Using the phase conventions of Jacob and Wick,[15] we have: Helicity

$$\lambda=1, \quad \epsilon_\nu=(\epsilon_0,\boldsymbol{\epsilon})=-(1/\sqrt{2})(0,1,i,0);$$

$$\lambda=0, \quad \epsilon_\nu=\frac{1}{\mu}(k,0,0,E)\underset{s\to\infty}{\longrightarrow}\frac{1}{\mu}(\tfrac{1}{2}P+Q)_\nu+O(s^{-1/2});$$

$$\lambda=-1, \quad \epsilon_\nu=(1/\sqrt{2})(0,1,-i,0);$$

$$\lambda'=1, \quad \epsilon_\mu'=(1/\sqrt{2})(0,1,-i,0);$$

$$\lambda'=0, \quad \epsilon_\mu'=\frac{1}{\mu}(-k,0,0,E)\underset{s\to\infty}{\longrightarrow}\frac{1}{\mu}(\tfrac{1}{2}P+Q')_\mu+O(s^{-1/2});$$

$$\lambda'=-1, \quad \epsilon_\mu'=-(1/\sqrt{2})(0,1,i,0),$$

where

$$s=4E^2=4(k^2+\mu^2),$$
$$t=-2k^2(1-\cos\theta),$$

and

$$Q_\mu=\tfrac{1}{2}(2E,k\sin\theta,0,k(1+\cos\theta)),$$
$$Q_\mu'=\tfrac{1}{2}(-2E,k\sin\theta,0,k(1+\cos\theta)),$$
$$P_\mu=(0,-k\sin\theta,0,k(1-\cos\theta)).$$

[15] J. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

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Covariant Version of the Bjorken Limit*

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Using local-field theory and the assumption that the commutator is not more singular than $\delta(x^2)$ and derivatives of $\delta(x^2)$, it is shown that the Bjorken limit $q_0\to\infty$, $\mathbf{q}$ fixed, can be generalized to $|q^2|\to\infty$, $|q_\mu|\to\infty$, by making the result of the Bjorken limit covariant. If Schwinger terms are present, the Bjorken limit does not determine the leading asymptotic behavior; in spite of this, however, it is possible to show that the leading asymptotic behavior can be obtained from the Bjorken limit if the coefficients of the Schwinger terms are known, and if the amplitude satisfies a divergence equation.

1. INTRODUCTION

Some time ago, Bjorken proposed[1] a method for calculating the (virtual) asymptotic behavior of matrix elements of time-ordered products. To illustrate this method, we consider the amplitude

$$M(q,\cdots)=-i\int dx\, e^{iqx}\langle\alpha|T[A(x)B(0)]|\beta\rangle, \quad (1)$$

where $|\alpha\rangle$ and $|\beta\rangle$ are arbitrary states and $A(x)$ and $B(x)$ are two arbitrary operators. The absorptive parts are given by

$$m(q,\cdots)=\int dx\, e^{iqx}\langle\alpha|A(x)B(0)|\beta\rangle,$$
$$\bar{m}(q,\cdots)=\int dx\, e^{iqx}\langle\alpha|B(0)A(x)|\beta\rangle. \quad (2)$$

Let us assume that $M(q,\cdots)$ satisfies an unsubtracted dispersion relation in q_0. Equation (1) then becomes

$$M(q,\cdots)=\int\frac{dq_0'}{2\pi}\left[\frac{m(q_0',\mathbf{q},\cdots)}{q_0-q_0'}-\frac{\bar{m}(q_0',-\mathbf{q},\cdots)}{q_0+q_0'}\right], \quad (3)$$

and assuming, furthermore, that

$$m\to O(1/q_0),\quad \bar{m}\to O(1/q_0),\quad \text{for}\quad q_0\to\infty$$

we find from Eqs. (2) and (3)

$$M(q,\cdots)\underset{q_0\to\infty}{\longrightarrow}\frac{1}{q_0}\int\frac{dq_0'}{2\pi}[m(q_0',\mathbf{q},\cdots)-\bar{m}(q_0',-\mathbf{q},\cdots)]$$
$$=\frac{1}{q_0}\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle\alpha|[A(0,\mathbf{x}),B(0)]|\beta\rangle. \quad (4)$$

Hence the asymptotic behavior of the time-ordered product is given by an equal-time commutator in the limit of high virtual masses ($q^2\to\infty$).

The Bjorken limit (4) is derived under the special assumption that $q_0\to\infty$, $\mathbf{q}$ finite, $q^2\to\infty$. In practical applications one is interested in the limit $|q^2|\to\infty$, and this limit can be achieved in various ways, e.g., by letting all components of q_μ go to infinity in such a way that $|q^2|\to\infty$. It is therefore of interest to investigate what happens in the limit $|q^2|\to\infty$, of which the Bjorken limit (4) is a special example.

One way to deal with the $|q^2|\to\infty$ limit is to start from expression (4), rewrite this expression in a covariant way, and then claim that the resulting expression is correct for $|q^2|\to\infty$ (but not necessarily $\mathbf{q}$ finite). This method is, however, based on the assump-

* Work supported in part by the U. S. Atomic Energy Commission.
[1] J. D. Bjorken, Phys. Rev. 148, 1467 (1966).

tion that the time-ordered product is covariant, and it is well known that if Schwinger terms are present, the time-ordered product is not covariant. (See Ref. 1 and references cited therein.)

Bjorken[1] develops a method for the identification of Schwinger terms. Suppose that the sums over intermediate states in Eqs. (2) are truncated. Then M always behaves as shown in Eq. (4) for $q_0 \to \infty$. The difference between the asymptotic behavior shown in Eq. (4) and the asymptotic behavior of the covariant amplitude $\tilde{M}$ (which we know must exist from the point of view of physics) is at most a polynomial in q_0 if we assume that M and $\tilde{M}$ have the same absorptive parts. Thus the physical amplitude $\tilde{M}$ behaves like

$$\tilde{M}(q,\cdots)=P_n(q_0)+\frac{1}{q_0}\int d^3x\,\langle\alpha|[A(\mathbf{x},0),B(0)]|\beta\rangle e^{-i\mathbf{q}\cdot\mathbf{x}}, \quad (5)$$

where $P_n(q_0)$ is a polynomial of the order n. If we make Eq. (5) covariant, the Schwinger terms in the equal-time commutator combine with $P_n(q_0)$ in such a way that M is covariant. This in general determines some of the coefficients in $P_n(q_0)$ from the coefficients of the Schwinger terms.

