics, the situation is entirely different as soon as the number of baryons exceeds unity. Indeed, the distinction between the Bose, Fermi, and para- forms of statistics is most easily visualized in terms of Young diagrams. While Bose and Fermi statistics, respectively, correspond to single-row and single-column structures in the basic units, one would expect the intermediate Young patterns or symmetries to have a suitable correspondence with other alternative forms of statistics, and of course parastatistics is general enough to accommodate the different varieties. Since, on the other hand, an increasing number of "boxes" gives rise to a rapidly increasing variety of Young patterns or symmetries, it is clearly necessary to restrict this number through additional selection rules. Indeed, the kind of quark symmetry associated with parastatistics that was proposed by Greenberg[2] was based essentially on such additional considerations. Thus he found that symmetrical $3Q$ functions for single baryons were the result of the requirement that the composite operators representing symmetrized products of three paraquark operators must satisfy anticommutation relations. This last condition is of course mandatory, since Fermi statistics for baryons as $3Q$ composites is not negotiable.

The above prescription provides the distinguishing criterion for the special type of parastatistics that is relevant to the multibaryon situation. It tells us, e.g.' that the Young pattern for a two-baryon system should be [3,3], in contrast to Bose statistics which gives the pattern [6]. It is this alternative [3,3] symmetry which parastatistics is able to provide, that physically acts as the main safeguard against collapse or condensation which would otherwise occur if quarks were allowed to be mere bosons. Further, since three boxes are the maximum number allowed in a row under parastatistics of order 3, one should physically expect energy saturation at a three-quark level for each baryon. It is thus clear that the most energetically favorable Young pattern for an $n$-baryon system under parastatistics is a column of $n$ rows, each row having three boxes (or quarks).

For multiquark systems with quark numbers $3n\pm1$, further prescriptions are necessary for obtaining the relation between parastatistics and Young patterns. For example, one possibility is to invoke the condition of maximum symmetry subject to the limitations of parastatistics, which does not allow more than three quarks in a given symmetric state. To put this requirement in a concrete mathematical form, it is probably most convenient to use an artifice first suggested by Katayama *et al.*,[10] viz., to express the paraquark operators ($q_\lambda$) in terms of certain fermion operators $f_{\lambda\alpha}$ and Pauli-type matrices $\omega_\alpha$ ($\alpha=1,2,3$), via the relation

$$q_\lambda=\sum_{\alpha=1}^{3}\omega_\alpha f_{\lambda\alpha}\,, \tag{1.1}$$

instead of the more conventional relation[2]

$$q_\lambda=\sum_{\alpha=1}^{3}t_{\lambda\alpha}\,, \tag{1.2}$$

where the suboperators $t_{\lambda\alpha}$ are supposed to obey the standard parastatistics relations. Now while the deeper physical interpretations of the two representations could be very different,[11] they are largely equivalent from the (more limited) point of view of constructing symmetries in multiquark wave functions consistent with parastatistics. In practice, the representation (1.1), in which $f_{\lambda\alpha}$'s are fermion operators, is a very convenient mathematical device for describing the symmetries generated by parastatistics, without having to use any ideas other than the familiar Fermi statistics (antisymmetrical wave functions). The price that one pays for this simplification is of course the introduction of an extra degree of freedom represented by the space of the $\omega$ matrices. In terms of Young patterns this means that a totally antisymmetric (singlet) structure in *all* the variables, including the $\omega$ matrices, is obtainable as a product of two conjugate Young patterns, one for the physical degrees of freedom (momenta, spin, and unitary spin) and the other for the fictitious $\omega$ degrees of freedom. Thus the $\omega$ matrices may be regarded as playing the role of "spurions" which are *formally* being used to construct antisymmetric wave functions.[12] In Sec. 2, we shall give a more elaborate description of this formalism, following the treatment of Katayama *et al.*,[10] and use it to obtain the Young patterns for multiquark systems. We shall also derive expressions for the energies of such systems up to $N=6$, so as to bring out explicitly how saturation is achieved for $N=3n$.

A second question which is closely related to the saturation of levels with $3n$ paraquarks is the problem of short-range repulsion between the baryons, as a consequence of [3,3] symmetry in the two-baryon wave function. Similar ideas have been discussed by various authors[13] using analogies with the $\alpha$-$\alpha$ model[14] through variational wave functions. We wish to show in this paper that parastatistics also provides a very convenient and practical framework for such a mechanism. To simplify the discussion, we shall consider a paraquark model of order 2, rather than the more realistic model of order 3. We shall show in Sec. 3 that a simple attractive $s$-wave force between each quark pair can lead to the idea of structural repulsion between two composites of two quarks each, provided that one considers a $4Q$ system of [2,2] symmetry. For this purpose we shall use a model four-body formalism in which the $Q$-$Q$ force is assumed separable, on the lines of a similar formalism developed a few years ago in connection with a model four-pion system.[15] The more realistic case of a two-baryon system, looked upon as a $6Q$ problem of [3,3] symmetry, should probably not contain any essentially new physical features, though we may well miss many finer details as a result of approximating the $6Q$ system by a $4Q$ one. In Sec. 4, we summarize our main conclusions.

## 2. PARAQUARK SYMMETRIES FOR $NQ$ SYSTEMS

We start by summarizing the essential features of parastatistics expressed in terms of creation and annihilation operators[2] and then present the alternative formulation of KUY[10] which maintains the same type of symmetries in the $N$-quark wave functions as generated by the former in conventional form.

