ing that axial-vector coupling is totally induced for N_2 and therefore f_{22} is expected to be of the order of, say, $f_{11}-1$, which is, in fact, the case in the solution (42). Such an observation would then imply that not only f_{22} but also f_{33} and f_{44} are both small compared with unity. We note that the solution (42) implies that, if f_{33} is small compared with unity, then f_{44} must also be small compared with unity, and vice versa. Thus it appears that the self-consistency conditions (33) and (36) suggest the very interesting conjecture that all f_{22}, f_{33}, and f_{44} are actually small compared with f_{11}.

(6) Concerning the use of the self-consistency conditions (33) and (36) for the purpose of bootstrapping the resonances, it is interesting to note that the condition (37) can never be satisfied unless the nucleon resonances with opposite parities both appear. This is exactly what is actually seen in experiments. Thus the self-consistency conditions are certainly useful in this respect. However, we also remark that as the number of the resonances increases, the total number of parameters that enter the self-consistency conditions increases faster than the total number of conditions contained in the self-consistency conditions. Therefore such self-consistency conditions are no longer very useful, when the number of the resonances is high, unless we have a great many experimental data.

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Third Sector of the Lee Model*

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We study the third sector of the Lee model. In the present work the model is augmented by a third static source U in addition to N and V, with the coupling $U\leftrightarrow V+\theta$. In the third sector the processes $U+\theta\leftrightarrow V+\theta+\theta\leftrightarrow N+\theta+\theta+\theta$ occur, providing a model enriched with a two-particle channel. Using the methods of dispersion theory, the dynamics are reduced to the solution of a Fredholm integral equation in one variable. A variational principle is given for the equation which yields the elastic scattering amplitude. Diagonalization of the second-sector connected S matrix plays an important part in the analysis. Finally, we discuss the relevance of the results to static models with crossing—specifically, to a three-meson solution of the charged-scalar static theory.

I. INTRODUCTION

THE Lee model has been extensively studied in the first and second sectors, but up to the present little work has been done on higher sectors.[1] However, higher sectors have the interesting feature that intermediate states containing many particles are present. In particular, the third sector has four-particle intermediate states, and hence it may provide hints as to how to incorporate four-particle unitarity in more interesting static models, namely those with crossing. The second sector of the Lee model served just this purpose with three-particle unitarity in the case of the charged-scalar theory.[2,3] Because we have models with crossing in mind, we study the Lee model by means of dispersion theory. Off-energy-shell methods are simpler in the case at hand, but they do not permit the inclusion of crossing, whereas dispersion methods do. It is also with more complicated models in mind that we add an elastic channel to the third sector. This is easily accomplished by adding a static source U to the Lee model with the coupling $U\leftrightarrow V+\theta$.[4] This coupling, together with the standard coupling $V\leftrightarrow N+\theta$, causes the states $U+\theta$, $V+\theta+\theta$, and $N+\theta+\theta+\theta$ to communicate in the third sector of the model. The usual Lee model, without the channel $U+\theta$, is recovered from our results by setting the $UV\theta$ coupling λ equal to zero.

In general approach, our work follows the classic paper of Amado on the second sector of the Lee model, which involves the states $V+\theta$ and $N+\theta+\theta$.[5] Amado found the $V\theta$ elastic amplitude by a scheme of contractions which avoids integrations over three-particle intermediate states in the dynamical equations. In spite of this, his elastic amplitude, being exact, naturally satisfies two- and three-particle unitarity equations. In

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[1] It is obvious that integral equations can be written for any sector which sum the Wigner-Brillouin perturbation series. Recently, D. I. Fivel [University of Maryland Report (unpublished)] has given a method, based on a dynamical algebra, for deriving equations in any sector.

[2] J. B. Bronzan, J. Math. Phys. 7, 1351 (1966).

[3] J.-P. Lebrun, McGill University Report (unpublished).

[4] J. B. Bronzan, Phys. Rev. 139, B751 (1965).

[5] R. D. Amado, Phys. Rev. 122, 696 (1961).

the third sector, Amado's scheme avoids four-particle intermediate states in the dynamical equations, but now the previously excluded three-particle states are present. This means that, as first written, the third-sector dynamical equations are integral equations in two variables. However, they may be converted to integral equations in one variable because of a factorization property of the S matrix in the second sector. Namely, the transition amplitude for $V+\theta(\omega)\to N+\theta(\omega_1)+\theta(\omega_2)$ (the ω's are meson energies) has the factorized form $f(\omega)g(\omega_1)g(\omega_2)$, and the connected amplitude for $N+\theta(\omega_1)+\theta(\omega_2)\to N+\theta(\omega_3)+\theta(\omega_4)$ has the form $h(\omega_1+\omega_2)g(\omega_1)g(\omega_2)g(\omega_3)g(\omega_4)$. It is precisely these second-sector amplitudes which appear in the kernals of the third-sector equations, and the factorization property permits the equations to be reduced to one-variable equations. There is a second, related consequence of factorization which is important in our analysis. It is that the second-sector scattering operator is the sum of diagonal and rank-one operators. This permits its explicit diagonalization.[6] In fact, if initial- and final-state rescattering factors are removed from the second-sector S matrix, the resulting connected S matrix is trivial. There are then precisely two eigenvectors with eigenvalues different from 1, and the infinity of orthogonal states have eigenvalues 1. As we shall see, it is the determinant of the second-sector connected S matrix which appears naturally in the third-sector equations.

We point out that the two-meson solution of the charged-scalar theory also has an S matrix which factorizes.[2] Hence, factorization is not a special property of the Lee model which could spoil it as a guide to a three-meson solution of the charged-scalar theory.

The final result of our work is to reduce the dynamical problem of the third sector to the solution of an integral equation in one variable. The equation is singular, but the singular term may be eliminated through use of the determinant of the connected S matrix; the equation then becomes Fredholm. The entire third-sector S matrix may be constructed in terms of the solution of this equation, although in this paper we give only elements involving $U\theta$ on one side. The elastic $U\theta$ scattering amplitude depends only on integrals over the solution of the fundamental integral equation. We express these integrals as functionals of the solutions of the fundamental integral equation and its adjoint. The stationary values of the functionals are the desired integrals, and the functionals are stationary with respect to errors in the solution of the fundamental equation and its adjoint.

As we have stated, we have studied the third sector of the Lee model partly to see if we can learn how to deal with four-particle states in other static models with crossing. We reserve to the conclusion comments on what we have learned in this direction.

[6] J. B. Bronzan, M. Cassandro, and M. Vaughn, Nuovo Cimento 46, 128 (1966).

II. FIRST AND SECOND SECTORS

When analyzed by dispersion theory, the Lee model has the feature that one must obtain first- and second-sector S matrices before formulating third-sector equations. Here we provide the required S matrices, together with a sketch of their derivation. We follow Amado except at one point near the end of the section.

The renormalized, momentum-space Hamiltonian for our extended Lee model is

$$H=mZ_U\psi_U^\dagger\psi_U+mZ_V\psi_V^\dagger\psi_V+m\psi_N^\dagger\psi_N+\sum_k \omega a_k^\dagger a_k +g[\psi_V^\dagger\psi_N A+\psi_N^\dagger A^\dagger\psi_V]+\lambda Z_1[\psi_U^\dagger\psi_V A+\psi_V^\dagger A^\dagger\psi_U] +\delta m_U Z_U\psi_U^\dagger\psi_U+\delta m_V Z_V\psi_V^\dagger\psi_V\,, \quad (1)$$

where

$$A=\sum_k \frac{u(\omega)}{(2\omega\Omega)^{1/2}}a_k\,,\quad [a_{k'},a_k^\dagger]=\delta_{k'k}\,,\quad \{\psi_U,\psi_{U'}\}=1/Z_U\,,$$

$$\{\psi_V,\psi_V^\dagger\}=1/Z_V\,,\quad \{\psi_N,\chi_N^\dagger\}=1\,,$$

and all other commutators (anticommutators in the case of two source operators) vanish. Ω is the volume of quantization, $u(\omega)$ is a cutoff function, Z_U and Z_V are wave-function renormalizations, and Z_1 is the $UV\theta$ coupling renormalization. For simplicity, and with an eye to models with crossing, we take all renormalized source masses to be m. Currents which appear are the meson current j, the V current f_V, and the U current f_U.

$$j(t)=\frac{(2\omega\Omega)^{1/2}}{u(\omega)}\left[-i\frac{d}{dt}+\omega\right]a_k(t)=\frac{(2\omega\Omega)^{1/2}}{u(\omega)}\times([H,a_k(t)]+\omega a_k(t)) =-g\psi_N^\dagger(t)\psi_V(t)-\lambda Z_1\psi_V^\dagger(t)\psi_U(t)\,,$$

$$f_V(t)=\left(-i\frac{d}{dt}+m\right)\psi_V(t)=-\delta m_V\psi_V(t)-\frac{g}{Z_V}\psi_N(t)A(t)\,,$$

$$f_U(t)=\left(-i\frac{d}{dt}-m\right)\psi_U(t)=-\delta m_U\psi_U(t)-\frac{\lambda Z_1}{Z_U}\psi_V(t)A(t)\,. \quad (2)$$

Expressions for the renormalization constants can be obtained from Ref. 4, and they are

$$Z_V=1-\frac{g^2}{\pi}\int_\mu^\infty\frac{d\omega}{\omega^2}\rho(\omega)\,,$$
$$Z_U=1+(\lambda^2/2g^2)(Z_V^2-1)\,, \quad (3)$$
$$Z_1=Z_V\,,$$

where we have set $\rho(\omega)=ku^2(\omega)/4\pi$.

