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Off-Mass-Shell Approach to Strong Decays of Baryon Resonances*

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It is shown that useful sets of self-consistency conditions on baryon resonances can be obtained from off-mass-shell dispersion relations satisfied by the decay amplitudes in which baryon resonances decay into a pion and other baryon resonances. The decay amplitudes are continued off the mass shell, not only with respect to pion in terms of divergence of axial-vector current, but also regarding baryon resonances using renormalized baryon fields. Self-consistency conditions are the off-mass-shell dispersion relations, with dispersion integrals saturated in terms of the baryon resonances which share the same quantum numbers, including spin. Use of the soft-pion technique and current-commutator algebra enables us to derive an additional set of self-consistency conditions. The details of the analysis are presented in the case of nucleon resonances with both spin and isospin $\frac{1}{2}$. The resulting self-consistency conditions can never be satisfied unless nucleon resonances with opposite parities are both assumed. It is shown that the self-consistency conditions are consistent with all the available experiments on four nucleons. Moreover, the self-consistency conditions imply that the decay of $N(1700)$ into $N(1400)$ and a pion has a decay width of $\sim$24 MeV, which can be tested by experiment. The self-consistency conditions are in favor of increasing the mass of $N(1400)$ considerably over 1400 MeV.

I. INTRODUCTION

ONE of the most complex phenomena observed in high-energy physics is the multitude of baryon resonances. In particular, it is very puzzling that different resonances that have the same quantum numbers, including spin, are actually seen in many cases. For example, it seems established by now that there are at least three baryon resonances with the baryon number, strangeness, isospin, and spin of the nucleon.

The purpose of the present paper is to point out that a sensible, yet quite simple way of understanding, or at least correlating, various baryon resonances may be provided by a study of the off-mass-shell decay amplitudes in which the baryon resonances decay into a pion and other baryon resonances. The off-mass-shell decay amplitudes which we study in this approach are the strong decay amplitudes continued off the mass shell, not only with respect to pion in terms of the divergences of the axial-vector currents, but also regarding baryon resonances using the renormalized baryon fields.

The basic assumption we make in this approach is that all the baryon resonances behave more or less like the baryons themselves. Thus we use the same renormalized baryon fields to describe all the baryon resonances which share the same quantum numbers, including spin. We also use the conventional method of defining the off-mass-shell decay amplitudes to define the off-mass-shell decay amplitudes regarding the baryon resonances.

We assume that the above off-mass-shell decay amplitudes are analytic in the usual sense in the respective external masses and, moreover, satisfy unsubtracted dispersion relations.[1] We study, in this approach, the dispersion relations regarding the external masses of the baryon resonances. In these dispersion relations the singularities are due to those intermediate states which have the same quantum numbers, including spin, as the baryon resonances in question. Thus these baryon resonances represent the most prominent nearby singularities when the dispersion relations are evaluated on the mass shells. Therefore we saturate the dispersion integrals in terms of the contributions from the same baryon resonances. The dispersion relations then reduce to a set of the self-consistency conditions that various resonance parameters must satisfy.

The number of resonance parameters that enter the above set of self-consistency conditions is in general much greater than the number of conditions contained in this set of self-consistency conditions. Therefore the above set of self-consistency conditions alone do not appear to be very useful unless we know a great deal of experimental results.

We show in this paper that an additional set of self-consistency conditions can be obtained using the soft-pion technique due to Adler[2] and also the current-commutator algebra due to Gell-Mann.[3] We assume in this approach a specific version of the current-commutator approach in which the time components of the axial-vector currents actually consist of the baryon fields.

The soft-pion technique, together with the above current-commutator algebra, enables us to evaluate the soft-pion limits of the off-mass-shell decay amplitudes. Thus we can write the once-subtracted dispersion relations without introducing additional parameters. These dispersion relations reduce, upon saturation mentioned earlier, to a new set of self-consistency conditions.

It is these sets of self-consistency conditions among various resonance parameters that are proposed in the present paper as being possibly very useful in under-

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[1] Decay amplitudes are three-point functions and there is as yet no particular reason why they cannot satisfy unsubtracted dispersion relations.