Parastatistics of order 3 is expressed in terms of the operators[2,16]

$$q_\lambda=\sum_{\alpha=1}^{3}q_\lambda{}^{(\alpha)}\,,\quad q_\lambda{}^\dagger=\sum_{\alpha=1}^{3}q_\lambda{}^{(\alpha)\dagger}\,, \tag{2.1}$$

[10] Y. Katayama, I. Umemura, and E. Yamada, Progr. Theoret. Phys. (Kyoto) Suppl., Yukawa No., 1965, hereafter referred to as KUY.

[11] For example, (1.1) predicts integral charges associated with the subquarks $f_{\lambda\alpha}$ in contrast to (1.2); see Ref. 10.

[12] The motivation is analogous, e.g., to the use of spurions for the construction of formally $SU(3)$-invariant structures even for $SU(3)$-violating (weak) interactions which are characterized by the selection rules.

[13] See, e.g., S. Otsaki, R. Tamagaki, and M. Yasuno, Progr. Theoret. Phys. (Kyoto) Suppl., Yukawa No., 1965, p. 578; and earlier references cited therein.

[14] H. Margenau, Phys. Rev. **59**, 37 (1941).

[15] A. N. Mitra and S. Ray, Phys. Rev. **137**, B982 (1965), hereafter referred to as MR.

[16] Our notation is slightly different from that of Ref. 2.

where

$$[q_\lambda^{(\alpha)}, q_\mu^{(\alpha)\dagger}]_+ = \delta_{\lambda\mu}, \tag{2.2}$$

$$[q_\lambda^{(\alpha)}, q_\mu^{(\beta)\dagger}]_- = 0, \quad \alpha \neq \beta, \tag{2.3}$$

and other pairs commute and anticommute according to whether $\alpha \neq \beta$ or $\alpha = \beta$, respectively. In this picture, a symmetrical combination of the form[2]

$$B_{\lambda\mu\nu}^\dagger = [[q_\lambda^\dagger, q_\mu^\dagger]_+, q_\nu^\dagger]_+ = 4 \sum_{\alpha\beta\gamma} q_\lambda^{(\alpha)\dagger} q_\mu^{(\beta)\dagger} q_\nu^{(\gamma)\dagger} \tag{2.4}$$

can be shown to satisfy anticommutation relations with similar objects, and this fact helps identify such composites as fermion operators appropriate for baryon states. Structures similar to (2.1) and (2.4) hold for antiquark operators ($r_\lambda$) and antibaryon operators ($\bar{B}$), respectively.

In the KUY picture, on the other hand, one uses the representation

$$q_\lambda = f_{\lambda\alpha}\omega_\alpha, \quad \text{(summation implied over } \alpha) \tag{2.5}$$

where the $\omega_\alpha$ ($\alpha = 1,2,3$) are Pauli spin matrices satisfying the relation

$$\omega_\alpha \omega_\beta = \delta_{\alpha\beta} + i\epsilon_{\alpha\beta\gamma}\omega_\gamma. \tag{2.6}$$

The operators $f_{\lambda\alpha}$ in this formalism are ordinary fermion operators satisfying

$$[f_{\lambda\alpha}, f_{\mu\beta}^\dagger]_+ = \delta_{\alpha\beta}\delta_{\lambda\mu}, \tag{2.7}$$

$$[f_{\lambda\alpha}, f_{\mu\beta}]_+ = [f_{\lambda\alpha}^\dagger, f_{\mu\beta}^\dagger]_+ = 0. \tag{2.8}$$

Similarly, for the antiquark operators $r_\lambda$ one has

$$r_\lambda = \omega_\alpha g_{\lambda\alpha}, \tag{2.9}$$

where the $g_{\lambda\alpha}$'s satisfy relations similar to (2.7) and (2.8). Since the theory is nonrelativistic, the quarks and antiquarks are entirely different objects, so that the vacuum has a dual structure

$$\Phi_0 = \begin{pmatrix} \Phi_0^{(+)} \\ \Phi_0^{(-)} \end{pmatrix}, \tag{2.10}$$

where $\Phi_0^{(\pm)}$ refers to the vacuum state with a two-valued degree of freedom associated with the Pauli matrices $\omega_\alpha$. In this theory, the states which satisfy ordinary statistics are, e.g.,

$$B_{\lambda\mu\nu}^\dagger \Phi_0^{(+)} \quad \text{or} \quad \bar{B}_{\lambda\mu\nu}^\dagger \Phi_0^{(-)}, \tag{2.11}$$

which according to (2.4) are given by

$$B_{\lambda\mu\nu}^\dagger \Phi_0^{(+)} = 4i\epsilon_{\alpha\beta\gamma} f_{\lambda\alpha}^\dagger f_{\mu\beta}^\dagger f_{\nu\gamma}^\dagger \Phi_0^{(+)}, \tag{2.12}$$

$$\bar{B}_{\lambda\mu\nu}^\dagger \Phi_0^{(-)} = 4i\epsilon_{\alpha\beta\gamma} g_{\lambda\alpha}^\dagger g_{\mu\beta}^\dagger g_{\nu\gamma}^\dagger \Phi_0^{(-)}. \tag{2.13}$$

As can be seen from these expressions, such states are characterized by (i) the absence of the $\omega$ matrices, and (ii) complete separation of the two vacuum states from each other.