In the first sector, only the elastic amplitude for $N\theta$ scattering appears. We denote the transition amplitude $T_{22}^1(\omega)$, where the superscript identifies the sector and the subscripts the particles in the initial and final states.

$$T_{22}^1(\omega)=\frac{(2\omega\Omega)^{1/2}}{u(\omega)}\langle N\theta_k\,\mathrm{out}|j^\dagger(0)|N\rangle. \quad (4)$$

This amplitude satisfies the Low equation obtained by contracting the meson

$$T_{22}^1(\omega)=-\frac{g^2}{\omega}+\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')|T_{22}^1(\omega')|^2}{\omega'-\omega-i\epsilon}. \quad (5)$$

The appropriate solution to this equation is the one without Castilleijo-Dalitz-Dyson (CDD) poles, as may be verified by solving Schrödinger's equation or summing the Wigner-Brillouin perturbation series for T_{32}^1. It is

$$T_{22}^1(\omega)=-g^2\Delta^1(\omega)/\omega\,,$$
$$\Delta^1(\omega)=\left[1+\frac{g^2\omega}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')}{\omega'^2(\omega'-\omega-i\epsilon)}\right]^{-1}. \quad (6)$$

The first-sector Omnès function Δ^1 is related to the $N\theta$ phase shift δ in the usual way:

$$\Delta^1(\omega)=\exp\left[\frac{\omega}{\pi}\int_\mu^\infty\frac{d\omega'\delta(\omega')}{\omega'(\omega'-\omega-i\epsilon)}\right], \quad (7)$$
$$e^{2i\delta(\omega)}=1+2i\rho(\omega)T_{22}^1(\omega).$$

The first-sector S matrix is

$$\langle N\theta_{k'}|S|N\theta_k\rangle=\langle N\theta_{k'}\,\mathrm{out}|N\theta_k\,\mathrm{in}\rangle$$
$$=\delta_{k'k}+2\pi i\delta(\omega'-\omega)\frac{u^2(\omega)}{2\omega\Omega}T_{22}^1(\omega). \quad (8)$$

We next consider the second sector and define scattering and production amplitudes.

$$T_{22}^2(\omega_1,\omega_2)=\frac{(2\omega\Omega)^{1/2}}{u(\omega)}\langle V\theta_k\,\mathrm{out}|j^\dagger(0)|V\rangle,$$
$$T_{32}^2(\omega_1,\omega_2)=\frac{(2\omega_1\Omega\times 2\omega_2\Omega)^{1/2}}{u(\omega_1)u(\omega_2)}\langle N\theta_{k_1}\theta_{k_2}\,\mathrm{out}|j^\dagger(0)|V\rangle. \quad (9)$$

We also define "associated" amplitudes which have disconnected parts removed, and differ from ordinary transition amplitudes only in having "out" states on both sides.

$$A_{23}^2(\omega_1,\omega_2)=\frac{(2\omega_1\Omega\times 2\omega_2\Omega)^{1/2}}{u(\omega_1)u(\omega_2)}[\langle V\theta_{k_1}\,\mathrm{out}|j^\dagger(0)|N\theta_{k_2}\,\mathrm{out}\rangle+g\delta_{k_1k_2}],$$

$$A_{33}^2(\omega_1,\omega_2,\omega_3)=\frac{(2\omega_1\Omega\times 2\omega_2\Omega\times 2\omega_3\Omega)^{1/2}}{u(\omega_1)u(\omega_2)u(\omega_3)}\Big[\langle N\theta_{k_1}\theta_{k_2}\mathrm{out}|j^\dagger(0)|N\theta_{k_3}\mathrm{out}\rangle$$
$$-\frac{1}{\sqrt{2}}\delta_{k_2k_3}\frac{u(\omega_1)}{(2\omega_1\Omega)^{1/2}}T_{22}^1(\omega_1)-\frac{1}{\sqrt{2}}\delta_{k_1k_3}\frac{u(\omega_2)}{(2\omega_2\Omega)^{1/2}}T_{22}^1(\omega_2)\Big]. \quad (10)$$

The associated amplitudes enter the theory in the following ways. First, by means of suitable contractions in Eqs. (9) and (10), we find that[2]

$$T_{32}(\omega_1,\omega_2)=(2)^{-1/2}A_{23}^2(\omega_1+\omega_2-i\epsilon,\,\omega_2)^*, \quad (11)$$

where $-i\epsilon$ means that A_{23}^2 must be continued in its first variable from the upper half-plane, where it is originally evaluated, around its threshold at μ and into the lower half-plane. Thus A_{23}^2 replaces T_{32}^2 in the theory. Second, A_{33}^2 gives us the $N\theta\theta\to N\theta\theta$ S-matrix element:

$$\langle N\theta_{k_1}\theta_{k_2}|S|N\theta_{k_3}\theta_{k_4}\rangle$$
$$=\langle N\theta_{k_1}\theta_{k_2}\,\mathrm{out}|N\theta_{k_3}\theta_{k_4}\,\mathrm{in}\rangle=\tfrac{1}{2}\langle N\theta_{k_1}\,\mathrm{out}|N\theta_{k_3}\,\mathrm{in}\rangle$$
$$\times\langle N\theta_{k_2}\,\mathrm{out}|N\theta_{k_4}\,\mathrm{in}\rangle+\tfrac{1}{2}\langle N\theta_{k_1}\,\mathrm{out}|N\theta_{k_4}\,\mathrm{in}\rangle$$
$$\times\langle N\theta_{k_2}\,\mathrm{out}|N\theta_{k_3}\,\mathrm{in}\rangle+\frac{2\pi i}{\sqrt{2}}\delta(\omega_1+\omega_2-\omega_3-\omega_4)$$
$$\times\frac{u(\omega_1)u(\omega_2)u(\omega_3)u(\omega_4)}{(2\omega_1\Omega\times 2\omega_2\Omega\times 2\omega_3\Omega\times 2\omega_4\Omega)^{1/2}}e^{2i\delta(\omega_4)}$$
$$\times A_{33}^2(\omega_1,\omega_2,\omega_4). \quad (12)$$

Equation (12) and the other S-matrix elements,

$$\langle V\theta_{k'}|S|V\theta_k\rangle$$
$$=\langle V\theta_{k'}\,\mathrm{out}|V\theta_k\,\mathrm{in}\rangle=\delta_{k'k}+2\pi i\delta(\omega'-\omega)\frac{u^2(\omega)}{2\omega\Omega}T_{22}^2(\omega),$$
$$\langle N\theta_{k_1}\theta_{k_2}|S|V\theta_k\rangle$$
$$=\langle V\theta_k|S|N\theta_{k_1}\theta_{k_2}\rangle=\langle N\theta_{k_1}\theta_{k_2}\,\mathrm{out}|V\theta_k\,\mathrm{in}\rangle$$
$$=2\pi i\delta(\omega_1+\omega_2-\omega)\frac{u(\omega_1)u(\omega_2)u(\omega)}{(2\omega_1\Omega\times 2\omega_2\Omega\times 2\omega\Omega)^{1/2}}T_{32}^2(\omega_1,\omega_2), \quad (13)$$

fill out the second-sector S matrix.

The advantage of the associated amplitudes is that Omnès-type equations can be obtained for them by contracting mesons on the right. These are

$$A_{23}^2(\omega_1,\omega_2)=\frac{g\omega_1[T_{22}^2(\omega_1)-T_{22}^1(\omega_1)]}{\omega_2(\omega_1-\omega_2+i\epsilon)}+\frac{1}{\pi}\int_\mu^\infty d\omega' e^{i\delta(\omega')}\sin\delta(\omega')\times A_{23}^2(\omega_1,\omega')\left[\frac{1}{\omega'-\omega_2+i\epsilon}+\frac{1}{\omega'-\omega_1+\omega_2-i\epsilon}\right] \quad (14)$$

and

$$A_{33}^2(\omega_1,\omega_2,\omega_3)=\frac{g(\omega_1+\omega_2)T_{32}^2(\omega_1,\omega_2)}{\omega_3(\omega_1+\omega_2-\omega_3+i\epsilon)}+\frac{1}{\pi}\int_\mu^\infty d\omega' e^{i\delta(\omega')}\sin\delta(\omega')\times A_{33}^2(\omega_1,\omega_2,\omega')\left[\frac{1}{\omega'-\omega_3+i\epsilon}+\frac{1}{\omega'-\omega_1-\omega_2+\omega_3-i\epsilon}\right]. \quad (15)$$