This procedure involves, however, an interchange of limits; we let $q_0 \to \infty$ in the truncated time-ordered product, and afterwards we neglect the truncation when we write the integral over the absorptive parts as an equal-time commutator [see Eq. (4)]. The amplitude $\tilde{M}(q,\cdots)$ defined by Eq. (5) need therefore not be equal to the physical, nontruncated amplitude. The procedure of making Eq. (5) covariant by replacing the $q_0 \to \infty$ limit by the $|q^2| \to \infty$ limit is, in general, not unique as far as the $1/q_0$ term is concerned. In fact, only the leading term in $P_n(q_0)$ is uniquely determined in the $|q^2| \to \infty$ limit.

In the present paper, we have investigated the covariant limit from the point of view of local-field theory. We make the following assumptions:

(a) The relevant commutator $[A(x),B(0)]$ is local, i.e.,

$$[A(x),B(0)]=0, \quad \text{for} \quad x^2<0.$$

(b) The strongest light-cone singularities of

$$\langle\alpha|[A(x),B(0)]|\beta\rangle$$

are $\delta(x^2)$ and derivatives of $\delta(x^2)$.

We then find that if $M(q,\cdots)$ satisfies

$$M(q,\cdots)=O(1/\sqrt{|q^2|}), \quad |q^2| \to \infty$$

then the asymptotic behavior of $M(q,\cdots)$ is given by the expression obtained by making the Bjorken-limit covariant.

If we allow Schwinger terms, we find an equation which in the limit $q_0 \to \infty$ can be written

$$\tilde{M}(q,\cdots)=P_n(q_0)+O(1/q_0).$$

The main tools in our proof is the Dyson representation[2] (discussed in Sec. 2) and a technique recently constructed by the author for investigating the asymptotic behavior of the Dyson representation[3] in the simple case where assumption (b) is satisfied.

In Sec. 3, we discuss the covariant $|q^2| \to \infty$ limit for a scalar amplitude M, and in Sec. 4, we discuss the same limit in the case when the amplitude is a tensor. In Sec. 5, we discuss the results and give a method for obtaining the covariant amplitude from the time-ordered product.

2. LOCAL REPRESENTATION OF THE COMMUTATOR

In this section, we shall construct a local representation of the commutator. To take a simple example, let us consider

$$F(x,p)=p_\mu\left\langle p\left|\left[\frac{\partial}{\partial x_\mu}\varphi(x),\varphi(0)\right]\right|p\right\rangle, \quad (6)$$

where $|p\rangle$ is a one-particle state and $\varphi(x)$ is a scalar field. According to assumption (b), the strongest singularities are of the type $\delta(x^2)$ and derivatives of $\delta(x^2)$. Let us assume that the highest derivative is the second (this assumption is not essential; the method can easily be generalized). Let us write

$$F(x,p)=F_0(x,p)+p_\mu\frac{\partial}{\partial x_\mu}F_1(x,p)+p_\mu p_\nu\frac{\partial^2}{\partial x_\mu\partial x_\nu}F_2(x,p), \quad (7)$$

where the strongest singularities in F_k are of the type $\delta(x^2)$ [with no derivatives of $\delta(x^2)$]. The function F_k depends on x^2 and px (we keep p^2 fixed), since locality guarantees that $\epsilon(x_0)$ is effectively the same as $\epsilon(px)$ (we take $|p\rangle$ to be a physical state, with p timelike). In writing down Eq. (7), we have used that

$$x^\mu\frac{\partial}{\partial x^\mu}\delta(x^2)=-2\delta(x^2), \qquad x_\mu x_\nu\frac{\partial^2}{\partial x_\mu\partial x_\nu}\delta(x^2)=6\delta(x^2), \quad (8)$$

so that no other derivatives than those written down in Eq. (7) are relevant. Consider one of the functions F_k, and let us use locality to write

$$F_k(x^2,px)=\int_0^\infty \tilde{F}_k(\mu^2,px)\delta(x^2-\mu^2)d\mu^2, \quad (9)$$

where $\tilde{F}_k=F_k$ for $x^2\geq 0$. Using the orthogonality

[2] F. J. Dyson, Phys. Rev. **110**, 1460 (1958).
[3] P. Olesen, Phys. Rev. **165**, 1682 (1968).

relation

$$\delta(x^2-\mu^2)=16\pi^2\int_0^\infty dm^2\,\bar{\Delta}(x^2,m^2)\bar{\Delta}(m^2,\mu^2)\,, \tag{10}$$

$$\bar{\Delta}(x^2,m^2)=-\tfrac{1}{2}\epsilon(x_0)\Delta(x,m^2)\,, \tag{10'}$$

we obtain

$$F_k(x^2,px)=\int_0^\infty dm^2\, f_k(m^2,xp)\Delta(x,m)\,, \tag{11}$$

where the spectral weight $f_k(m^2,xp)$ is given by

$$f_k(m^2,xp)=-8\pi^2\int_0^\infty d\mu^2\,\bar{\Delta}(m^2,\mu^2)\epsilon(px)\tilde{F}_k(\mu^2,px)\,. \tag{12}$$

The Dyson representation[2] goes much further than Eq. (10) by introducing the assumption of a "reasonable" mass spectrum. However, we only need the information (i.e., locality) contained in Eq. (11).

Using Eqs. (7) and (11), we can write

$$F(x,p)=\int_0^\infty dm^2\left[\rho_0(m^2,px)+\rho_1(m^2,px)p_\mu\frac{\partial}{\partial x_\mu}+i\rho_2(m^2,px)p_\mu p_\nu\frac{\partial^2}{\partial x_\mu\partial x_\nu}\right]\Delta(x,m)\,, \tag{13}$$

where the new spectral weights are given in terms of the old spectral weights by the equations

$$\begin{aligned}\rho_0&=f_0+p_\mu(\partial f_1/\partial x_\mu)+p_\mu p_\nu(\partial^2 f_2/\partial x_\mu\partial x_\nu)\,,\\ \rho_1&=f_1+2(p_\mu\partial f_2/\partial x_\mu)\,,\\ i\rho_2&=f_2\,.\end{aligned} \tag{14}$$

The spectral weights ρ_i are functions only of m^2 and px (they do not depend on x^2), as one can see from Eqs. (12) and (14).