The difference of this formulation from that of usual parastatistics is that quarks by themselves are not eigenstates of charge, so that the question of whether or not fractional charges like $\frac{2}{3}$, $-\frac{1}{3}$, etc., are true eigenvalues is not relevant in this theory. On the other hand, baryon and antibaryon states like (2.12) and (2.13), as well as meson states like

$$[q_\mu^\dagger, r_\nu^\dagger]_- \Phi_0 = 2 f_{\mu\alpha}^\dagger g_{\nu\alpha}^\dagger \Phi_0, \tag{2.14}$$

are eigenstates of integral charge and hypercharge. Even the subquark states like $f_{\mu\alpha}^\dagger \Phi_0$ have integral charges.

As was already mentioned in Sec. 1, the deeper implications of these two formulations may well be very different, yet from the (limited) point of view of obtaining the correct symmetries in the quark wave functions, the two formulations are totally equivalent. From the practical point of view, the latter is of course preferable since it gives a means of generating these symmetries via the (extended) $\omega$ degrees of freedom which are now available for explicit construction of totally antisymmetric wave functions in the "Fock space" spanned by the operators $f_{\mu\alpha}, g_{\nu\beta}$. The Fock amplitudes, which must be fully antisymmetrized products (singlets in the Young patterns) of the physical wave functions depending on momentum, spin, and $SU(3)$ spin, and the (fictitious) functions of the $\omega$ matrices, now give a means of deducing the Young patterns for the physical wave functions as conjugates of the corresponding patterns generated by the products of the $\omega$ matrices. For example, with a fully antisymmetric ($A$) product $\epsilon_{\alpha\beta\gamma}$ of three $\omega$ factors, one must associate a fully symmetric ($S$) physical wave function. Similarly, with a mixed symmetric ($M$) product like $\delta_{\alpha\beta}\omega_\gamma$, one must have a physical wave function of [2,1] symmetry.

To extend this idea to symmetries involving larger numbers of quarks, one clearly requires additional selection rules. The natural selection is of course governed by the consideration that the state of minimum energy for a given multiquark configuration be one with the maximum possible symmetry subject to the restriction of not having more than three boxes in a row. This immediately tells us that for a $3n$-quark configuration one must pick from the expression

$$\omega_{\alpha_1}\omega_{\alpha_2}\cdots\omega_{\alpha_{3n}} \tag{2.15}$$

a term which depends only on alternating tensors like $\epsilon_{\alpha\beta\gamma}$. Since the number of such factors is exactly $n$ in this case, one obtains a term like

$$\epsilon_{123}\epsilon_{456}\cdots\epsilon_{3n-2,3n-1,3n}, \tag{2.16}$$

which goes with the physical state of maximum symmetry consistent with the constraints of parastatistics. Thus for $3n$-quark configurations, the selection rule is uniquely prescribed by the above choice. For $(3n+1)$- or $(3n+2)$-quark configurations on the other hand, one cannot fulfill the above requirement in a maximal manner, since either a single $\omega_\alpha$ (for $3n+1$) or a product like $\omega_\alpha\omega_\beta$ (for $3n+2$) will be left unassociated. Thus in such cases one expects somewhat smaller symmetry in the physical wave functions. For a $(3n+1)$-quark configuration, the single $\omega$ factor must give rise to an additional box in the first column of an otherwise filled Young pattern of $3n$ boxes with three boxes in each row. For a $(3n+2)$-quark configuration, we have a choice of two terms in the expansion (2.6). The first term $\delta_{\alpha\beta}$ corresponds to a Young pattern with $(n+2, n, n)$ boxes in the three respective columns, while the second one $i\epsilon_{\alpha\beta\gamma}\omega_\gamma$ corresponds to the numbers $(n+1, n+1, n)$. Since the second pattern has greater symmetry, we supplement our selection rule by the prescription that for a product of two $\omega$ matrices the appropriate term to pick is $i\epsilon_{\alpha\beta\gamma}\omega_\gamma$ and *not* $\delta_{\alpha\beta}$.

The above selection rule would have been intuitively obvious, since the object is to obtain Young patterns with the maximum number of filled rows. The advantage of the KUY formalism is that it provides a practical means of automatically ensuring this objective, via the $\omega$ matrices. The essential point is to sort out as many antisymmetric triplets like $\epsilon_{\alpha\beta\gamma}$ as are available.

The energies for the various multiquark configurations can be simply worked out in terms of two-body forces (nonrelativistic). Thus if we assume that the $Q$-$Q$ potential has an even part $V_s$ (operative in spatially symmetric states) and an odd part $V_a$ (operative in spatially antisymmetric states), the respective energies [taking account of the quark mass $(Q)$] from $N=1$ to $N=6$ are given by

$$E_1=Q, \tag{2.17}$$

$$E_2=V_s+2Q=\tfrac{1}{3}B+Q, \tag{2.18}$$

$$E_3=3V_s+3Q=B, \tag{2.19}$$

$$E_4=2V_a+4V_s+4Q=\tfrac{4}{3}B+2V_a, \tag{2.20}$$

$$E_5=6V_s+4V_a+5Q=2B-Q+4V_a, \tag{2.21}$$

$$E_6=9V_s+6V_a+6Q=3V_s+6V_a+2B. \tag{2.22}$$

Here we have introduced the baryon mass $B$ for convenience. It is clearly seen that if $Q\gg B$, then $E_1\sim E_2\gg E_3$. Further, if we assume the empirical result that $E_6\approx 2E_3$ (as in the deuteron), then we have approximately