These equations have unique solutions[2,5]:

$$A_{23}^2(\omega_1,\omega_2)=\frac{g\omega_1[T_{22}^2(\omega_1)-T_{22}^1(\omega_1)]}{\omega_2(\omega_1-\omega_2+i\epsilon)\Delta^1(\omega_1+i\epsilon)}\Delta^1(\omega_1-\omega_2+i\epsilon)\Delta^1(\omega_2-i\epsilon),$$

$$T_{32}^2(\omega_1,\omega_2)=\frac{g(\omega_1+\omega_2)[T_{22}^2(\omega_1+\omega_2)-T_{22}^1(\omega_1+\omega_2)]}{\sqrt{2}\omega_1\omega_2\Delta^1(\omega_1+\omega_2+i\epsilon)}\times\Delta^1(\omega_1+i\epsilon)\Delta^1(\omega_2+i\epsilon), \quad (16)$$

$$A_{33}^2(\omega_1,\omega_2,\omega_3)=\frac{g^2(\omega_1+\omega_2)^2[T_{22}^2(\omega_1+\omega_2)-T_{22}^1(\omega_1+\omega_2)]}{\sqrt{2}\omega_1\omega_2\omega_3(\omega_1+\omega_2-\omega_3+i\epsilon)[\Delta^1(\omega_1+\omega_2+i\epsilon)]^2}\times\Delta^1(\omega_1+i\epsilon)\Delta^1(\omega_2+i\epsilon)\Delta^1(\omega_3-i\epsilon)\times\Delta^1(\omega_1+\omega_2-\omega_3+i\epsilon).$$

We now have the complete second-sector S matrix in terms of the elastic amplitude. To obtain this amplitude, we deviate for the first and only time from Amado's prescription, and contract the meson in Eq. (9). We remark that Amado's procedure, which we shall follow when we treat the third sector, results in a linear algebraic equation for the elastic amplitude. Our contraction results in the Low equation, which is a nonlinear integral equation for the elastic amplitude. In spite of its complexity, we can solve the Low equation by a trick introduced in the solution of the charged-scalar theory.[2] We proceed as we do because Amado's procedure is unwieldy when a single-particle state is present (the U particle in the second sector), whereas the nonlinearity of the Low equation is unwieldy when four-particle unitarity must be considered (as in the third sector). The Low equation for T_{22}^2 is

$$T_{22}^2(\omega)=\frac{g^2-\lambda^2}{\omega}+\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')|T_{22}^2(\omega')|^2}{\omega'-\omega-i\epsilon}+\frac{1}{\pi^2}\int_\mu^\infty\int_\mu^\infty\frac{d\omega' d\omega''\rho(\omega')\rho(\omega'')|T_{32}(\omega',\omega'')|^2}{\omega'+\omega''-\omega-i\epsilon}. \quad (17)$$

We solve Eq. (17) by considering the function

$$f(\omega)=(2\omega/g)\{[T_{22}^2(\omega)]^{-1}-[T_{22}^1(\omega)]^{-1}\}^{-1}. \quad (18)$$

We can study the analyticity of f by means of Eqs. (5) and (17), and the discontinuity across the cut once we use Eq. (16) to eliminate the production amplitude in Eq. (17). We find that

$$f(0)=K^{-1}=2(g^2-\lambda^2)/(2g^2-\lambda^2).$$

Also, $f(\omega)$ has no cut beginning at μ, and the discontinuity across the cut beginning at 2μ is a known function:

$$f(\omega+i\epsilon)-f(\omega-i\epsilon)=4ig^2\omega I(\omega),$$

$$I(\omega)=\frac{g^2}{2\pi}\int_\mu^{\omega-\mu}\frac{d\omega'\rho(\omega')\rho(\omega-\omega')}{\omega'^2(\omega-\omega')^2}\times|\Delta^1(\omega')\Delta^1(\omega-\omega')|^2. \quad (19)$$

The discontinuity of f vanishes at ∞. We assume that f has no poles, and that it approaches a constant at ∞. Then

$$f(\omega)=K^{-1}+\frac{2g^2\omega}{\pi}\int_{2\mu}^\infty\frac{d\omega' I(\omega')}{\omega'-\omega-i\epsilon}=K^{-1}+\omega C(\omega). \quad (20)$$

From Eq. (18),

$$T_{22}^2(\omega)=T_{22}^1(\omega)\frac{1+K\omega C(\omega)}{1+K\omega C(\omega)-2K\Delta^1(\omega)}. \quad (21)$$

This expression agrees with that obtained by summing perturbation theory.[4]

III. DIAGONALIZATION OF THE CONNECTED SECOND-SECTOR S MATRIX

In the Lee model, interactions occur only in s waves. The S matrix is therefore unity in all other angular momentum states, and in diagonalizing it we need consider only the s-wave states

$$|V\omega\rangle=\left[\frac{k\omega\Omega}{32\pi^4}\right]^{1/2}\int d\Omega_k|V\theta_k\rangle\,,$$
$$|N\omega_1\omega_2\rangle=\left[\frac{k_1\omega_1\Omega}{32\pi^4}\frac{k_2\omega_2\Omega}{32\pi^4}\right]^{1/2}\int\int d\Omega_{k_1}d\Omega_{k_2}|N\theta_{k_1}\theta_{k_2}\rangle\,, \tag{22}$$

which are normalized to δ functions of energy:

$$\langle V\omega'|V\omega\rangle=\delta(\omega'-\omega)\,,$$
$$\langle N\omega_1'\omega_2'|N\omega_1\omega_2\rangle=\tfrac{1}{2}\delta(\omega_1'-\omega_1)\delta(\omega_2'-\omega_2)+\tfrac{1}{2}\delta(\omega_1'-\omega_2)\delta(\omega_2'-\omega_1)\,. \tag{23}$$

In the present work we need the connected S matrix, which we obtain through the use of the diagonal, unitary, disconnected operator S_D. It is defined by its matrix elements, which are

$$\langle V\omega'|S_D|V\omega\rangle=\delta(\omega'-\omega)\,,\quad \langle N\omega_1\omega_2|S_D|V\omega\rangle=0\,,$$
$$\langle N\omega_1\omega_2|S_D|N\omega_3\omega_4\rangle=\tfrac{1}{2}[\delta(\omega_1-\omega_3)\delta(\omega_2-\omega_4)+\delta(\omega_1-\omega_4)\delta(\omega_2-\omega_3)]\times e^{-i\delta(\omega_3)-i\delta(\omega_4)}\,. \tag{24}$$

The connected S matrix is then

$$S_C=S_DSS_D\,. \tag{25}$$

In our s-wave basis, the matrix elements of S_C are

$$\langle V\omega'|S_C|V\omega\rangle=\delta(\omega'-\omega)[1+2i\rho(\omega)T_{22}^2(\omega)]\,,$$
$$\langle N\omega_1\omega_2|S_C|V\omega\rangle=\langle V\omega|S_C|N\omega_1\omega_2\rangle=i(2/\pi)^{1/2}\delta(\omega_1+\omega_2-\omega)\times[\rho(\omega)\rho(\omega_1)\rho(\omega_2)]^{1/2}\frac{g\omega[T_{22}^2(\omega)-T_{22}^1(\omega)]}{\omega_1\omega_2\Delta^1(\omega+i\epsilon)}\times|\Delta^1(\omega_1)\Delta^1(\omega_2)|\,, \tag{26}$$
$$\langle N\omega_1\omega_2|S_C|N\omega_3\omega_4\rangle=\tfrac{1}{2}[\delta(\omega_1-\omega_3)\delta(\omega_2-\omega_4)+\delta(\omega_1-\omega_4)\delta(\omega_2-\omega_3)]+i\pi^{-1}\delta(\omega_1+\omega_2-\omega_3-\omega_4)[\rho(\omega_1)\rho(\omega_2)\rho(\omega_3)\rho(\omega_4)]^{1/2}\times\frac{g^2(\omega_1+\omega_2)^2[T_{22}^2(\omega_1+\omega_2)-T_{22}^1(\omega_1+\omega_2)]}{\omega_1\omega_2\omega_3\omega_4[\Delta^1(\omega_1+\omega_2+i\epsilon)]^2}\times|\Delta^1(\omega_1)\Delta^1(\omega_2)\Delta^1(\omega_3)\Delta^1(\omega_4)|\,.$$

Eigenvectors of S_C with energy ω must satisfy the equation

$$S_C|\omega,\lambda\rangle=\lambda|\omega,\lambda\rangle\,, \tag{27}$$

and have the form

$$|\omega,\lambda\rangle=\alpha(\omega,\lambda)|V\omega\rangle+\int_\mu^{\omega-\mu}d\omega_1\beta(\omega_1,\omega,\lambda)|N\omega_1,\omega-\omega_1\rangle\,. \tag{28}$$

When Eq. (28) is substituted into Eq. (27), the factorization property of S_C allows us to derive the eigenvalue equation

$$\lambda^2-2\lambda\left\{1+i\rho(\omega)T_{22}^2(\omega)+\frac{\omega^2i}{[\Delta^1(\omega+i\epsilon)]^2}\times[T_{22}^2(\omega)-T_{22}^1(\omega)]I(\omega)\right\}+2e^{2i\delta(\omega)}\frac{\omega^2i}{[\Delta^1(\omega+i\epsilon)]^2}\times[T_{22}^2(\omega)-T_{22}^1(\omega)]I(\omega)+1+2i\rho(\omega)T_{22}^2(\omega)=0\,. \tag{29}$$