Equation (13) leads to the following equal-time commutation relation:

$$p_\mu\left\langle p\left|\left[\frac{\partial}{\partial x_\mu}\varphi(x),\varphi(0)\right]\right|p\right\rangle_{x_0=0}=p_0c_1\delta(\mathbf{x})+2ip_0c_2p_k\times\frac{\partial}{\partial x_k}\delta(\mathbf{x})-2ip_0\mathbf{p}^2c_2'\delta(\mathbf{x})\,, \tag{15}$$

where

$$\begin{aligned}c_1&=\int_0^\infty dm^2\,\rho_1(m^2,0)\,,\\ c_2&=\int_0^\infty dm^2\,\rho_2(m^2,0)\,,\\ c_2'&=\int_0^\infty dm^2\,\frac{\partial\rho_2(m^2,\mathbf{px})}{\partial\mathbf{p}\cdot\mathbf{x}}\bigg|_{\mathbf{x}=0}.\end{aligned} \tag{16}$$

In deriving Eqs. (16), we have used locality and the assumption (b), which implied Eq. (13) with regular spectral functions. We could, of course, add higher derivatives of $\Delta(x,m)$, leading to higher-order Schwinger terms in the equal-time commutator (15). This corresponds to adding more terms to Eq. (7). These higher-order Schwinger terms provide no complications in principle, and in order to save writing, we take a minimal number of Schwinger terms. Finally, we mention that in a normal "canonical" theory, $c_2'=0$ (and the Schwinger terms are also absent).

In addition to the somewhat academical scalar amplitude (6), we also consider the physically more interesting case of a tensor amplitude,

$$F_{\mu\nu}{}^{\alpha\beta}(x,p)=\langle p|[j_\mu{}^\alpha(x),j_\nu{}^\beta(0)]|p\rangle\,. \tag{17}$$

Following arguments similar to the arguments (6)–(13), we write the following local representation:

$$\begin{aligned}F_{\mu\nu}{}^{\alpha\beta}(x,p)=\int_0^\infty dm^2\bigg[&i\rho_1{}^{\alpha\beta}(m^2,xp)\frac{\partial^2}{\partial x_\mu\partial x_\nu}\\ &+\rho_2{}^{\alpha\beta}(m^2,xp)p_\mu\frac{\partial}{\partial x_\nu}+\rho_3{}^{\alpha\beta}(m^2,xp)p_\nu\frac{\partial}{\partial x_\mu}\\ &+\rho_4{}^{\alpha\beta}(m^2,xp)g_{\mu\nu}p_\lambda\frac{\partial}{\partial x_\lambda}+\rho_5{}^{\alpha\beta}(m^2,xp)p_\mu p_\nu\\ &+\rho_6{}^{\alpha\beta}(m^2,xp)g_{\mu\nu}\bigg]\Delta(x,m)\,,\end{aligned} \tag{18}$$

where the ρ's are invariant spectral functions. The equal-time commutator computed from Eq. (18) is given by

$$\begin{aligned}F_{\mu\nu}{}^{\alpha\beta}(0,\mathbf{x},p)=&\,p_\mu\delta_{\nu0}c_2{}^{\alpha\beta}\delta(\mathbf{x})+p_\nu\delta_{\mu0}c_3{}^{\alpha\beta}\delta(\mathbf{x})\\ &+g_{\mu\nu}p_0c_4{}^{\alpha\beta}\delta(\mathbf{x})+i(\delta_{\mu0}g_{\nu k}+\delta_{\nu0}g_{\mu k})c_1{}^{\alpha\beta}\frac{\partial}{\partial x_k}\delta(\mathbf{x})\\ &-i(\delta_{\mu0}g_{\nu k}+\delta_{\nu0}g_{\mu k})p_kc_1{}'^{\alpha\beta}\delta(\mathbf{x})\,,\end{aligned} \tag{19}$$

$$\begin{aligned}c_i{}^{\alpha\beta}&=\int_0^\infty dm^2\,\rho_i{}^{\alpha\beta}(m^2,0)\,,\\ c_i{}'^{\alpha\beta}&=\int_0^\infty dm^2\,\frac{\partial\rho_i{}^{\alpha\beta}(m^2,\mathbf{px})}{\partial\mathbf{p}\cdot\mathbf{x}}\bigg|_{\mathbf{x}=0}.\end{aligned} \tag{20}$$

Thus, in addition to three "canonical" terms, we have a Schwinger term as well as a term proportional to $\delta(\mathbf{x})$ originating from the "Schwinger part" of Eq. (18). In writing down Eqs. (18) and (19), we have used assumption (b), i.e., we have assumed that the various spectral functions are regular. We can add higher-order derivatives in Eq. (18), leading to higher-order Schwinger terms. This type of generalization turns out to be trivial.

If the currents $j_\mu^\alpha(x)$ belong to an algebra, we have

$$[j_0^\alpha(x),j_\nu^\beta(0)]_{x_0=0}=c^{\alpha\beta\gamma}j_\nu^\gamma(0)\delta(\mathbf{x}) + \text{Schwinger terms}. \quad (21)$$

This relation can be obtained from Eq. (19) if we make the identifications

$$c_1'^{\alpha\beta}=0\,,\quad c_4^{\alpha\beta}=-c_2^{\alpha\beta}\,,\quad c_2^{\alpha\beta}=c_3^{\alpha\beta}\,,\quad p_\mu c_2^{\alpha\beta}=c^{\alpha\beta\gamma}\langle p|j_\mu^\gamma(0)|p\rangle. \quad (22)$$

The solution (22) corresponds to current algebra,[4] where the space-space part of the equal-time commutator is nonvanishing. We can also find a solution which corresponds to the algebra of gauge fields[5] (where the space-space part of the equal-time commutator vanishes), namely,

$$c_4^{\alpha\beta}=0\,,\quad c_2^{\alpha\beta}=c_3^{\alpha\beta}\,,\quad p_\mu c_2^{\alpha\beta}=c^{\alpha\beta\gamma}\langle p|j_\mu^\gamma(0)|p\rangle\,,\quad c_1'^{\alpha\beta}=ic_2^{\alpha\beta}. \quad (23)$$

It is very interesting that *the possibility of having vanishing space-space commutators is intimately connected to the presence of Schwinger terms*. Without Schwinger terms the local representation (18) is inconsistent with a vanishing space-space commutator.

In Eq. (18), we could include derivatives of the form $p_\mu p_\nu p_\lambda \partial/\partial x_\lambda$, $p_\mu p_\nu (p_\lambda\partial/\partial x_\lambda)^2$, etc. However, since Eq. (18) gives the current-algebra commutators, it is not very interesting to include such terms, and in order to keep the algebra at a reasonable level, we do not include these possible terms. In passing, we mention that a derivative of the form $p_\mu p_\nu (p_\lambda\partial/\partial x_\lambda)^2$ would lead to a Schwinger term of the type $p_\mu p_\nu p_k \partial_k \delta(\mathbf{x})$.