$$V_s\approx -2V_a<0, \tag{2.23}$$

which tells us that the even potential is about twice as attractive as the odd one is repulsive. Using (2.23), we have

$$E_4=B+Q=E_1+B, \tag{2.24}$$

$$E_5=\tfrac{4}{3}B+Q=E_2+B, \tag{2.25}$$

$$E_6=2B. \tag{2.26}$$

The list, which could clearly be extended beyond $N=6$, shows how saturation of levels for $N=3n$ can be brought about through the peculiar symmetries generated by parastatistics. This result also depends crucially on the empirical relation (2.23), which, however, we have no direct means of establishing through more fundamental considerations. However, if we accept (2.23), we get the general results

$$E_{3n}\approx nB, \tag{2.27}$$

$$E_{3n+1}\approx nB+Q, \tag{2.28}$$

$$E_{3n+2}\approx nB+\tfrac{1}{3}B+Q. \tag{2.29}$$

This result for $E_{3n}$ is in accord with what we know of the nuclear energy levels, provided that we remember that the nuclear binding energies are too small compared to their rest masses. This fact, in a way, has been incorporated in the picture of quark-quark forces essentially through a relation of form (2.23); the (small) differences between various nuclear masses from their mass numbers should probably be interpreted as manifestations of small violations of this relation, and presumably also of various nonadditive corrections to the above expressions for the energies.

## 3. MODEL CALCULATION FOR THE HARD CORE

We discuss here a simple model, which is designed to bring out how a short-range repulsion between two baryons is related to the parastatistics symmetry of the multiquark wave function representing the two-baryon state. As stated earlier, we shall for this purpose consider the paraquarks to be of order 2, rather than 3, so that the two-baryon or four-quark wave function has [2,2] symmetry under parastatistics. It is clearly sufficient to ignore the spin and $SU(3)$ degrees of freedom and consider only the orbital structure, since the case of physical interest is one in which the single baryon wave function in these variables is *symmetric* (counterpart of the **56** representation). The main question is one of [2,2] symmetry in the orbital wave function for a four-quark system, so that for the purpose of this investigation the quarks may be considered to be spinless and $SU(3)$-singlet objects, but obeying parastatistics.

The kinematics of the nonrelativistic $4Q$ system can be developed on lines similar to Ref. 15. We shall also make free use of several results of MR, both in respect of the formalism as well as details of algebraic calculations whenever applicable to the present situation.

In the notation of MR, the momenta $\mathbf{P}_i$ of the quarks have the c.m. constraint

$$\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3+\mathbf{P}_4=0. \tag{3.1}$$

For the [2,2] symmetry of the symmetric group, the various permutation matrices $P(ij)$ admit of a $2\times 2$

representation[15,17] which we take as

$$P(12)=P(34)=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \tag{3.2}$$

$$P(23)=P(14)=\begin{pmatrix}\frac{1}{2} & +\frac{1}{2}\sqrt{3}\\ +\frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix}, \tag{3.3}$$

$$P(31)=P(24)=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\sqrt{3}\\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix}. \tag{3.4}$$

We note that $P(ij)$ and $P(k4)$ have identical representations, where $(ijk)$ are cyclic permutations of (1,2,3). The "vector" for the above representation is of the form

$$\begin{pmatrix}\psi'\\ \psi''\end{pmatrix}, \tag{3.5}$$

where the wave functions $(\psi',\psi'')$ are, respectively, antisymmetric and symmetric in the indices 1 and 2 (as well as 3 and 4). The construction of these functions can be made as follows.

Define the momentum variables

$$2\mathbf{p}_{ij}\equiv\mathbf{P}_i-\mathbf{P}_j\,,\quad 2\mathbf{p}_{k4}\equiv\mathbf{P}_k-\mathbf{P}_4\,, \tag{3.6}$$

$$2\mathbf{Q}_k\equiv\mathbf{P}_i+\mathbf{P}_j-\mathbf{P}_k-\mathbf{P}_4=2(\mathbf{P}_i+\mathbf{P}_j)\,, \tag{3.7}$$

where $(ijk)$ are cyclic permutations of (1,2,3). In terms of these variables, the kinetic energy $K$ can be expressed as

$$MK=\tfrac{1}{2}(P_1^2+P_2^2+P_3^2+P_4^2)=p_{ij}^2+p_{k4}^2+\tfrac{1}{2}Q_k^2\,. \tag{3.8}$$

The following crossing relations between the various momenta may also be noted:

$$\mathbf{p}_{31},\mathbf{p}_{24}=\mp\tfrac{1}{2}\mathbf{Q}_3-\tfrac{1}{2}(\mathbf{p}_{12}-\mathbf{p}_{34})\,, \tag{3.9}$$

$$\mathbf{p}_{23},\mathbf{p}_{14}=\tfrac{1}{2}\mathbf{Q}_3\mp\tfrac{1}{2}(\mathbf{p}_{12}+\mathbf{p}_{34})\,, \tag{3.10}$$

$$\mathbf{Q}_1,\mathbf{Q}_2=\mp\mathbf{p}_{12}+\mathbf{p}_{34}\,. \tag{3.11}$$

The $4Q$ wave functions depend on three independent relative momentum coordinates which may be taken as $\mathbf{p}_{ij}$, $\mathbf{p}_{k4}$, and $\mathbf{Q}_k$ in any cyclical form, but which can all be expressed, via (3.9)–(3.11), in terms of the set $(\mathbf{p}_{12},\mathbf{p}_{34},\mathbf{Q}_3)$ alone. If we now define the functions