We see immediately that there are exactly two nontrivial eigenvalues. All the other eigenvalues are 1. The constant term is the product of the two eigenvalues. When the relationships of I to $\operatorname{Im}f$, and f to the elastic amplitudes are used, we find for the product

$$\det S_C=\lambda_1\lambda_2=\left[\frac{T_{22}^2(\omega)}{T_{22}^1(\omega)}-1\right]\left[\frac{T_{22}^2(\omega)^*}{T_{22}^1(\omega)^*}-1\right]^{-1}. \tag{30}$$

Equation (30) permits us to evaluate the Omnès function related to the sum of the eigenvalues. We define

$$\lambda_1=e^{2i\theta_1},\quad \lambda_2=e^{2i\theta_2},$$

and the Omnès function for the second sector

$$\Delta^2(\omega)=\exp\left[\frac{\omega}{\pi}\int_\mu^\infty\frac{d\omega'[\theta_1(\omega')+\theta_2(\omega')]}{\omega'(\omega'-\omega-i\epsilon)}\right]=\exp\left[\frac{\omega}{2i\pi}\int_\mu^\infty\frac{d\omega'\ln\lambda_1(\omega')\lambda_2(\omega')}{\omega'(\omega'-\omega-i\epsilon)}\right]=\exp\left[\frac{\omega}{2i\pi}\int_C\frac{d\omega'}{\omega'(\omega'-\omega-i\epsilon)}\ln\left\{\frac{T_{22}^2(\omega')}{T_{22}^1(\omega')}-1\right\}\right], \tag{31}$$

where C is a contour which circles the cut from μ to ∞ in a clockwise fashion. By the calculus of residues,

$$\Delta^2(\omega)=\frac{g^2}{\lambda^2-2g^2}\left[\frac{T_{22}^2(\omega)}{T_{22}^1(\omega)}-1\right]. \tag{32}$$

Finally, we shall later encounter the sum of the eigenvalues in the form

$$e^{i(\theta_1+\theta_2-\delta)}\sin(\theta_1+\theta_2-\delta)=\frac{\lambda_1\lambda_2 e^{-2i\delta}-1}{2i}=\rho(\omega)[T_{22}{}^2(\omega)-T_{22}{}^1(\omega)]e^{-2i\delta(\omega)}+\frac{\omega^2[T_{22}{}^2(\omega)-T_{22}{}^1(\omega)]}{[\Delta^1(\omega+i\epsilon)]^2}I(\omega).\quad(33)$$

IV. THIRD SECTOR

The elastic scattering amplitude in the third sector is

$$T_{22}{}^3(\omega)=\frac{(2\omega\Omega)^{1/2}}{u(\omega)}\langle U\theta_k\,\mathrm{out}|j^\dagger(0)|U\rangle.\quad(34)$$

We follow Amado's prescription and contract the U particle on the right. This results in the representation

$$T_{22}{}^3(\omega)=-\frac{\lambda V_{21}{}^2(\omega)}{\omega}+\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')}{\omega'}A_{23}{}^3(\omega,\omega')V_{21}{}^2(\omega')+\frac{1}{\pi^2}\int_\mu^\infty\int_\mu^\infty\frac{d\omega'd\omega''\rho(\omega')\rho(\omega'')}{\omega'+\omega''}\times A_{24}{}^3(\omega,\omega',\omega'')V_{31}{}^2(\omega',\omega''),\quad(35)$$

where

$$V_{21}{}^2(\omega)=\frac{(2\omega\Omega)^{1/2}}{u(\omega)}\langle V\theta_k\,\mathrm{out}|f_U{}^\dagger(0)|0\rangle,$$

$$V_{31}{}^2(\omega_1,\omega_2)=\frac{(2\omega_1\Omega\times2\omega_2\Omega)^{1/2}}{u(\omega_1)u(\omega_2)}\langle N\theta_{k_1}\theta_{k_2}\,\mathrm{out}|f_U{}^\dagger(0)|0\rangle,$$

$$A_{23}{}^3(\omega_1,\omega_2)=\frac{(2\omega_1\Omega\times2\omega_2\Omega)^{1/2}}{u(\omega_1)u(\omega_2)}\times[\langle U\theta_{k_1}\,\mathrm{out}|j^\dagger(0)|V\theta_{k_2}\,\mathrm{out}\rangle+\lambda\delta_{k_1k_2}],\quad(36)$$

$$A_{24}{}^3(\omega_1,\omega_2,\omega_3)=\frac{(2\omega_1\Omega\times2\omega_2\Omega\times2\omega_3\Omega)^{1/2}}{u(\omega_1)u(\omega_2)u(\omega_3)}\times\langle U\theta_{k_1}\,\mathrm{out}|j^\dagger(0)|N\theta_{k_2}\theta_{k_3}\,\mathrm{out}\rangle.$$

We remark that the disconnected parts of $A_{24}{}^3$ are proportional to $[(2\omega\Omega)^{1/2}/u(\omega)]\times\langle U|j^\dagger(0)|N\theta_k\,\mathrm{out}\rangle$. This amplitude satisfies an integral equation with the same kernel as Eq. (14), but with no inhomogeneous term. However, Eq. (14) has no homogeneous solutions, and consequently $A_{24}{}^3$ has no disconnected parts to be removed in its definition.

We first study the second-sector vertex functions appearing in Eq. (35). Contracting mesons, we find they satisfy a pair of coupled singular integral equations.

$$V_{21}{}^2(\omega)=-\frac{\lambda Z_V}{Z_U}+\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')T_{22}{}^2(\omega')^*V_{21}{}^2(\omega')}{\omega'-\omega-i\epsilon}+\frac{1}{\pi^2}\int_\mu^\infty\int_\mu^\infty\frac{d\omega'd\omega''\rho(\omega')\rho(\omega'')T_{32}{}^2(\omega',\omega'')^*V_{31}{}^2(\omega',\omega'')}{\omega'+\omega''-\omega-i\epsilon},$$

$$V_{31}{}^2(\omega_1,\omega_2)=-\frac{g\lambda Z_V\Delta^1(\omega_2)}{Z_U\omega\sqrt2}+\frac{gV_{21}{}^2(\omega_2)}{\omega_1\sqrt2}+\frac{1}{\pi\sqrt2}\int_\mu^\infty\frac{d\omega'\rho(\omega')A_{23}{}^2(\omega',\omega_2)^*V_{32}{}^2(\omega')}{\omega'-\omega_1-\omega_2-i\epsilon}+\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')T_{22}{}^1(\omega')^*V_{31}{}^2(\omega',\omega_2)}{\omega'-\omega_1-i\epsilon}+\frac{1}{\pi^2}\int_\mu^\infty\int_\mu^\infty\frac{d\omega'd\omega''\rho(\omega')\rho(\omega'')A_{33}{}^2(\omega',\omega'',\omega_2)V_{31}{}^2(\omega',\omega'')}{\omega'+\omega''-\omega_1-\omega_2-i\epsilon}.\quad(37)$$

The first term on the right of the second equation is an equal-time commutator term involving the first-sector vertex function $V_{21}{}^1(\omega)=[(2\omega\Omega)^{1/2}/u(\omega)]\langle N\theta_k\,\mathrm{out}|f_V{}^\dagger(0)|0\rangle$. In Ref. 7 it is shown that $V_{21}{}^1(\omega)=-g\Delta^1(\omega)$. We simplify these equations by means of the kernel transformation developed in the Appendix. For $V_{21}{}^2$ we add and subtract the term

$$\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')T_{21}{}^1(\omega')^*V_{21}{}^1(\omega')}{\omega'-\omega-i\epsilon}$$

on the right, and eliminate the term we have added by means of the transformation. For $V_{31}{}^2$ we eliminate the second integral on the right by means of the kernel transformation. We also substitute the integral equation for $V_{21}{}^2(\omega_2)$ in the second inhomogeneous term for $V_{31}{}^2$. The resulting equations show that

$$V_{31}{}^2(\omega_1,\omega_2)=\frac{g(\omega_1+\omega_2)V_{21}{}^2(\omega_1+\omega_2)\Delta^1(\omega_1+i\epsilon)\Delta^1(\omega_2+i\epsilon)}{\sqrt2\omega_1\omega_2\Delta^1(\omega_1+\omega_2+i\epsilon)},\quad(38)$$

so the problem now involves only one function of one variable. This function is conveniently chosen to be

$$W(\omega)=V_{21}{}^2(\omega)/\Delta^1(\omega+i\epsilon).\quad(39)$$

[7] M. L. Goldberger and S. B. Treiman, Phys. Rev. 113, 1663 (1959).