[2] S. L. Adler, Phys. Rev. **140**, B736 (1965).

[3] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, 63 (1964).

standing, or at least correlating, various baryon resonances, which may not otherwise be feasible.

The details of the aforementioned analysis are presented in the present paper in the case of the nucleon resonances with both spin and isospin $\frac{1}{2}$. The off-nucleon mass-shell decay amplitudes are defined in Sec. II. The soft-pion limits are discussed in Sec. III. Both sets of self-consistency conditions are derived in Sec. IV. One of the interesting features of these conditions is that they can never be satisfied unless nucleons with opposite parities are both assumed, as are actually seen in experiments.

In Sec. V, we show that these self-consistency conditions are actually consistent with all the available experiments concerning the four nucleons. Moreover, we show in Sec. V that these self-consistency conditions determine all the unknown decay parameters in such a way that further experimental tests of these self-consistency conditions are possible. Section VI summarizes such further experimental tests and also some of the interesting implications of these self-consistency conditions.

The extension of the present analysis to the nucleon resonances of the higher spin and/or isospin will be reported in a future communication.

II. OFF-MASS-SHELL DECAY AMPLITUDES

Let N_i be one of the nucleon resonances (including the nucleon) with mass m_i and parity ϵ_i which is $+1$ or -1, depending upon whether the parity of N_i is even or odd, respectively. The dimensionless decay amplitude F_{ij} for N_j decaying into N_i and a pion is given by

$$\left(\frac{2k_0q_0p_0}{m_im_j}\right)^{1/2}\langle\pi(k),N_i(q)|N_j(p)\rangle=i(2\pi)^4\delta(p-q-k)\times(\bar{u}(q)\Gamma_i i\gamma_5\Gamma_j u(p))F_{ij},\quad(1)$$

where p, q, and k are the four-momenta[4] of the respective particles, the u's are the corresponding free Dirac spinors, and the Γ's are

$$\Gamma_i=1\text{ or }i\gamma_5,\quad(2)$$

depending upon whether the parity of N_i is even or odd, respectively. It is useful to note that

$$\Gamma_i\Gamma_i^\dagger=1,\quad\Gamma_i^2=\epsilon_i.\quad(3)$$

We define the off-pion-mass-shell decay amplitude by

$$\left(\frac{q_0p_0}{m_im_j}\right)^{1/2}\langle N_i(q)|j_\pi(0)|N_j(p)\rangle=(\bar{u}(q)\Gamma_i i\gamma_5\Gamma_j u(p))F_{ij}(-(q-p)^2),\quad(4)$$

where

$$j_\pi(x)=\frac{(q-p)^2+m_\pi^2}{C_\pi}\partial_\mu A_\mu(x)\quad(5)$$

and $A_\mu(x)$ is the axial-vector current participating in the weak Hamiltonian of $V-A$ type. The off-mass-shell decay amplitude reduces on the mass shell to the physical decay amplitude as

$$F_{ij}(m_\pi^2)=F_{ij},\quad(6)$$

when the constant C_π in (5) is determined by the pion lifetime as

$$C_\pi=0.94m_\pi^3.\quad(7)$$

We note that $F_{ii}(m_\pi^2)$ given by (4) is the renormalized coupling constant of pion to N_i.

We assume in this approach that all the nucleon resonances behave more or less like the nucleon. Thus we assume that $|N_i(q)\rangle$ transforms under time reversal T as

$$T|N_i(q)\rangle=\epsilon_i\eta_i^*|N_i(-\mathbf{q},-\mathbf{s})\rangle,\quad(8)$$

where the phase factor η_i is due to the relation

$$\eta_i u^*(q)=\gamma_1\gamma_3 u(-\mathbf{q},-\mathbf{s}).\quad(9)$$

In Eqs. (8) and (9), $\mathbf{q}$ and $\mathbf{s}$ are the three-momentum and spin vectors, respectively. It then follows that the decay amplitude $F_{ij}(-(q-p)^2)$ defined by (4) is real and therefore symmetric under the interchange of i and j, since the amplitude can be defined with respect to the neutral pion.