$$A_k=f(\mathbf{p}_{ij},\mathbf{p}_{k4},\mathbf{Q}_k)\,, \tag{3.12}$$

$$B_k=f(\mathbf{p}_{k4},\mathbf{p}_{ij},\mathbf{Q}_k)\,, \tag{3.13}$$

the pair $(\psi',\psi'')$ should have the structure[15]

$$\psi'=\tfrac{1}{4}\sqrt{3}(A_2+B_2-A_1-B_1)\,, \tag{3.14}$$

$$\psi''=-\tfrac{1}{2}(A_3+B_3)+\tfrac{1}{4}(A_2+B_2+A_1+B_1)\,. \tag{3.15}$$

The dynamics governing the structure of the $(A_k,B_k)$ functions is expressed by the Schrödinger equation

$$D_E(\mathbf{P}_i)\begin{pmatrix}\psi'\\ \psi''\end{pmatrix}=-M(V_{12}+V_{23}+V_{31}+V_{14}+V_{24}+V_{34})\begin{pmatrix}\psi'\\ \psi''\end{pmatrix}, \tag{3.16}$$

where

$$D_E(\mathbf{P}_i)=\tfrac{1}{2}(P_1^2+P_2^2+P_3^2+P_4^2)-EM \tag{3.17}$$

and the pairwise interaction $V_{ij}$ has the representation

$$\langle\mathbf{P}_i\mathbf{P}_j|V_{ij}|\mathbf{P}_i'\mathbf{P}_j'\rangle=\delta(\mathbf{P}_{ij}-\mathbf{P}_{ij}')\langle\mathbf{p}_{ij}|V_{ij}|\mathbf{p}_{ij}'\rangle\,, \tag{3.18}$$

$$\mathbf{P}_{ij}=\mathbf{P}_i+\mathbf{P}_j\,. \tag{3.19}$$

The total energy $E$ of the system depends of course on the boundary condition of the problem. Thus for our present problem of baryon-baryon scattering through $Q$-$Q$ forces, the appropriate boundary condition is one in which the bound state of two quarks scatters against a similar object. If $\pm\mathbf{k}$ are the c.m. momenta of these composites at infinite separation, then one has

$$E=k^2m_B^{-1}-2\alpha_B^2M^{-1}\,, \tag{3.20}$$

where $m_B$ is the baryon mass given by

$$m_B=2M-\alpha_B^2M^{-1} \tag{3.21}$$

and $\alpha_B^2M^{-1}$ is the binding energy for each baryon.

Now while the usual quark model requires $\alpha_B^2M^{-1}$ to be comparable with $M$,[18] such a condition appears to be too stringent in our two-quark model of the baryon which puts a much heavier strain on the strength of the $Q$-$Q$ force than does a three-quark model. Indeed, an inspection of the expression (3.17) in terms of (3.8) and (3.20), viz.,

$$D_E(\mathbf{P}_i)=(p_{ij}^2+\alpha_B^2)+(p_{k4}^2+\alpha_B^2)+\tfrac{1}{2}(Q_k^2-2Mk^2m_B^{-1})\,, \tag{3.22}$$

shows that at the baryon poles defined by

$$p_{ij}^2=-\alpha_B^2\,,\quad p_{k4}^2=-\alpha_B^2\,, \tag{3.23}$$

the separation momentum $\mathbf{Q}_k$ has a pole

$$Q_k^2=2Mk^2m_B^{-1}\,. \tag{3.24}$$

Since, on the other hand, this pole should occur at $Q_k^2=k^2$, one must demand that

$$2M\approx m_B\,, \tag{3.25}$$

which implies that the binding energy is small compared to the rest energy.

To probe further into the structure of the functions $(\psi',\psi'')$, one must now use a specific model of the interaction. Since we are mainly interested in the symmetric wave functions of single baryons, and since we know that this is most easily brought about by an attractive $s$-wave force,[5] we shall assume the same in this investigation. To simplify the problem further we shall consider

[17] M. Grynberg and Z. Koba, Phys. Letters 1, 130 (1962).

[18] A. N. Mitra, Phys. Rev. 142, 1119 (1966).

the interaction to be separable in momentum space, as had been done earlier for the single baryon problem as a 3$Q$ system.[4,5] Thus we take

$$M\langle \mathbf{p}|V|\mathbf{p}'\rangle=-\lambda u(p)u(p'), \qquad (3.26)$$

$$u(p)=\exp(-\tfrac{1}{2}p^2\beta^{-2}). \qquad (3.27)$$

Using such an interaction, the single-baryon binding energy $\alpha_B{}^2M^{-1}$ can be expressed in terms of the strength parameter $\lambda$ and inverse range parameter $\beta$ through the relation

$$\lambda^{-1}=\int d\mathbf{q}\, u^2(q)(q^2+\alpha_B{}^2)^{-1}. \qquad (3.28)$$

If we assume[18] that $\alpha_B{}^2\gg\beta^2$, this equation simplifies to

$$\alpha_B{}^2\approx\sigma\beta^2, \qquad (3.29)$$

where

$$\sigma=\lambda\beta\pi^{3/2}. \qquad (3.30)$$

Taking account of the conditions (3.20) and (3.29), one can at most maintain the inequalities

$$M^2\gg\alpha_B{}^2\gg\beta^2\sim k^2, \qquad (3.31)$$

for the nonrelativistic 4$Q$ problem. The first of these inequalities is of course against the spirit of the usual quark model, but this seems to be the best that can be done without giving up the nonrelativistic assumption in a multiquark problem which simultaneously involves bound and scattering states.