W satisfies the singular equation

$$W(\omega)=-\frac{\lambda Z_V^2}{Z_U}+\frac{1}{\pi}\int_\mu^\infty \frac{d\omega'\rho(\omega')[T_{22}^2(\omega')^*-T_{22}^1(\omega')^*]e^{2i\delta(\omega')}W(\omega')}{\omega'-\omega-i\epsilon}$$
$$+\frac{1}{\pi}\int_{2\mu}^\infty \frac{d\omega'\omega'^2[T_{22}^2(\omega')^*-T_{22}^1(\omega')^*]I(\omega')W(\omega')}{[\Delta^1(\omega'-i\epsilon)]^2(\omega'-\omega-i\epsilon)}. \quad (40)$$

Using Eq. (33), this may be written as the Omnès equation

$$W(\omega)=-\frac{\lambda Z_V^2}{Z_U}+\frac{1}{\pi}\int_\mu^\infty \frac{d\omega' e^{-i[\theta_1(\omega')+\theta_2(\omega')-\delta(\omega')]}\sin[\theta_1(\omega')+\theta_2(\omega')-\delta(\omega')]W(\omega')}{\omega'-\omega-i\epsilon}. \quad (41)$$

The solution is

$$W(\omega)=-\lambda\frac{\Delta^2(\omega+i\epsilon)}{\Delta^1(\omega+i\epsilon)},$$
$$V_{21}^2(\omega)=-\lambda\Delta^2(\omega+i\epsilon), \quad (42)$$
$$V_{31}^2(\omega)=-\frac{\lambda g(\omega_1+\omega_2)\Delta^2(\omega_1+\omega_2+i\epsilon)\Delta^1(\omega_1+i\epsilon)\Delta^1(\omega_2+i\epsilon)}{\sqrt{2}\omega_1\omega_2\Delta^1(\omega_1+\omega_2+i\epsilon)}.$$

Here we have made use of the asymptotic values

$$\Delta^1(\infty)=1/Z_V,\quad \Delta^2(\infty)=Z_V/Z_U. \quad (43)$$

In Eq. (35) the dynamics of the third sector is contained in the associated amplitudes A_{23}^3 and A_{24}^3. Contracting mesons on the right, we again obtain coupled integral equations for these amplitudes.

$$A_{23}^3(\omega_1,\omega_2)=\lambda[T_{22}^3(\omega_1)-T_{22}^2(\omega_1)]\left[\frac{1}{\omega_2-i\epsilon}+\frac{1}{\omega_1-\omega_2+i\epsilon}\right]+\frac{1}{\pi}\int_\mu^\infty d\omega'\rho(\omega')T_{22}^2(\omega')A_{23}^3(\omega_1,\omega')$$
$$\times\left[\frac{1}{\omega'-\omega_2+i\epsilon}+\frac{1}{\omega'-\omega_1+\omega_2-i\epsilon}\right]+\frac{1}{\pi^2}\int_\mu^\infty\int_\mu^\infty d\omega'd\omega''\rho(\omega')\rho(\omega'')T_{32}^2(\omega',\omega'')A_{24}^3(\omega_1,\omega',\omega'')$$
$$\times\left[\frac{1}{\omega'+\omega''-\omega_2+i\epsilon}+\frac{1}{\omega'+\omega''-\omega_1+\omega_2-i\epsilon}\right],$$
$$A_{24}^3(\omega_1,\omega_2,\omega_3)=\frac{1}{\sqrt{2}}[gA_{23}^3(\omega_1,\omega_2)-\lambda A_{23}^2(\omega_1,\omega_2)]\left[\frac{1}{\omega_3-i\epsilon}+\frac{1}{\omega_1-\omega_2-\omega_3+i\epsilon}\right]+\frac{1}{\pi}\int_\mu^\infty d\omega'\rho(\omega')T_{22}^1(\omega')A_{24}^3(\omega_1,\omega_2,\omega')$$
$$\times\left[\frac{1}{\omega'-\omega_3+i\epsilon}+\frac{1}{\omega'-\omega_1+\omega_2+\omega_3-i\epsilon}\right]+\frac{1}{\pi\sqrt{2}}\int_\mu^\infty d\omega'\rho(\omega')A_{23}^2(\omega',\omega_2)A_{23}^3(\omega_1,\omega')$$
$$\times\left[\frac{1}{\omega'-\omega_2-\omega_3+i\epsilon}+\frac{1}{\omega'-\omega_1+\omega_3-i\epsilon}\right]+\frac{1}{\pi^2\sqrt{2}}\int_\mu^\infty\int_\mu^\infty d\omega'd\omega''\rho(\omega')\rho(\omega'')A_{33}^2(\omega',\omega'',\omega_2)A_{24}^3(\omega_1,\omega',\omega'')$$
$$\times\left[\frac{1}{\omega'+\omega''-\omega_2-\omega_3+i\epsilon}+\frac{1}{\omega'+\omega''-\omega_1+\omega_3-i\epsilon}\right]. \quad (44)$$

In the first equation we add and subtract the term

$$\frac{1}{\pi}\int_\mu^\infty d\omega'\rho(\omega')T_{22}^1(\omega')A_{23}^3(\omega_1,\omega')\left[\frac{1}{\omega'-\omega_2+i\epsilon}+\frac{1}{\omega'-\omega_1+\omega_2-i\epsilon}\right],$$

and use the kernel transformation given in the Appendix to eliminate the term we have added. Similarly, we eliminate the first integral in the second equation by means of the transformation. The resulting equations may be written in terms of the single function

$$\psi(\omega_1,\omega_2)=\frac{1}{\pi}\int_\mu^\infty \frac{d\omega'\rho(\omega')[T_{22}^2(\omega')-T_{22}^1(\omega')]A_{23}^3(\omega_2,\omega')}{(\omega'-\omega_1)\Delta^1(\omega'+i\epsilon)\Delta^1(\omega_2-\omega'+i\epsilon)}+\frac{g}{\pi^2\sqrt{2}}\int_{2\mu}^\infty \frac{d\omega'\omega'[T_{22}^2(\omega')-T_{22}^1(\omega')]}{(\omega'-\omega_1)[\Delta^1(\omega'+i\epsilon)]^2\Delta^1(\omega_2-\omega'+i\epsilon)}$$
$$\times\int_\mu^{\omega'-\mu}\frac{d\omega''\rho(\omega'')\rho(\omega'-\omega'')}{\omega''(\omega'-\omega'')}\Delta^1(\omega''+i\epsilon)\Delta^1(\omega'-\omega''+i\epsilon)A_{24}^3(\omega_2,\,\omega'',\,\omega'-\omega''). \quad (45)$$

These equations are

$$A_{23}^3(\omega_1,\omega_2)=\Delta^1(\omega_2-i\epsilon)\Delta^1(\omega_1-\omega_2+i\epsilon)\left\{\frac{\lambda\omega_1[T_{22}^3(\omega_1)-T_{22}^2(\omega_1)]}{\omega_2(\omega_1-\omega_2+i\epsilon)\Delta^1(\omega_1+i\epsilon)}+\psi(\omega_2-i\epsilon,\,\omega_1)+\psi(\omega_1-\omega_2+i\epsilon,\,\omega_1)\right\},$$

$$A_{24}^3(\omega_1,\omega_2,\omega_3)=\frac{g}{\sqrt{2}}\Delta^1(\omega_2-i\epsilon)\Delta^1(\omega_3-i\epsilon)\Delta^1(\omega_1-\omega_2-\omega_3+i\epsilon)\left\{\frac{\lambda\omega_1[T_{22}^3(\omega_1)+T_{22}^1(\omega_1)-2T_{22}^2(\omega_1)]}{\omega_2\omega_3(\omega_1-\omega_2-\omega_3+i\epsilon)\Delta^1(\omega_1+i\epsilon)}\right.$$
$$\left.+\frac{(\omega_1-\omega_2)\psi(\omega_1-\omega_2+i\epsilon,\,\omega_1)}{\omega_3(\omega_1-\omega_2-\omega_3+i\epsilon)}+\frac{(\omega_1-\omega_3)\psi(\omega_1-\omega_3+i\epsilon,\,\omega_1)}{\omega_2(\omega_1-\omega_2-\omega_3+i\epsilon)}+\frac{(\omega_2+\omega_3)\psi(\omega_2+\omega_3-i\epsilon,\,\omega_1)}{\omega_2\omega_3}\right\}. \quad (46)$$

If we operate on these equations with the integral operator indicated in Eq. (45), we can derive a singular integral equation in one variable for ψ. From this integral equation we see that ψ has the representation

$$\psi(\omega_1,\omega_2)=\frac{\lambda\omega_2[T_{22}^3(\omega_2)-T_{22}^2(\omega_2)]}{\Delta^1(\omega_2+i\epsilon)}\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')[T_{22}^2(\omega')-T_{22}^1(\omega')]e^{-2i\delta(\omega')}}{\omega'(\omega_2-\omega'+i\epsilon)}\psi(\omega_1,\omega',\omega_2)$$
$$+\frac{\lambda\omega_2[T_{22}^3(\omega_2)+T_{22}^1(\omega_2)-2T_{22}^2(\omega_2)]}{\Delta^1(\omega_2+i\epsilon)}\frac{1}{\pi}\int_{2\mu}^\infty\frac{d\omega'\omega'[T_{22}^2(\omega')-T_{22}^1(\omega')]I(\omega')\psi(\omega_1,\omega',\omega_2)}{[\Delta^1(\omega'+i\epsilon)]^2(\omega_2-\omega'+i\epsilon)}, \quad (47)$$

where the new function $\psi(\omega_1,\omega_0,\omega_2)$ satisfies the equation

$$\psi(\omega_1,\omega_0,\omega_2)=\frac{1}{\omega_0-\omega_1}+\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'}{\omega'-\omega_1}\psi(\omega'-i\epsilon,\,\omega_0,\,\omega_2)e^{i[\theta_1(\omega')+\theta_2(\omega')-\delta(\omega')]}\sin[\theta_1(\omega')+\theta_2(\omega')-\delta(\omega')]$$
$$+\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')}{\omega'-\omega_1}[T_{22}^2(\omega')-T_{22}^1(\omega')]e^{-2i\delta(\omega')}\psi(\omega_2-\omega'+i\epsilon,\,\omega_0,\,\omega_2)+\frac{1}{\pi}\int_{2\mu}^\infty\frac{d\omega'\omega'[T_{22}^2(\omega')-T_{22}^1(\omega')]}{(\omega'-\omega_1)[\Delta^1(\omega'+i\epsilon)]^2}$$
$$\times\frac{g^2}{\pi}\int_\mu^{\omega'-\mu}\frac{d\omega''\rho(\omega'')\rho(\omega'-\omega'')}{\omega''(\omega'-\omega'')}|\Delta^1(\omega'')\Delta^1(\omega'-\omega'')|^2\psi(\omega_2-\omega''+i\epsilon,\,\omega_0,\,\omega_2)\left[\frac{1}{\omega'-\omega''}+\frac{1}{\omega_2-\omega'+i\epsilon}\right]. \quad (48)$$