Note that in the limit $p\to 0$, the local representations (13) and (18) are identical to the usual Källen representation for the two-point function.

In Sec. 3 we treat the covariant $|q^2|\to\infty$ limit for the scalar case (6), and in Sec. 4, we treat the tensor case (17).

3. COVARIANT $|q^2|\to\infty$ LIMIT (SCALAR CASE)

We now consider the amplitude

$$T(p,q)=-i\int dx\, e^{iqx}\theta(x_0)p_\mu \times\left\langle p\left|\left[\frac{\partial\varphi(x)}{\partial x_\mu},\varphi(0)\right]\right|p\right\rangle. \quad (24)$$

Instead of the time-ordered product, we consider the retarded commutator; this is, however, only a technical point which makes the algebra a bit simpler. The Bjorken limit $q_0\to\infty$, $\mathbf{q}$ fixed, gives for the amplitude

$$T(p,q)\to\frac{1}{q_0}\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}p_\mu \times\left\langle p\left|\left[\frac{\partial\varphi(x)}{\partial x_\mu},\varphi(0)\right]\right|p\right\rangle_{x_0=0}. \quad (25)$$

Introducing the equal-time commutator (15), we obtain

$$T(p,q)\to(1/q_0)(p_0c_1-2ip_0\mathbf{p}^2c_2'-2c_2p_0\mathbf{p}\cdot\mathbf{q})\,,\quad q_0\to\infty\,,\quad \mathbf{q}\text{ finite}. \quad (26)$$

If $c_2=$ zero, it is obvious that $T(p,q)$ can be made covariant by writing

$$T(p,q)\to\frac{qp}{q^2}c_1+\frac{2ic_2'}{q^2}pq\left[p^2-\frac{(pq)^2}{q^2}\right] \quad (27)$$

for $|q^2|\to\infty$. However, if $c_2\neq 0$, the amplitude $T(p,q)$ cannot be made invariant, indicating that the time-ordered product is not covariant in the presence of Schwinger terms. Hence $T(p,q)$ is different from the physical amplitude $\tilde{T}(p,q)$. According to Bjorken's prescription,[1] we should then add a polynomial in q_0 to obtain the physical amplitude, i.e.,

$$\tilde{T}(p,q)\to a+(1/q_0)(p_0c_1-2ip_0\mathbf{p}^2c_2'-2c_2p_0\mathbf{p}\cdot\mathbf{q}) \to\frac{1}{q^2}\left\{pqc_1+2ic_2'pq\left[p^2-\frac{(pq)^2}{q^2}\right]+q_0(aq_0-2c_2\mathbf{p}\cdot\mathbf{q}p_0)\right\}. \quad (28)$$

This is covariant if $a=c_2p_0^2$, in which case we get

$$\tilde{T}(p,q)\to\frac{1}{q^2}\left\{c_2(pq)^2+c_1pq+2ic_2'pq\left[p^2-\frac{(pq)^2}{q^2}\right]\right\} \quad (29)$$

for $|q^2|\to\infty$. The above procedure requires that one can make substitutions like $p_0q_0\to p_0q_0-\mathbf{p}\cdot\mathbf{q}$, which is valid in the Bjorken limit. The derivation of the result (29) is somewhat naive, of course. However, using the local representation (13), we shall show below that we do indeed get a leading term given by $a=c_2p_0^2$, with a reasonable definition of the physical amplitude. In passing, we mention that Eq. (27) can only be expected to be correct for the truncated amplitude as far as the nonleading $O(1/q)$ term is concerned. Also, the polynomial is in this case given by

$$P_0(q_0)=c_2p_0^2=\tilde{T}(p,q)-T(p,q). \quad (30)$$

To see what happens in the covariant limit $|q^2|\to\infty$, we introduce the local representation (13) in the ampli-

[4] M. Gell-Mann, Physics 1, 63 (1964).

[5] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters 18, 1029 (1967).

tude (24). We have

$$T=T^{(2)}(p,q)+T^{(1)}(p,q)+T^{(0)}(p,q)\,,$$

$$T^{(2)}(p,q)=\int dx\,e^{iqx}\theta(x_0)\int_0^\infty dm^2\,\rho_2(m^2,px)p_\mu p_\nu\times\frac{\partial^2\Delta(x,m)}{\partial x_\mu\partial x_\nu}\,,$$

$$T^{(1)}(p,q)=-i\int dx\,e^{iqx}\theta(x_0)\int_0^\infty dm^2\,\rho_1(m^2,px)p_\mu\times\frac{\partial\Delta(x,m)}{\partial x_\mu}\,,\qquad(31)$$

$$T^{(0)}(p,q)=-i\int dx\,e^{iqx}\theta(x_0)\int_0^\infty dm^2\,\rho_0(m^2,px)\times\Delta(x,m)\,.$$

We begin the treatment of the quantities (31) by considering $T^{(0)}(p,q)$, since all other amplitudes can be expressed in terms of this quantity. The function $\Delta(x,m)$ consists of a part $\Delta(x, m=0)$ which is singular on the light cone and a part which is nonsingular on the light cone. The nonsingular contribution to the integral Eq. (31) can be shown to be smaller than the singular contribution[3] in the limit $|q^2|\to\infty$ (just as in the Bjorken limit) by a factor of at least $1/|q^2|$. For the reader's convenience, we have shown this in the Appendix. We therefore only have to consider the singular contributions on the light cone (this statement is a generalization of the Riemann-Lebesque lemma[6]). The singular contribution to $T^{(0)}(p,q)$ is ($q_0\to q_0+i\epsilon$),

$$T^{(0)}(p,q)\to -i\int dx\,e^{iqx}\Delta(x,\,m=0)\theta(x_0)\times\int_0^\infty dm^2\,\rho_0(m^2,px)+R(p,q)\,,\qquad(32)$$

where $R(p,q)$ denotes the nonleading contributions (down by a factor $1/q^2$ relative to the leading term). Inserting the explicit form of $\Delta(x, m=0)$, we find

$$T^{(0)}(p,q)\to -i\int dx\,e^{iqx}\frac{1}{4\pi|\mathbf{x}|}\delta(x_0-|\mathbf{x}|)\times\int_0^\infty dm^2\,\rho_0(m^2,\,p_0|\mathbf{x}|-\mathbf{p}\cdot\mathbf{x})$$

$$=-i\int d^3x\,e^{iq_0|\mathbf{x}|-i\mathbf{q}\cdot\mathbf{x}}\frac{e^{-\epsilon|\mathbf{x}|}}{4\pi|\mathbf{x}|}\times\int_0^\infty dm^2\,\rho_0(m^2,\,p_0|\mathbf{x}|-\mathbf{p}\cdot\mathbf{x})\,.\qquad(33)$$