Inserting the interaction (3.26) into the Schrödinger equation (3.16), one can express the structure of the $A_k$ and $B_k$ functions as[15]

$$A_k=u(p_{ij})G(\mathbf{p}_{k4},\mathbf{Q}_k), \qquad (3.32)$$

$$B_k=u(p_{k4})G(\mathbf{p}_{ij},\mathbf{Q}_k), \qquad (3.33)$$

where $G$ is an even function of its two arguments.

The function $G(\mathbf{p}_{34},\mathbf{Q}_3)$, which may be called the "spectator" function for a four-particle system, should be interpreted as the (three-particle) wave function for the system consisting of the particles 3 and 4, and the c.m. of 1 and 2 by straightforward extension of the corresponding concept in the three-body problem with separable potentials.[19] It satisfies the integral equation

$$G(\mathbf{p}_{34},\mathbf{Q}_3)[1-\lambda h(p_{34}{}^2+\tfrac{1}{2}Q_3{}^2-EM)]$$
$$=\lambda\int d\mathbf{p}_{12}'u(p_{12}')(p_{12}'^2+p_{34}{}^2+\tfrac{1}{2}Q_3{}^2EM)^{-1}$$
$$\times[u(p_{34})G(\mathbf{p}_{12}',\mathbf{Q}_3)-u(p_{31}')G(\mathbf{p}_{24}',\mathbf{Q}_2')$$
$$-u(p_{24}')G(\mathbf{p}_{31}',\mathbf{Q}_2')], \qquad (3.34)$$

where

$$\lambda h(z)=\lambda\int d\mathbf{q}\, u^2(q)(q^2+z)^{-1}, \qquad (3.35)$$

[19] A. N. Mitra, Nucl. Phys. **32**, 529 (1962).

and $\mathbf{p}_{31}',\mathbf{p}_{24}',\mathbf{Q}_2'$ are given by (3.9)–(3.11) with the replacement of $\mathbf{p}_{12}$ by the integration variable $\mathbf{p}_{12}'$. The integral $h(z)$ under the conditions $z>0$, $z\gg\beta^2$ can be expressed, as for (3.28), by

$$\lambda h(z)\approx\alpha_B{}^2[z+O(\beta^2)]^{-1}, \qquad (3.36)$$

where we have used (3.29) and (3.30).

Before proceeding further with the simplification of Eq. (3.34), we first examine the interpretation of the various terms in this equation. While in a three-body problem[19] a corresponding structure of the left- and right-hand sides would have already represented a connected equation for a (2+1) system, Eq. (3.34) for the four-body problem represents only one of the steps towards the final goal of obtaining a properly connected equation. Thus on the right-hand side of (3.34), the first term still corresponds to unconnected graphs in which particles 3 and 4 are uncorrelated with 1 and 2. The second and third terms, on the other hand, represent rearrangements of the (3,4) indices with the (1,2) indices, and hence give rise to properly connected graphs. Therefore, to make the integral equation (3.34) correspond to a true (2+2) scattering problem, one must transfer the first term to the left-hand side and use the resolvant so obtained to define a new four-body kernel[20] with only connected terms. In such a form, one would expect to see a pole structure in the variable $Q_3{}^2$ at the point $Q_3{}^2=K^2$ since $\mathbf{Q}_3$ is the relative momentum variable between the two composites (1,2) and (3,4).

The reduction of Eq. (3.34) follows on lines very similar to MR, especially to Appendix II of that paper. Indeed, the present problem is somewhat simpler because of the inequalities (3.31). We sketch only the essential steps, leaving the details to the corresponding discussion in MR. We write

$$G(\mathbf{p}_{12},\mathbf{Q}_3)=F(\mathbf{p}_{12},\mathbf{Q}_3)\exp[-\tfrac{1}{2}(p_{12}{}^2+\tfrac{1}{2}Q_3{}^2)\beta^{-2}], \qquad (3.37)$$

which gives, for the $F$-equation, the structure

$$[1-\lambda h(p_{34}{}^2+\tfrac{1}{2}Q_3{}^2-EM)]F(\mathbf{p}_{34},\mathbf{Q}_3)$$
$$=\lambda\int d\mathbf{p}_{12}'\exp(-p_{12}'^2/\beta^2)[p_{12}'^2+p_{34}{}^2+\tfrac{1}{2}Q_3{}^2-EM]^{-1}$$
$$\times[F(\mathbf{p}_{12}',\mathbf{Q}_3)-F(\mathbf{p}_{24}',\mathbf{Q}_2')-F(\mathbf{p}_{31}',\mathbf{Q}_2')], \qquad (3.38)$$

where we have exploited the permutation symmetry of the kinetic energy operator $K$ of Eq. (3.8), which appears in the exponential of every term, to simplify the right-hand side. In this form, one expects the dependence of $F$ on the first argument to be much weaker than on the second. This approximation can be simulated by writing

$$F(\mathbf{p}_{34},\mathbf{Q}_3)\approx F(0,\mathbf{Q}_3), \qquad (3.39)$$

[20] For a fuller discussion of how to obtain the kernel for a connected four-body problem, see A. N. Mitra, J. Gillespie, R. Sugar, and N. Panchapakesan, Phys. Rev. **140**, B1336 (1965).