The presence of the last two integrals in Eq. (48) prevents us from solving this equation in the same way that we solved the second-sector vertex-function equation. These new terms come from the fact that all the mesons can scatter off the source; that is, they are required by Bose symmetry. The first integral operator in Eq. (48) is singular, and can be eliminated by a transformation given in the Appendix. The resulting Fredholm equation takes its simplest form when written in terms of the function

$$\chi(\omega_1,\omega_0,\omega_2)=\frac{\Delta^2(\omega_0+i\epsilon)\Delta^1(\omega_2-\omega_1+i\epsilon)}{\Delta^2(\omega_2-\omega_1+i\epsilon)\Delta^1(\omega_0+i\epsilon)}\psi(\omega_2-\omega_1+i\epsilon,\,\omega_0,\,\omega_2). \quad (49)$$

The equation is

$$\chi(\omega_1,\omega_0,\omega_2)=\frac{1}{\omega_0+\omega_1-\omega_2-i\epsilon}+(2g^2-\lambda^2)\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')|\Delta^1(\omega')|^2\Delta^2(\omega_2-\omega'+i\epsilon)}{\omega'(\omega'+\omega_1-\omega_2-i\epsilon)\Delta^1(\omega_2-\omega'+i\epsilon)}\chi(\omega',\omega_0,\omega_2)$$
$$+(2g^2-\lambda^2)\frac{1}{\pi}\int_{2\mu}^\infty\frac{d\omega'}{\omega'+\omega_1-\omega_2-i\epsilon}\frac{g^2}{\pi}\int_\mu^{\omega'-\mu}\frac{d\omega''\rho(\omega'')\rho(\omega'-\omega'')}{\omega''(\omega'-\omega'')}|\Delta^1(\omega'')\Delta'(\omega'-\omega'')|^2$$
$$\times\frac{\Delta^2(\omega_2-\omega''+i\epsilon)}{\Delta^1(\omega_2-\omega''+i\epsilon)}\chi(\omega'',\omega_0,\omega_2)\left[\frac{1}{\omega'-\omega''}+\frac{1}{\omega_2-\omega'+i\epsilon}\right]. \quad (50)$$

The Cauchy denominators in Eq. (50) are nonsingular as long as $\omega_2<2\mu$, and the equation is Fredholm. For $\omega_2>2\mu$, the equation is essentially Fredholm in that it is the boundary value (in ω_2) of a Fredholm equation. In fact, for $\omega_2>2\mu$ the equation can be made explicitly Fredholm by a slight deformation of the contour of integration. This technique has been used in the numerical solution of integral equations in Lee-type models (with recoil) of the three-nucleon system.[8] If use is made of the representation

$$\Delta^1(\omega)=\frac{1}{Z_V}-\frac{g^2}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')|\Delta^1(\omega')|^2}{\omega'(\omega'-\omega-i\epsilon)},\tag{51}$$

Eq. (50) takes the final form

$$\chi(\omega_1,\omega_0,\omega_2)=\frac{1}{\omega_0+\omega_1-\omega_2-i\epsilon}+(2g^2-\lambda^2)\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')|\Delta^1(\omega')|^2\Delta^2(\omega_2-\omega'+i\epsilon)}{\omega'\Delta^1(\omega_2-\omega'+i\epsilon)}\chi(\omega',\omega_0,\omega_2)$$
$$\times\left\{\frac{\Delta^1(\omega_2-\omega'+i\epsilon)}{\omega_1}+\Delta^1(\omega_2-\omega_1-\omega'+i\epsilon)\left[\frac{1}{\omega'+\omega_1-\omega_2-i\epsilon}-\frac{1}{\omega_1}\right]\right\}.\tag{52}$$

We now have reduced the third-sector dynamics to the solution of a Fredholm integral equation in one variable. In view of Eqs. (46) and (47), Eq. (35) is a linear algebraic equation for $T_{22}{}^3$ once we have obtained ψ. Thus, all S-matrix elements involving the $U\theta$ channel are determined by ψ. In fact, the complete third-sector S matrix can be given in terms of ψ. Rather than display these other amplitudes, we shall concentrate on recovering $T_{22}{}^3$ in terms of ψ. In view of Eq. (47), an integral over $\psi(\omega_1,\omega_0,\omega_2)$ is required to obtain $A_{23}{}^3$ and $A_{24}{}^3$. Then, in Eq. (35) either one or two more integrations are required to obtain $T_{22}{}^3$. This procedure yields a very complicated expression for $T_{22}{}^3$, and we therefore proceed to simplify it as much as possible. We begin by operating on both sides of Eq. (48), written for $\psi(\omega_2-\omega'+i\epsilon,\,\omega_0,\,\omega_2)$, with the operator

$$\frac{g^2}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')|\Delta^1(\omega')|^2}{\omega'}.$$

We make use of Eq. (51) to obtain the relation

$$\frac{g^2}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')}{\omega'}|\Delta^1(\omega')|^2\psi(\omega_2-\omega'+i\epsilon,\omega_0,\omega_2)+\Delta^1(\omega_2-\omega_0+i\epsilon)+\frac{1}{Z_V}\lim_{\omega\to\infty}\omega\psi(\omega,\omega_0,\omega_2)$$
$$=-\frac{1}{\pi}\int_\mu^\infty d\omega'\Delta^1(\omega_2-\omega'+i\epsilon)\rho(\omega')[T_{22}{}^2(\omega')-T_{22}{}^1(\omega')]e^{-2i\delta(\omega')}[\psi(\omega'-i\epsilon,\,\omega_0,\,\omega_2)+\psi(\omega_2-\omega'+i\epsilon,\,\omega_0,\,\omega_2)]$$
$$-\frac{1}{\pi}\int_{2\mu}^\infty\frac{d\omega'\Delta^1(\omega_2-\omega'+i\epsilon)\omega'[T_{22}{}^2(\omega')-T_{22}{}^1(\omega')]}{[\Delta^1(\omega'+i\epsilon)]^2}\left\{\omega' I(\omega')\psi(\omega'-i\epsilon,\,\omega_0,\,\omega_2)\right.$$
$$\left.+\frac{g^2}{\pi}\int_\mu^{\omega'-\mu}\frac{d\omega''\rho(\omega'')\rho(\omega'-\omega'')}{\omega''(\omega'-\omega'')}|\Delta^1(\omega'')\Delta^1(\omega'-\omega'')|^2\psi(\omega_2-\omega''+i\epsilon,\,\omega_0,\,\omega_2)\left[\frac{1}{\omega'-\omega''}+\frac{1}{\omega_2-\omega'+i\epsilon}\right]\right\}.\tag{53}$$

Now when Eq. (47) is substituted into Eq. (46), and Eq. (46) is substituted into Eq. (35), the right side of Eq. (53) appears. Thus, the expression for $T_{22}{}^3$ is already simplified by using Eq. (53). The limit on the left side of Eq. (53) may be evaluated by means of Eq. (52), leading to the further relation

$$\frac{g^2}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')}{\omega'}|\Delta^1(\omega')|^2\psi(\omega_2-\omega'+i\epsilon,\,\omega_0,\,\omega_2)+\Delta^1(\omega_2-\omega_0+i\epsilon)+\frac{1}{Z_V}\lim_{\omega\to\infty}\omega\psi(\omega,\omega_0,\omega_2)$$
$$=\Delta^1(\omega_2-\omega_0+i\epsilon)-\frac{1}{Z_UZ_V}\frac{\Delta^1(\omega_0+i\epsilon)}{\Delta^2(\omega_0+i\epsilon)}+\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')}{\omega'}|\Delta^1(\omega')|^2$$
$$\times\psi(\omega_2-\omega'+i\epsilon,\,\omega_0,\,\omega_2)\left\{g^2-\frac{(2g^2-\lambda^2)}{Z_UZ_V}\Delta^1(\omega_2-\omega'+i\epsilon)\right\}.\tag{54}$$

[8] R. Aaron (private communication).