In practice, the state $|p\rangle$ is a physical one-particle state, so that p is timelike, and we can introduce a frame of reference where $\mathbf{p}=0$. The angular integration is then trivial and gives

$$T^{(0)}(p,q)\to\frac{-i}{|\mathbf{q}|}\int_0^\infty d|\mathbf{x}|\,e^{-\epsilon|\mathbf{x}|}e^{iq_0|\mathbf{x}|}\sin(|\mathbf{q}|\,|\mathbf{x}|)\times\int_0^\infty dm^2\,\rho_0(m^2,p_0|\mathbf{x}|)\,.\qquad(34)$$

In the limit $|q^2|\to\infty$, $|q_0|\to\infty$, $|\mathbf{q}|\to\infty$, Eq. (34) can be written

$$T^{(0)}(p,q)\to\frac{-i}{|\mathbf{q}|q_0}\int_0^\infty du\,e^{-\epsilon u}e^{iu}\sin\left(\frac{|\mathbf{q}|}{q_0}u\right)\times\int_0^\infty dm^2\left\{\rho_0(m^2,0)+\frac{p_0u}{q_0}\times\left[\frac{\partial\rho_0(m^2,\alpha)}{\partial\alpha}\right]_{\alpha=0}\right\}$$

$$\to\frac{i}{q^2}\int_0^\infty dm^2\,\rho_0(m^2,0)-2\frac{pq}{(q^2)^2}\times\int_0^\infty dm^2\left[\frac{\partial\rho_0(m^2,\alpha)}{\partial\alpha}\right]_{\alpha=0}+R(p,q)\,,\qquad(35)$$

$$q^2(\sqrt{q^2})R(p,q)\to 0\quad\text{for}\quad |q^2|\to\infty\,.\qquad(36)$$

The first term on the right-hand side of this formula is easy to remember if we take $\rho_0(m^2,px)=\delta(m^2)$. Equation (35) then simply states that the Fourier transform of the free-field retarded commutator $\theta(x_0)\Delta(x, m=0)$ is $\sim 1/q^2$. However, the main point in the arguments (31)–(36) is that $\rho_0(m^2,px)$ is an arbitrarily complicated function subject only to the condition that it is regular on the light cone. The reason why we get the simple result (35) is that

$$-\frac{q_0^2-\mathbf{q}^2}{4\pi|\mathbf{x}|}e^{iq_0|\mathbf{x}|-i\mathbf{q}\cdot\mathbf{x}}e^{-\epsilon|\mathbf{x}|}\to\delta(\mathbf{x})\qquad(37)$$

for $|q^2|\to\infty$. Therefore for large values of $|q^2|$ the function (37) acts as a δ function if it is multiplied by a function which is *independent* of q. It is a consequence of the local representation (13) that ρ_0 is independent of q.

The amplitude $T^{(1)}$ in (31) can be calculated by the same methods. Keeping only the leading term,

[6] See, e.g., I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic Press Inc., New York, 1964), Vol. I.

we get

$$T^{(1)}(p,q) \to -i\int dx\, e^{iqx} p_\mu \frac{\partial\Delta(x, m=0)}{\partial x_\mu}\theta(x_0) \times \int_0^\infty dm^2\, \rho_1(m^2,px)$$

$$= -qp\int dx\, e^{iqx}\Delta(x, m=0)\theta(x_0) \times \int_0^\infty dm^2\, \rho_1(m^2,px) + i\int dx\, e^{iqx}\Delta(x, m=0)\theta(x_0) \times \int_0^\infty dm^2\, p_\mu \frac{\partial\rho_1(m^2,px)}{\partial x_\mu} \to \frac{qp}{q^2}\int_0^\infty dm^2\, \rho_1(m^2,0) + R(p,q), \quad (38)$$

$$(\sqrt{q^2})^{1+\alpha}R(p,q) \to 0, \quad 0<\alpha<1, \; |q^2| \to \infty. \quad (39)$$

The fact that the time-ordered product is not covariant in the presence of Schwinger terms shows up when we calculate $T^{(2)}(p,q)$. Using

$$\theta(x_0)\frac{\partial^2\Delta(x,m)}{\partial x_\mu \partial x_\nu} = \frac{\partial^2}{\partial x_\mu \partial x_\nu}[\theta(x_0)\Delta(x,m)] - g_{\mu 0}g_{\nu 0}\delta(x), \quad (40)$$

we can write $T^{(2)}$ as

$$T^{(2)}(p,q) = p_\mu p_\nu \int dx\, e^{iqx}\theta(x_0)\int_0^\infty dm^2 \times \left\{ -q_\mu q_\nu \rho_2(m^2,px) + 2iq_\nu p_\mu \frac{\partial\rho_2(m^2,px)}{\partial(px)} + p_\mu p_\nu \frac{\partial^2\rho_2(m^2,px)}{\partial(px)^2}\right\}\Delta(x,m) - p_0^2\int_0^\infty dm^2\, \rho_2(m^2,0). \quad (41)$$

By use of Eq. (35) we can easily find the asymptotic behavior, and the result is

$$T^{(2)}(p,q) = -p_0^2\int_0^\infty dm^2\, \rho_2(m^2,0) + \frac{(pq)^2}{q^2}\int_0^\infty dm^2\, \rho_2(m^2,0) + 2i\frac{(pq)^3}{(q^2)^2}\int_0^\infty dm^2 \left.\frac{\partial\rho_2(m^2,px)}{\partial(px)}\right|_{px=0} - 2i\frac{p^2 pq}{q^2}\int_0^\infty dm^2 \left.\frac{\partial\rho_2(m^2,px)}{\partial(px)}\right|_{px=0} + R(p,q), \quad (42)$$

$$(\sqrt{q^2})^{1+\alpha}R(p,q) \to 0, \quad 0<\alpha<1, \quad |q^2| \to \infty. \quad (43)$$

Collecting our results, we get

$$T(p,q) = -p_0^2 c_2 + \frac{(pq)^2}{q^2}c_2 + \frac{qp}{q^2}c_1 + 2ic_2'\frac{pq}{q^2}\left[p^2 - \frac{(pq)^2}{q^2}\right] + R(p,q), \quad (44)$$

where

$$R(p,q) \to O(1/q^2) \quad \text{for} \quad |q^2| \to \infty. \quad (45)$$

The main features of this result are the following: If Schwinger terms are absent, we get

$$T(p,q) = c_1\frac{pq}{q^2} + 2ic_2'\frac{pq}{q^2}\left[p^2 - \frac{(pq)^2}{q^2}\right], \quad (46)$$

where the leading terms are exactly the same as in Eq. (27), which was obtained by making the Bjorken limit covariant. This is of course necessary, since in this case the time-ordered product is covariant.