and transferring the first term to the left (after putting $\mathbf{p}_{34}\approx 0$). The same approximation in the last two terms, however, leaves

$$F(0,\mathbf{Q}_2')=F(0,\mathbf{p}_{12}'+\mathbf{p}_{34})$$

by virtue of Eq. (3.11). If, in consonance with our stated approximation of neglecting $\mathbf{p}_{34}$ in the $F$ function, we write

$$F(0,\,\mathbf{p}_{12}'+\mathbf{p}_{34})\approx F(0,\mathbf{p}_{12}')\,,$$

we eventually obtain the truncated integral equation

$$\tfrac{1}{2}F(\mathbf{Q}_3)(Q_3{}^2-k^2)\approx -4\lambda\alpha_B{}^2\int d\mathbf{q}(q^2+\tfrac{1}{3}Q_3{}^2-EM)^{-1} \times F(\mathbf{q})\exp(-q^2\beta^{-2})\,, \quad (3.40)$$

where we have made use of (3.36) to simplify the first term on the right-hand side of (3.38) and noted equal contributions from the last two terms.

Equation (3.40) has the desired pole structure characteristic of a (2+2) scattering problem. Because of the neglect of several angular correlations, especially between the momenta $\mathbf{p}_{34}$, $\mathbf{p}_{12}'$, and $Q_2'$, this equation seems to describe only $s$-wave scattering, but this defect could certainly have been remedied by a more adequate treatment of the angles between the various momentum vectors. The kernel is clearly repulsive, thus showing explicitly how this formalism is capable of describing structural repulsion. The necessary boundary condition for scattering is

$$F(\mathbf{Q}_3)=(2\pi)^3\delta(\mathbf{Q}_3-\mathbf{k})+4\pi a(Q_3)(Q_3{}^2-k^2-i\epsilon)^{-1}, \quad (3.41)$$

which gives

$$2\pi a(\mathbf{Q}_3)\approx -4\lambda\alpha_B{}^2(2\pi)^3e^{-K^2/\beta^2}(k^2+2\alpha_B{}^2)^{-1} -16\pi\lambda\alpha_B{}^2\int d\mathbf{q}\,e^{-q^2/\beta^2}a(\mathbf{q})(q^2-k^2-i\epsilon)^{-1} \times(q^2+\tfrac{1}{2}Q_3{}^2-EM)^{-1}. \quad (3.42)$$

Under the condition (3.31), this equation reduces approximately to

$$a(\mathbf{Q}_3)\approx -8\pi^2\lambda e^{-K^2/\beta^2} -4\lambda\int d\mathbf{q}\,e^{-q^2/\beta^2}a(\mathbf{q})(q^2-k^2-i\epsilon)^{-1}, \quad (3.43)$$

having a solution, for $Q_3{}^2=k^2$, in the form

$$a(\mathbf{k})=-8\pi^2\lambda e^{-k^2/\beta^2} \times 1\left[+4\lambda\int d\mathbf{q}\,e^{-q^2/\beta^2}(q^2-k^2-i\epsilon)^{-1}\right]^{-1}. \quad (3.44)$$

This shows explicitly how the amplitude for (2+2) scattering arises out of an effective $s$-wave repulsion between the two composites. The repulsion is rather momentum-dependent, unlike the rigid, hard-core, features of Ref. 13.

## 4. DISCUSSION AND CONCLUSIONS

In this section we try to put in perspective the scope and results of the present investigation in relation to similar approaches in the past. The essential ideas of saturation of baryon energy levels and short-range repulsion of baryons as consequences of parastatistics were pointed out by Dalitz.[3] A related theory which gives similar predictions is the three-triplet model of Han and Nambu,[21] who also discussed the problem of saturation. Subsequently, the saturation problem was studied in the three-triplet model by Peres[22] in a semi-quantitative fashion. He postulated the existence of a massive vector field whose coupling to an appropriate quark current was held responsible for the low masses of zero-triality states. A phenomenological analysis of saturation with the help of two-body forces was also given by Greenberg and Zwanziger[23] through a comparative study of several models, viz., the Fermi-quark, two-triplet, three-triplet, and paraquark models. These authors explicitly showed the equivalence of the last two models with respect to saturation.

One of the new results of the present investigation is the empirical relation $V_s\approx -2V_a$, deduced from the observed magnitudes of the deuteron and nucleon masses. Since this relation is purely observational, it was probably not noticed in the earlier (purely theoretical) investigations.[22,23] We believe that small violations of this relation are probably responsible for the deviations in the various nuclear masses from their respective mass numbers.