When Eqs. (53) and (54) are used, Eq. (35) becomes

$$T_{22}{}^3(\omega)=\frac{\lambda^2[T_{22}{}^2(\omega)-T_{22}{}^1(\omega)]}{(\lambda^2-2g^2)\Delta^1(\omega+i\epsilon)}+\frac{\lambda^2\omega[T_{22}{}^3(\omega)-T_{22}{}^2(\omega)]}{\Delta^1(\omega+i\epsilon)}\frac{1}{\pi}\int_\mu^\infty\frac{d\omega_0\rho(\omega_0)}{\omega_0{}^2}\frac{\Delta^1(\omega_0-i\epsilon)\Delta^2(\omega_0+i\epsilon)}{\omega_0-\omega-i\epsilon}$$
$$\times\left\{\frac{1}{Z_UZ_V}\frac{\Delta^1(\omega_0+i\epsilon)}{\Delta^2(\omega_0+i\epsilon)}-\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')}{\omega'}|\Delta^1(\omega')|^2\psi(\omega-\omega'+i\epsilon,\omega_0,\omega)\left[g^2-\frac{(2g^2-\lambda^2)}{Z_UZ_V}\Delta^1(\omega-\omega'+i\epsilon)\right]\right\}$$
$$+\frac{\lambda^2\omega[T_{22}{}^3(\omega)+T_{22}{}^1(\omega)-2T_{22}{}^2(\omega)]}{\Delta^1(\omega+i\epsilon)}\frac{1}{\pi}\int_{2\mu}^\infty\frac{d\omega_0\Delta^2(\omega_0+i\epsilon)I(\omega_0)}{\Delta^1(\omega_0+i\epsilon)(\omega_0-\omega-i\epsilon)}\left\{\frac{1}{Z_UZ_V}\frac{\Delta^1(\omega_0+i\epsilon)}{\Delta^2(\omega_0+i\epsilon)}\right.$$
$$\left.-\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')}{\omega'}|\Delta^1(\omega')|^2\psi(\omega-\omega'+i\epsilon,\omega_0,\omega)\left[g^2-\frac{(2g^2-\lambda^2)}{Z_UZ_V}\Delta^1(\omega-\omega'+i\epsilon)\right]\right\}. \quad (55)$$

The integrations over ω_0 in Eq. (55) can be performed once we display the dependence on this variable of the terms involving ψ. To this end we introduce the "adjoint" function ϕ, which satisfies the integral equation

$$\phi(\omega_1,\omega_2)=g^2-(2g^2-\lambda^2)\left[\frac{1}{Z_UZ_V}-\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')|\Delta^1(\omega')|^2}{\omega'^2\Delta^1(\omega_2-\omega'+i\epsilon)}\Delta^2(\omega_2-\omega'+i\epsilon)\phi(\omega',\omega_2)\right]\Delta^1(\omega_2-\omega_1+i\epsilon)$$
$$+(2g^2-\lambda^2)\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')|\Delta^1(\omega')|^2\Delta^2(\omega_2-\omega'-i\epsilon)\Delta^1(\omega_2-\omega_1-\omega'+i\epsilon)\phi(\omega',\omega_2)}{\omega'\Delta^1(\omega_2-\omega'+i\epsilon)}\left[\frac{1}{\omega'+\omega_1-\omega_2-i\epsilon}-\frac{1}{\omega'}\right]. \quad (56)$$

This equation is Fredholm. Operating on Eqs. (52) and (56) with the appropriate integral operators, we find

$$\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')}{\omega'}|\Delta^1(\omega')|^2\psi(\omega-\omega'+i\epsilon,\omega_0,\omega)\left[g^2-\frac{(2g^2-\lambda^2)}{Z_UZ_V}\Delta^1(\omega-\omega'+i\epsilon)\right]$$
$$=\frac{\Delta^1(\omega_0+i\epsilon)}{\Delta^2(\omega_0+i\epsilon)}\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')|\Delta^1(\omega')|^2\Delta^2(\omega-\omega'+i\epsilon)\phi(\omega',\omega)}{\Delta^1(\omega-\omega'+i\epsilon)\omega'(\omega'+\omega_0-\omega-i\epsilon)}. \quad (57)$$

When we substitute Eq. (57) into Eq. (55), the integration over ω_0 can be carried out. The result is

$$T_{22}{}^3(\omega)=\lambda^2\left\{-\frac{\Delta^2(\omega+i\epsilon)}{\omega}+\left[\frac{g^2}{2g^2-\lambda^2}-\frac{g^2\Delta^1(\omega+i\epsilon)}{(2g^2-\lambda^2)\Delta^2(\omega+i\epsilon)}-\frac{\lambda^2}{g^2}\Delta^2(\omega+i\epsilon)-\frac{(2g^2-\lambda^2)}{g^2}\Delta^1(\omega+i\epsilon)\Delta^2(\omega+i\epsilon)\right.\right.$$
$$\left.+2\Delta^1(\omega+i\epsilon)\right]\frac{J_1(\omega)}{\omega}+\left[\frac{g^2}{2g^2-\lambda^2}-\frac{\lambda^2}{g^2}\Delta^2(\omega+i\epsilon)\right]J_2(\omega)-\frac{(2g^2-\lambda^2)}{g^2}\Delta^2(\omega+i\epsilon)J_3(\omega)+\left[-\frac{g^2}{2g^2-\lambda^2}+2\Delta^2(\omega+i\epsilon)\right]$$
$$\left.\times J_4(\omega)\right\}\left\{1-\frac{\lambda^2}{(2g^2-\lambda^2)\Delta^1(\omega+i\epsilon)}\left[\left(1-\frac{\Delta^1(\omega+i\epsilon)}{\Delta^2(\omega+i\epsilon)}\right)J_1(\omega)+\omega J_2(\omega)-\omega J_4(\omega)\right]\right\}^{-1}, \quad (58)$$

where

$$J_1(\omega)=\frac{1}{Z_UZ_V}-\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')|\Delta^1(\omega')|^2\Delta^2(\omega-\omega'+i\epsilon)\phi(\omega',\omega)}{\omega'^2\Delta^1(\omega-\omega'+i\epsilon)},$$
$$J_2(\omega)=-\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')|\Delta^1(\omega')|^2\Delta^2(\omega-\omega'+i\epsilon)\phi(\omega',\omega)}{\omega'^2\Delta^1(\omega-\omega'+i\epsilon)(\omega'-\omega-i\epsilon)},$$
$$J_3(\omega)=-\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')|\Delta^1(\omega')|^2\Delta^2(\omega-\omega'+i\epsilon)\phi(\omega',\omega)}{\omega'^2(\omega'-\omega-i\epsilon)}, \quad (59)$$
$$J_4(\omega)=-\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\rho(\omega')|\Delta^1(\omega')|^2\phi(\omega',\omega)}{\omega'^2(\omega'-\omega-i\epsilon)}.$$

Since these integrals are taken over the variable in which we have an equation for ϕ, we expect we can find variational principles for them. We define the four functionals $F_i(i=1,\cdots,4)$ as follows:

$$F_i(\phi,\chi_i)=J_i(\phi)+\frac{1}{\pi}\int_\mu^\infty \frac{d\omega'\rho(\omega')|\Delta^1(\omega')|^2\Delta^2(\omega-\omega'+i\epsilon)}{\omega'\Delta^1(\omega-\omega'+i\epsilon)}\chi_i(\omega',\omega)\left[\phi(\omega',\omega)-g^2+\frac{(2g^2-\lambda^2)}{Z_UZ_V}\Delta^1(\omega-\omega'+i\epsilon)\right]$$
$$-(2g^2-\lambda^2)\frac{1}{\pi^2}\int_\mu^\infty\int_\mu^\infty \frac{d\omega' d\omega''\rho(\omega')\rho(\omega'')|\Delta^1(\omega')\Delta^1(\omega'')|^2}{\omega'\omega''\Delta^1(\omega-\omega'+i\epsilon)\Delta^1(\omega-\omega''+i\epsilon)}\Delta^2(\omega-\omega'+i\epsilon)\Delta^2(\omega-\omega''+i\epsilon)\chi_i(\omega',\omega)\phi(\omega'',\omega)$$
$$\times\left\{\frac{\Delta^1(\omega-\omega'+i\epsilon)}{\omega''}+\Delta^1(\omega-\omega'-\omega''+i\epsilon)\left[\frac{1}{\omega'+\omega''-\omega-i\epsilon}-\frac{1}{\omega''}\right]\right\}. \quad (60)$$

Then $\delta F_i/\delta\chi_i(\omega',\omega)=0$ yields the integral equation for $\phi(\omega',\omega)$ and $\delta F_i/\delta\phi(\omega',\omega)=0$ shows that $\chi_i(\omega',\omega)$ satisfies the same integral equation as $\chi(\omega',\omega_0,\omega)$ except that the inhomogeneous term $B=(\omega'+\omega_0-\omega-i\epsilon)^{-1}$ is replaced by

$$B_1=1/\omega',\quad B_2=1/\omega'(\omega'-\omega-i\epsilon),$$
$$B_3=\Delta^1(\omega-\omega'+i\epsilon)/\omega'(\omega'-\omega-i\epsilon), \quad (61)$$
$$B_4=\Delta^1(\omega-\omega'-i\epsilon)/\omega'(\omega'-\omega-i\epsilon)\Delta^2(\omega-\omega'-i\epsilon).$$