In order to define the off-nucleon-mass-shell decay amplitudes, we introduce the renormalized nucleon field $\psi(x)$ which transforms under T as

$$T\psi(x)T^{-1}=\gamma_1\gamma_3\psi(\mathbf{x},-x_0).\quad(10)$$

Lorentz covariance implies that

$$(q_0/m_i)^{1/2}\langle 0|\psi(x)|N_i(q)\rangle=C_i\Gamma_i u(q)e^{iqx},\quad(11)$$

where Γ_i is given by (2) and the constant C_i is real because of (8) and unity for the nucleon. The constant C_i (or C_i^2) is the wave-function renormalization constant of the nucleon resonance N_i with respect to the renormalized nucleon field. The matrix element (11) implies that $C_i^{-1}\Gamma_i^\dagger\psi(x)$ can be used as the field operator which describes N_i, the free equation of motion of which is the usual Dirac equation. Thus the conventional reduction technique allows us to rewrite (4) as

$$(\bar{u}(q)\Gamma_i i\gamma_5\Gamma_j u(p))F_{ij}(-(q-p)^2)=\frac{i}{C_j}\int\left(\frac{q_0}{m_i}\right)^{1/2}\langle N_i(q)|[j_\pi(0),\bar{\psi}(x)\Gamma_j^\dagger]|0\rangle\theta(-x_0)\times\left(-\gamma\frac{\overleftarrow{\partial}}{\partial x}+m_j\right)u(p)e^{ipx}dx,\quad(12)$$

and similarly with respect to $\langle N_i(q)|$.

We now define the off-nucleon-mass-shell decay amplitudes by

$$\bar{u}(q)\Gamma_i i\gamma_5[F_{ij}(-(q-p)^2,-p^2)+(i\gamma p+\epsilon_j m_j)F_{ij}'(-(q-p)^2,-p^2)]=\frac{i}{C_j}\int e^{ipx}dx\left(\frac{q_0}{m_i}\right)^{1/2}\times\langle N_i(q)|[j_\pi(0),\bar{\psi}(x)]|0\rangle\theta(-x_0)(i\gamma p+\epsilon_j m_j),\quad(13)$$

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[4] The fourth components of these four-momenta are pure imaginary, and k_0, q_0, and p_0 represent (real) energies.

where $F_{ij}(-(q-p)^2,-p^2)$ becomes, on the mass shell,

$$F_{ij}(-(q-p)^2,m_j^2)=F_{ij}(-(q-p)^2) \qquad (14)$$

given by (4), but $F_{ij}'(-(q-p)^2,-p^2)$ does not reduce to the physical decay amplitude. Similarly, we define another set of the off-nucleon-mass-shell decay amplitudes by

$$i\gamma_5[F_{ij}(-(q-p)^2,-q^2)+(-i\gamma q+\epsilon_i m_i)F_{ij}' \times(-(q-p)^2,-q^2)]\Gamma_j u(p)=\frac{i}{C_i}\int e^{-iqx}dx(i\gamma q+\epsilon_i m_i) \times\theta(x_0)\left(\frac{p_0}{m_j}\right)^{1/2}\langle 0|[\psi(x),j_\pi(0)]|N_j(p)\rangle, \qquad (15)$$

where only $F_{ij}(-(q-p)^2,-q^2)$ reduces to the physical decay amplitude as

$$F_{ij}(-(q-p)^2,m_i^2)=F_{ij}(-(q-p)^2). \qquad (16)$$

We assume that the off-mass-shell decay amplitudes defined by (13) and (15) are analytic except for the dynamical singularities in the usual sense. Thus the imaginary parts of these amplitudes in the upper-p^2 plane are given by