If Schwinger terms are present, the amplitude (44) consists of a covariant term plus a noncovariant term. If we compare with our discussion of the Bjorken limit in Eq. (30), we see that the noncovariant term is indeed the same in Eqs. (44) and (30). Adding the polynomial in Eq. (30) to Eq. (44), we obtain the covariant amplitude

$$\tilde{T}(p,q) = \frac{(pq)^2}{q^2}c_2 + \frac{qp}{q^2}c_1 + 2ic_2'\frac{pq}{q^2}\left[p^2 - \frac{(pq)^2}{q^2}\right] + R(p,q), \quad (47)$$

in complete accordance with our discussion (28)–(30). Thus we see that there is a complete correspondence between the Bjorken limit $q_0 \to \infty$, $\mathbf{q}$ finite, and the covariant limit $|q^2| \to \infty$.

This brings us to our final point, which is the observation that we could have obtained the covariant limit (47) directly from $T^{(2)}(p,q)$ defined in Eq. (31) by making the replacement

$$\theta(x_0)\frac{\partial^2\Delta(x,m)}{\partial x_\mu \partial x_\nu} \to \frac{\partial^2\theta(x_0)\Delta(x,m)}{\partial x_\mu \partial x_\nu}. \quad (48)$$

The right-hand side is covariant, while the left-hand side is not [see also Eq. (40)]. This result is equivalent to the following recipe for dealing with covariant amplitudes in the presence of Schwinger terms: *Construct a local representation of the type (13) which is in accordance with all known equal-time commutation relations. In constructing the covariant amplitude from the noncovariant amplitude, make replacements of the type (48), and the resulting amplitude is then the correct amplitude.* It is trivial that this recipe gives a covariant amplitude, since the right-hand side of Eq. (48) is covariant. It is also trivial that only Schwinger terms are influenced by (48) (but not "canonical" terms). The only remaining question is whether $\tilde{T}(p,q)$ is also the physical covariant amplitude, since *a priori* one could add an arbitrary constant to $\tilde{T}$ to obtain another covariant amplitude. We would like to argue that $\tilde{T}(p,q)$ is also the physical amplitude. The reason is that the only feature which prevents $T(p,q)$ from being a physical amplitude is the noncovariant term exhibited in Eq. (44), and this noncovariant term has been taken care of in $\tilde{T}(p,q)$. Because of the lack of any reason for adding a further constant, we conjecture that Eq. (47) gives the physical amplitude for the case when the equal-time commutation relation is given by Eq. (15). How to construct the physical amplitude in the tensor case will be discussed in the next section.

4. COVARIANT $|q^2|\to\infty$ LIMIT (TENSOR CASE)

In this section, we shall discuss the more interesting tensor amplitude

$$T_{\mu\nu}{}^{\alpha\beta}(p,q)=-i\int dx\, e^{iqx}\theta(x_0)\times\langle p|[j_\mu{}^\alpha(x),j_\nu{}^\beta(0)]|p\rangle, \quad (49)$$

for which the Bjorken limit is [see Eq. (19)]

$$\begin{aligned}T_{\mu\nu}{}^{\alpha\beta}(p,q)&\to\frac{1}{q_0}\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle p|[j_\mu{}^\alpha(0,\mathbf{x}),j_\nu{}^\beta(0)]|p\rangle\\&=(1/q_0)[p^\mu\delta_{\nu0}c_2{}^{\alpha\beta}+p_\nu\delta_{\mu0}c_3{}^{\alpha\beta}+g_{\mu\nu}p_0c_4{}^{\alpha\beta}\\&\quad-i(\delta_{\mu0}g_{\nu k}+\delta_{\nu0}g_{\mu k})p_kc_1{}'^{\alpha\beta}]\\&\quad-(1/q_0)c_1{}^{\alpha\beta}(\delta_{\mu0}g_{\nu k}+\delta_{\nu0}g_{\mu k})q_k. \quad (50)\end{aligned}$$

If Schwinger terms are absent, $c_1{}^{\alpha\beta}=0$, the covariant generalization of Eq. (50) is obviously

$$T_{\mu\nu}{}^{\alpha\beta}(p,q)\to(1/q^2)(p_\mu q_\nu c_2{}^{\alpha\beta}+p_\nu q_\mu c_3{}^{\alpha\beta}+g_{\mu\nu}pqc_4{}^{\alpha\beta})-i\frac{c_1{}'^{\alpha\beta}}{q^2}\left(q_\nu p_\mu+q_\mu p_\nu-2\frac{pq}{q^2}q_\nu q_\mu\right). \quad (51)$$

In current algebra this gives, according to Eq. (22),

$$T_{\mu\nu}{}^{\alpha\beta}(p,q)\to(1/q^2)(p_\mu q_\nu+p_\nu q_\mu-g_{\mu\nu}pq)c^{\alpha\beta\gamma}\times\langle p|I^\gamma|p\rangle, \quad (52)$$

with

$$\langle p|j_\mu{}^\gamma(0)|p\rangle=2p_\mu\langle p|I^\gamma|p\rangle. \quad (53)$$

If Schwinger terms are present, we add a polynomial to Eq. (50) to get the covariant amplitude. It is easily seen that if we add

$$c_1{}^{\alpha\beta}\delta_{\mu0}\delta_{\nu0}, \quad (54)$$

we get an amplitude which is covariant ($g_{kr}=-\delta_{kr}$):

$$\begin{aligned}\tilde{T}_{\mu\nu}{}^{\alpha\beta}&\to c_1{}^{\alpha\beta}\delta_{\mu0}\delta_{\nu0}-(c_1{}^{\alpha\beta}/q_0)(\delta_{\mu0}g_{\nu k}+\delta_{\nu0}g_{\mu k})q_k\\&\qquad+\text{“canonical terms”}\\&\to\frac{c_1{}^{\alpha\beta}}{q^2}q_\mu q_\nu+\frac{1}{q^2}(p_\mu q_\nu c_2{}^{\alpha\beta}+p_\nu q_\mu c_3{}^{\alpha\beta}+g_{\mu\nu}c_4{}^{\alpha\beta})\\&\qquad-i\frac{c_1{}'^{\alpha\beta}}{q^2}\left[q_\nu p_\mu+q_\mu p_\nu-2\frac{pq}{q^2}q_\nu q_\mu\right], \quad (55)\end{aligned}$$

and in the algebra of fields this becomes, according to Eq. (23),

$$\tilde{T}_{\mu\nu}{}^{\alpha\beta}(p,q)\to\frac{c_1{}^{\alpha\beta}}{q^2}q_\mu q_\nu+\frac{2}{q^2}\left[p_\mu q_\nu+p_\nu q_\mu-\frac{pq}{q^2}q_\nu q_\mu\right]\times c^{\alpha\beta\gamma}\langle p|I^\gamma|p\rangle. \quad (56)$$