It follows that the χ_i are either integrals over ω_0 of $\chi(\omega',\omega_0,\omega)$ or they are limiting values of $\chi(\omega',\omega_0,\omega)$. This is because $\chi(\omega',\omega_0,\omega)$ has an inhomogeneous term $B=(\omega_0+\omega'-\omega-i\epsilon)^{-1}$, and the B_i are analytic functions of $\omega-\omega'$. Thus, in all the functionals ϕ and χ have adjoint roles. The stationary value of F_i is J_i, and if approximate functions are used for ϕ and χ_i, the error in the resulting approximation for J_i is second order, namely, $\delta\phi\delta\chi_i$. In practice, it is convenient to replace the F_i by functionals which yield J_i, but are independent of the normalizations of ϕ and χ_i. This modification can be performed in a standard way.[9] Then the simple trial functions

$$\phi(\omega',\omega)=a+\Delta^1(\omega-\omega'+i\epsilon),$$
$$\chi_i(\omega',\omega)=B_i \quad (62)$$

furnish what is probably a good approximation to the J_i, and hence to $T_{22}{}^3$. The parameter a is determined by $0=\partial F_i/\partial a$, a linear algebraic equation.

V. CONCLUSIONS

We now want to discuss how the methods we have used to study the Lee model can be applied to static models with crossing. We assume that we have at our disposal both one- and two-meson solutions of the model under investigation. At present, one-meson solutions are available for the neutral-scalar, charged-scalar, symmetric-scalar, and neutral-pseudoscalar models[10,11]; a two-meson solution is known for the charged-scalar model.[2] The one-meson solution replaces $T_{22}{}^1$, the two-meson solution replaces $T_{22}{}^2$, and as in the present paper we wish to construct a three-meson solution. The amplitudes to be obtained should satisfy two-, three-, and four-particle unitarity, like the Lee-model amplitudes, and they should satisfy appropriate dispersion relations with intermediate states truncated in a way suggested by the Lee-model dispersion relations. The essential new requirement is that the elastic amplitudes satisfy crossing. As in Sec. IV one first derives dispersion relations for the analogs of $A_{23}{}^3$ and $A_{24}{}^3$. In a model with crossing, these equations have the same structure as in the Lee model, and therefore they can be reduced to a Fredholm equation in one variable. This means that a three-meson solution in closed form cannot be achieved, though we hasten to point out that the reduction of the dynamics to the solution of a Fredholm equation in one variable represents an enormous simplification. Since the elastic scattering amplitudes are to be crossing-symmetric, we can no longer use Amado's representation for them [Eq. (35)]. Instead we must use the Low equation, which is explicitly crossing-symmetric. This means we must proceed as in Sec. II, and what is wanted is a method, analogous to the introduction of f [Eq. (18)], for solving the Low equation in the three-meson approximation. Here again we can learn from the Lee model. We first write the Low equation for the third sector of the Lee model, and note that in Sec. IV we have obtained all the amplitudes which appear in the equation.[12] Further, we know that these amplitudes, being exact, must satisfy the equation. The task is to see in detail how this happens. Once this is done, we may be able to solve the third-sector Low equation directly, without recourse of Amado's contraction. If so, there is every reason to hope that the same method can be applied to models with crossing, specifically the charged-scalar model. Our optimism is based on the fact that the function f, which we used in the second-sector Lee-model problem, is also the function which leads to the two-meson charged-scalar solution. It is simple to include crossing in f once it is realized that this is the proper object to study.

[9] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Co., New York, 1953), p. 1111.

[10] L. Castillejo, R. Dalitz, and F. Dyson, Phys. Rev. **101**, 453 (1956).

[11] G. Wanders, Nuovo Cimento **23**, 817 (1962).

[12] In order to obtain one- and two-meson production amplitudes from $A_{23}{}^3$ and $A_{24}{}^3$, an analytic continuation analogous to Eq. (11) is required.

APPENDIX

Here we derive three kernel transformations which we have used in the text. The first applies to Eq. (37) for the second-sector vertex functions. These equations have the form (possibly after adding and subtracting the required term)

$$V(\omega,\cdots)=\frac{1}{\pi}\int_{\mu}^{\infty}\frac{d\omega'\rho(\omega')T_{22}{}^{1}(\omega')^{*}V(\omega',\cdots)}{\omega'-\omega-i\epsilon}+\int\frac{d\omega_0\sigma(\omega_0,\cdots)}{\omega_0-\omega-i\epsilon},\quad\text{(A1)}$$

where σ involves δ functions and V. From the linearity of the equation,

$$V(\omega,\cdots)=\int d\omega_0 f(\omega_0,\omega)\sigma(\omega_0,\cdots),\quad\text{(A2)}$$

where $f(\omega_0,\omega)$ satisfies

$$f(\omega_0,\omega)=\frac{1}{\omega_0-\omega-i\epsilon}+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{d\omega'\rho(\omega')T_{22}{}^{1}(\omega')^{*}f(\omega_0,\omega')}{\omega'-\omega-i\epsilon}.\quad\text{(A3)}$$

This is an Omnès-type equation with the single solution

$$f(\omega_0,\omega)=\frac{\Delta^{1}(\omega+i\epsilon)}{(\omega_0-\omega-i\epsilon)\Delta^{1}(\omega_0-i\epsilon)}.\quad\text{(A4)}$$

Since σ contains V, Eq. (A2) is a new integral equation for V.

The second transformation applies to Eq. (44) for the third-sector associated amplitude. These equations are written in the form

$$A(\omega,\bar\omega,\cdots)=\frac{1}{\pi}\int_{\mu}^{\infty}d\omega'\rho(\omega')T_{22}{}^{1}(\omega')A(\omega',\bar\omega)\times\left[\frac{1}{\omega'-\omega+i\epsilon}+\frac{1}{\omega'-\bar\omega+\omega-i\epsilon}\right]+\int d\omega_0\sigma(\omega_0,\bar\omega)\times\left[\frac{1}{\omega_0-\omega+i\epsilon}+\frac{1}{\omega_0-\bar\omega+\omega-i\epsilon}\right],\quad\text{(A5)}$$

where σ involves δ functions and A. Then

$$A(\omega,\bar\omega,\cdots)=\int d\omega_0 f(\omega_0,\omega,\bar\omega)\sigma(\omega_0,\bar\omega,\cdots),\quad\text{(A6)}$$

with

$$f(\omega_0,\omega,\bar\omega)=\frac{1}{\omega_0-\omega+i\epsilon}+\frac{1}{\omega_0-\bar\omega+\omega-i\epsilon}+\frac{1}{\pi}\int_{\mu}^{\infty}d\omega'\rho(\omega')T_{22}{}^{1}(\omega')f(\omega_0,\omega',\bar\omega)\times\left[\frac{1}{\omega'-\omega+i\epsilon}+\frac{1}{\omega'-\bar\omega+\omega-i\epsilon}\right].\quad\text{(A7)}$$

Equation (A7) is an Omnès equation with a crossed cut. It has the unique solution[2,5]

$$f(\omega_0,\omega,\bar\omega)=\left[\frac{1}{\omega_0-\omega+i\epsilon}+\frac{1}{\omega_0-\bar\omega+\omega-i\epsilon}\right]\times\frac{\Delta^{1}(\omega-i\epsilon)\Delta^{1}(\bar\omega-\omega+i\epsilon)}{\Delta^{1}(\omega_0+i\epsilon)\Delta^{1}(\bar\omega-\omega_0+i\epsilon)}.\quad\text{(A8)}$$

Equation (A6) is the transformed integral equation.

The third transformation applies to Eq. (48), which has the form

$$\psi(\omega,\cdots)=\frac{1}{\pi}\int_{\mu}^{\infty}\frac{d\omega'}{\omega'-\omega}\psi(\omega'-i\epsilon,\cdots)e^{i[\theta_1(\omega')+\theta_2(\omega')-\delta(\omega')]}\times\sin[\theta_1(\omega')+\theta_2(\omega')-\delta(\omega')]+\int\frac{d\omega_0\sigma(\omega_0,\cdots)}{\omega_0-\omega}.\quad\text{(A9)}$$

Then

$$\psi(\omega,\cdots)=\int d\omega_0 f(\omega_0,\omega)\sigma(\omega_0,\cdots),\quad\text{(A10)}$$

where

$$f(\omega_0,\omega)=\frac{1}{\omega_0-\omega}+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{d\omega'}{\omega'-\omega}f(\omega'-i\epsilon,\,\omega_0)\times e^{i[\theta_1(\omega')+\theta_2(\omega')-\delta(\omega')]}\times\sin[\theta_1(\omega')+\theta_2(\omega')-\delta(\omega')].\quad\text{(A11)}$$

The solution is

$$f(\omega_0,\omega)=\frac{\Delta^{2}(\omega)\Delta^{1}(\omega_0+i\epsilon)}{(\omega_0-\omega)\Delta^{1}(\omega)\Delta^{2}(\omega_0+i\epsilon)}.\quad\text{(A12)}$$