The formalism discussed in Sec. 2 is in principle adequate for a discussion of saturation for excited baryon states as well, but the definitions of the quantities $V_s$ and $V_a$ need some elaboration. These quantities were defined in Sec. 2 as the respective even and odd parts of the spatial $Q$-$Q$ potential. A more accurate definition is that $V_s$ and $V_a$ are the expectation values of the complete $Q$-$Q$ potential [including spin and $SU(3)$ degrees of freedom] evaluated for symmetric and antisymmetric pairs, respectively, in the Young diagram. Now this modified definition is completely equivalent to the earlier (less rigorous) one, as long as only **56** baryons with totally symmetric wave functions are considered, such as was done in Sec. 2, since for such cases the $SU(6)$ degrees of freedom do not play any explicit role and can be effectively suppressed. However, the situation becomes different when one also considers other representations of $SU(6)$, **70**, which are believed to be relevant for the excited baryon states. Indeed, according to present ideas,[3] the negative-parity states

[21] M. Y. Han and Y. Nambu, Phys. Rev. **139**, B1006 (1965).
[22] A. Peres, Phys. Rev. **149**, 1131 (1966).
[23] O. W. Greenberg and D. Zwanziger, Phys. Rev. **150**, 1177 (1966).

are supposed to belong to the (**70**,1⁻) representation of $SU(6)\times O(3)^P$, with an (internal) orbital excitation of $L^P=1^-$. If we now consider an (excited) baryon of this type, its Young pattern is still given by a row of three boxes, but it should be remembered that each of these "boxes" carries *both* the spatial and the $SU(6)$ labels. Thus the expectation value, say, $V_s'$, of the $Q$-$Q$ potential for a symmetrical $Q$-$Q$ pair in an excited (**70**) baryon state is not the same as the corresponding quantity $V_s$ for a **56** baryon; likewise, for the expectation values $V_a'$ and $V_a$ for antisymmetric pairs in **70** and **56** states, respectively. The important thing now is to remember that there is no geometrical relationship between the two sets $(V_s,V_a)$ and $(V_s',V_a')$ since the spatial structures of the wave functions for **56** and **70** states are by no means geometrically related to each other.[24] Of course, one could formally obtain a "saturation requirement" for a **70** state by equating its mass $B'$ to $3Q+3V_s'$, in exact analogy with Eq. (2.19) for a **56** mass $B=3Q+3V_s$. However, the quantity $B'$ *cannot* be expressed in any simple way in terms of $B$ and $Q$ without further knowledge about $V_s'$. Mathematically, one would require a specific model to be able to calculate the quantities $V_{s,a}$ and $V_{s,a}'$ in terms of a common set of parameters. Experimentally, our knowledge about excited baryon states is nowhere near that for a collection of nucleons, so that any empirical analog of the "nuclear" relation $V_s\approx -2V_a$ for excited baryon states is almost out of the question. We note in this connection that the earlier investigations on saturation[21,23] did not throw any light on its status for excited baryon states.

The second point concerns the method of evaluation of the short-range repulsion in relation to contemporary investigations. In this respect, we were motivated not so much by a desire to predict quantitative numbers, as by a desire to offer a qualitatively correct picture of core repulsion through a simple model which incorporates the essential features of a short-range two-body force. As was suggested by Dalitz,[3] one should expect a short-range repulsion to arise from the peculiar overlapping of wave functions which characterizes the paraquark model. Now it appears that a property as delicate as core repulsion due to the structure of the wave functions should depend sensitively on internal polarizations of the individual baryons when these two composites come sufficiently close to each other. The internal polarizations, in turn, cannot be adequately taken into account without considering the effects of virtual excited states of the baryons in addition to their ground states.

[24] This statement is true only in an $SU(6)$ or $SU(6)\times O(3)$ symmetry, where the relation between the **56** or **70** wave functions is at most dynamical. However, there might well exist geometrical relationships between such states in a higher-symmetry group.

To incorporate the above features it was believed necessary to resort to a multiquark approach to the entire problem, preferably in a nonvariational manner, even at the expense of certain simplifying assumptions. The two simplifying assumptions of (i) parastatistics of order 2 and of (ii) separable form of interaction, that were used in Sec. 3 were mainly suggested from considerations of practical feasibility without having to give up the main physical features of the problem. It seems to us that within the premises of the nonrelativistic quark model, the separable assumption admits of adequate justification which has already been discussed elsewhere.[4,5] Since for the present problem one requires mainly an $s$-wave short-range force, the separable assumption is a useful alternative to the (more usual) Yukawa interaction. Its great advantage is that it not only makes the three-body problem exactly soluble,[19] but even reduces the four-body problem to approximate solubility with few extra assumptions.[15] This explains our choice of paraquarks of order 2, so that the solution of the resultant four-body problem with separable potentials could be facilitated in terms of the previous investigations on the problem[15,25] which did take account of the effect of internal polarization, virtual excited states, etc.

In this respect, the scope of the present investigation is claimed to be considerably wider than those of variational treatments which have mostly been confined to ground states of the composites. The calculations of Otsuki *et al.*,[13] on the lines of the $\alpha$-$\alpha$ model[14] taking Yukawa-type interactions between quarks, also follow the general variational pattern of neglecting the effect of excited states. While these authors obtain sufficiently encouraging numerical results for the phase shifts through suitable parametrization of the basic forces, the precise roles of the various substructures in the complete wave function in bringing out the desired types of "overlaps" seem to be completely hidden. The present objective, on the other hand, was more limited, viz., not to predict accurate numbers, but to demonstrate in a simple way how a successive reduction of the full wave function in terms of suitable substructures could ultimately bring out a repulsive interaction between the units eventually identified as baryons. This last feature (we believe) did need an explicit demonstration, even through a simple model.

## ACKNOWLEDGMENTS

This work was performed while one of us (A.N.M.) was a Summer Visitor at the UCLA. He is grateful to Professor A. Wright and to Professor H. Ticho for the hospitality of the Department of Physics at UCLA.

[25] N. Panchapakesan, Phys. Rev. **140**, B20 (1965).