In addition to (54), we can of course add any tensor of order 1, e.g., $g_{\mu\nu}$, $p_\mu p_\nu$, etc. Because of this freedom, we postpone a discussion of whether the covariant amplitude $\tilde{T}_{\mu\nu}{}^{\alpha\beta}$ is also the physical amplitude. We shall now show that Eqs. (50)–(56) also follow from the local representation (18).

Using Eq. (18) as well as the methods developed in Sec. 3, it is easily seen that

$$T_{\mu\nu}{}^{\alpha\beta}(p,q)=\sum_{k=1}^{6}T_{\mu\nu}{}^{\alpha\beta}(p,q)^{(k)},$$

$$T_{\mu\nu}{}^{\alpha\beta}(p,q)^{(1)}=\int dx\, e^{iqx}\theta(x_0)\int_0^\infty dm^2\,\rho_1{}^{\alpha\beta}(m^2,xp)\frac{\partial^2\Delta(x,m)}{\partial x_\mu\partial x_\nu}\to-g_{\mu0}g_{\nu0}c_1{}^{\alpha\beta}+\frac{q_\mu q_\nu}{q^2}c_1{}^{\alpha\beta}-i\left(q_\nu p_\mu+q_\mu p_\nu-2\frac{pq}{q^2}q_\mu q_\nu\right)\frac{c_1{}'^{\alpha\beta}}{q^2}, \quad (57)$$

$$T_{\mu\nu}{}^{\alpha\beta}(p,q)^{(2)}=-i\int dx\, e^{iqx}\theta(x_0)\int_0^\infty dm^2\,\rho_2{}^{\alpha\beta}(m^2,px)p_\mu\frac{\partial\Delta(x,m)}{\partial x_\nu}\to\frac{p_\mu q_\nu}{q^2}c_2{}^{\alpha\beta}, \quad (58)$$

$$T_{\mu\nu}{}^{\alpha\beta}(p,q)^{(3)}\to(p_\nu q_\mu/q^2)c_3{}^{\alpha\beta}, \quad (59)$$

$$T_{\mu\nu}{}^{\alpha\beta}(p,q)^{(4)}=-i\int dx\, e^{iqx}\theta(x_0)\int_0^\infty dm^2\,\rho_4{}^{\alpha\beta}(m^2,px)g_{\mu\nu}p_\lambda\frac{\partial\Delta(x,m)}{\partial x_\lambda}\to g_{\mu\nu}(pq/q^2)c_4{}^{\alpha\beta}, \quad (60)$$

$$T_{\mu\nu}{}^{\alpha\beta}(p,q)^{(5)}+T_{\mu\nu}{}^{\alpha\beta}(p,q)^{(6)}=-i\int dx\, e^{iqx}\theta(x_0)\int_0^\infty dm^2\,\Delta(x,m)[\rho_5{}^{\alpha\beta}(m^2,px)p_\mu p_\nu+\rho_6{}^{\alpha\beta}(m^2,px)g_{\mu\nu}]\to\frac{i}{q^2}\times\int_0^\infty dm^2\,[\rho_5{}^{\alpha\beta}(m^2,0)p_\mu p_\nu+\rho_6{}^{\alpha\beta}(m^2,0)g_{\mu\nu}]. \quad (61)$$

The amplitude then behaves like

$$T_{\mu\nu}{}^{\alpha\beta}(p,q)=-g_{\mu0}g_{\nu0}c_1{}^{\alpha\beta}+(q_\mu q_\nu/q^2)c_1{}^{\alpha\beta}+(1/q^2)(p_\mu q_\nu c_2{}^{\alpha\beta}+p_\nu q_\mu c_3{}^{\alpha\beta}+g_{\mu\nu}pqc_4{}^{\alpha\beta})-i\frac{c_1'^{\alpha\beta}}{q^2}\left[q_\nu p_\mu+q_\mu p_\nu-2\frac{pq}{q^2}q_\mu q_\nu\right]+R(p,q), \quad (62)$$

$$R(p,q)=O(1/q^2),\quad \text{for}\quad |q^2|\to\infty. \quad (63)$$

If Schwinger terms are absent, we get the covariant version (51) of the Bjorken limit, as we would expect, since the retarded commutator is covariant in this case.

If Schwinger terms are present, the amplitude is not covariant. The noncovariance has its origin in the noncovariance of

$$\theta(x_0)\frac{\partial^2\Delta(x,m)}{\partial x_\mu\partial x_\nu}=\frac{\partial^2\theta(x_0)\Delta(x,m)}{\partial x_\mu\partial x_\nu}-g_{\mu0}g_{\nu0}\delta(x), \quad (64)$$

which enters in Eq. (57). The important point is that we are able to isolate the noncovariant term because we understand the origin of this term [Eqs. (57) and (64)], a fact which is due to the local representation (18). We can then define a covariant amplitude by adding the constant $g_{\mu0}g_{\nu0}c_1{}^{\alpha\beta}$ just as we did in Eqs. (54) and (55), and get the covariant amplitude $T_{\mu\nu}{}^{\alpha\beta}$ in Eq. (55); this procedure is again equivalent to make the replacement

$$\theta(x_0)\frac{\partial^2\Delta(x,m)}{\partial x_\mu\partial x_\nu}\to\frac{\partial^2\theta(x_0)\Delta(x,m)}{\partial x_\mu\partial x_\nu} \quad (65)$$

in the local representation (57)–(61).

The remaining question is whether the covariant amplitude, defined uniquely by (65), is also the physical amplitude. This is in contrast to the scalar case, not so obvious. First of all, in our discussion we have ignored seagull terms. Fortunately, the effect of these terms can be taken into account through the theorem[7] that in the divergence of the physical $T_{\mu\nu}$ the seagull cancel the Schwinger terms. If we take the divergence of the covariant $\tilde{T}_{\mu\nu}$ in Eq. (55) and use the current algebra of the algebra of fields, we get

$$q^\mu\tilde{T}_{\mu\nu}{}^{\alpha\beta}\to c_1{}^{\alpha\beta}q_\nu+p_\nu c^{\alpha\beta\gamma}\langle p|I^\gamma|p\rangle. \quad (66)$$

To avoid the contribution (proportional to $c_1{}^{\alpha\beta}$) from the Schwinger terms, we must define the physical amplitude

$$\tilde{T}_{\mu\nu}{}^{\alpha\beta,ph}\to-(g_{\mu\nu}-q_\mu q_\nu/q^2)c_1{}^{\alpha\beta}+\text{"canonical terms"}, \quad (67)$$

which satisfies the usual divergence condition. The covariant contribution from the seagull term is then $-g_{\mu\nu}c_1{}^{\alpha\beta}$.

We therefore end up with the following recipe for dealing with tensor amplitudes: *Construct a local representation of the type (18) which is in accordance with all known equal-time commutation relations. Construct a covariant amplitude from the noncovariant amplitude by making replacements of the type (65). To obtain the physical amplitude, include seagull terms and require that these terms cancel the Schwinger terms in the divergence. The resulting amplitude is then the physical amplitude.*[7]

5. CONCLUSION

In Secs. 3 and 4, we have discussed special examples. We shall now briefly indicate how the method can be extended to more complicated examples. It is obvious that the work in Sec. 4 is valid not only for diagonal matrix elements of the type (17), but also for the matrix element

$$\langle p|[j_\mu{}^\alpha(x),j_\nu{}^\beta(0)]|0\rangle.$$

If we consider a matrix element of the type

$$\langle p'|[j_\mu{}^\alpha(x),j_\nu{}^\beta(0)]|p\rangle,$$

it is again possible to write down a local representation of the type (18), where the spectral functions then depend on m^2, px, and $p'x$, and where we have more terms corresponding to $p_\mu p_\nu'(p_\lambda\partial/\partial x_\lambda)$, etc. However, in spite of these complications, we obtain similar results. For matrix elements

$$\langle p_1\cdots p_n|[j_\mu{}^\alpha(x),j_\nu{}^\beta(0)]|q_1\cdots q_m\rangle$$

we can also write down local representations, and again the result is similar to what we have obtained in Sec. 4.

The main assumption used in deriving the covariant version of the Bjorken limit is that the light-cone singularities are not stronger than $\delta(x^2)$ and derivatives of $\delta(x^2)$. It is perhaps reasonable to expect that this limitation corresponds to renormalizable theories, since if the light-cone commutator is not bounded by derivatives of $\delta(x^2)$, it is likely that the corresponding amplitude is not bounded by a polynomial in momentum space. This argument is, of course, not rigorous.

[7] L. S. Brown, Phys. Rev. **150**, 1338 (1966); D. G. Boulware and L. S. Brown, *ibid.* **156**, 1724 (1967); R. P. Feynman, in *Proceedings of the 1967 International Conference on Particles and Fields* (Interscience Publishers, Inc., New York, 1967), p. 111.

The results obtained in Secs. 3 and 4 show that if the equal-time commutator contains only "canonical terms," the Bjorken limit is valid and can be made covariant by replacing the limit $q_0 \to \infty$, $\mathbf{q}$ finite, by the limit $|q^2| \to \infty$.

If Schwinger terms are present, one can obtain the correct asymptotic behavior by the following *practical recipe*: Construct a local representation of the type (18) which is in accordance with the equal-time commutation relations. Construct the covariant amplitude by making replacements of the type

$$\theta(x_0)\frac{\partial^2\Delta(x,m)}{\partial x_\mu \partial x_\nu} \to \frac{\partial^2\theta(x_0)\Delta(x,m)}{\partial x_\mu \partial x_\nu}.$$

The physical amplitude is then given by the covariant amplitude plus seagull terms. The latter are determined by demanding that the seagulls cancel the Schwinger terms in the divergence of the physical amplitude.[7]

APPENDIX

In this Appendix we shall show that the nonsingular part of the commutator does not contribute in the $|q^2| \to \infty$ limit. The proof is analogous to the arguments in Ref. 3.

Consider the expression [f does not depend on q]

$$T(q,\cdots) = -i\int dx\, e^{iqx}\theta(x_0) f(x,\cdots), \qquad \text{(A1)}$$

where $f(x,\cdots)$ is local (and nonsingular on the light cone) and

$$f(x,\cdots)=0, \quad \text{for} \quad x^2<0. \qquad \text{(A2)}$$

Then we get by a partial integration

$$T(q,\cdots) = -i\int d^3x \int_{|\mathbf{x}|}^{\infty} dx_0\, e^{iqx} f(x,\cdots)$$
$$= \frac{1}{q_0}\int d^3x\, e^{iq_0|\mathbf{x}| - i\mathbf{q}\cdot\mathbf{x}} f(x,\cdots)_{x_0=|\mathbf{x}|}, \qquad \text{(A3)}$$

where we have neglected a term which is smaller than the leading term. The angular integration gives

$$\int_{-1}^{+1} d\alpha\, e^{-i|\mathbf{q}||\mathbf{x}|\alpha} f(x_0=|\mathbf{x}|, \mathbf{x}, \cdots)$$
$$= \frac{1}{|\mathbf{q}||\mathbf{x}|}\int_{-|\mathbf{q}||\mathbf{x}|}^{|\mathbf{q}||\mathbf{x}|} du\, e^{-iu} f\left(\frac{u}{|\mathbf{q}|\alpha}, \frac{\mathbf{u}}{|\mathbf{q}|\alpha}, \cdots\right), \qquad \text{(A4)}$$

where $\mathbf{u} = u\mathbf{x}/|\mathbf{x}|$. Hence in the limit $|\mathbf{q}| \to \infty$ we get

$$T(q,\cdots) = \frac{-2}{q_0|\mathbf{q}|}\int_0^{\infty} |\mathbf{x}| e^{iq_0|\mathbf{x}|} \sin(|\mathbf{q}||\mathbf{x}|) f(0,\cdots)$$
$$= O(1/q^2), \quad \text{since } f(0,\cdots)=0 \qquad \text{(A5)}$$

since $f(x,\cdots)$ is assumed to be nonsingular on the light cone and the contribution at $x_\mu=0$ is singular [$\delta(\mathbf{x})$ or derivatives of $\delta(\mathbf{x})$]. The nonleading terms are of the order $1/q^2$, as one can see by further partial integrations.

The above theorem can also be proved more rigorously by using the Riemann-Lebesque lemma,[6] which shows that asymptotically the Fourier transform is determined by its most singular terms.