$$\bar{u}(q)\Gamma_i i\gamma_5[\mathrm{Im}F_{ij}(-(q-p)^2,-p^2)+(i\gamma p+\epsilon_j m_j) \times\mathrm{Im}F_{ij}'(-(q-p)^2,-p^2)]=\frac{\pi}{C_j}\sum_n(2\pi)^3\delta(p-p_n) \times\left(\frac{q_0}{m_i}\right)^{1/2}\langle N_i(q)|j_\pi(0)|n\rangle\langle n|\bar{\psi}(0)|0\rangle(i\gamma p+\epsilon_j m_j) \qquad (17)$$

and by a similar expression in the $-q^2$ plane. The intermediate states $|n\rangle$ on the right-hand side of (17) must have, because of the operator $\bar{\psi}(0)$, the same set of quantum numbers as the nucleon, including spin but not parity. Thus all the nucleon resonances contribute to the imaginary parts. These contributions are expressed, because of the definitions (4) and (11), as

$$\begin{aligned}
&\mathrm{Im}F_{ij}(-(q-p)^2,-p^2)\\
&\quad=\frac{\pi}{C_j}\sum_n\delta(p^2+m_n^2)(m_j^2-m_n^2)C_nF_{in}(-(q-p)^2),\\
&\mathrm{Im}F_{ij}'(-(q-p)^2,-p^2)\\
&\quad=\frac{\pi}{C_j}\sum_n\delta(p^2+m_n^2)(\epsilon_n m_n-\epsilon_j m_j)C_nF_{in}(-(q-p)^2),\\
&\mathrm{Im}F_{ij}(-(q-p)^2,-q^2)\\
&\quad=\frac{\pi}{C_i}\sum_n\delta(q^2+m_n^2)(m_i^2-m_n^2)C_nF_{nj}(-(q-p)^2),\\
&\mathrm{Im}F_{ij}'(-(q-p)^2,-q^2)\\
&\quad=\frac{\pi}{C_i}\sum_n\delta(q^2+m_n^2)(\epsilon_n m_n-\epsilon_i m_i)C_nF_{nj}(-(q-p)^2),
\end{aligned} \qquad (18)$$

where all the F's on the right-hand sides are the decay amplitudes on the mass shell.

III. SOFT-PION LIMITS

It is possible to evaluate the limits of both sides of Eq. (13) as $p\to q$ and also those of Eq. (15) as $q\to p$. These limits may be called the soft-pion limits, since the external pion mass $-(q-p)^2$ becomes zero in these limits.

In order to analyze the limit of the right-hand side of (13), this expression is first put into the form

$$i\frac{(q-p)^2+m_\pi^2}{C_\pi C_j}\int e^{-i(q-p)x}dx\left(\frac{q_0}{m_i}\right)^{1/2} \times\{i(p-q)_\mu\theta(x_0)\langle N_i(q)|[A_\mu(x),\bar{\psi}(0)]|0\rangle -\delta(x_0)\langle N_i(q)|[A_0(x),\bar{\psi}(0)]|0\rangle\}(i\gamma p+\epsilon_j m_j), \qquad (19)$$

where the expression (5) has been used, followed by partial integration[5] with respect to the divergence of the axial-vector current. Then the technique due to Adler[2] allows us to evaluate the limit of the first term of (19), which is expressed in terms of the matrix element of $A_\mu(0)$ given by

$$\left(\frac{q_0p_0}{m_im_j}\right)^{1/2}\langle N_i(q)|A_\mu(0)|N_j(p)\rangle =(\bar{u}(q)\Gamma_i[i\gamma_\mu\gamma_5A_{ij}(-(q-p)^2)+\cdots]\Gamma_ju(p)), \qquad (20)$$

where the dots stand for the terms which vanish as $p\to q$. The relations (4) and (5) imply that

$$A_{ij}(0)=\frac{C_\pi}{m_\pi^2}\frac{F_{ij}(0)}{\epsilon_im_i+\epsilon_jm_j}, \qquad (21)$$

where $A_{ii}(0)$ is the renormalized axial-vector coupling constant of N_i. Therefore the limit of the first term of (19) is proportional to $F_{ii}(0)$, the renormalized coupling constant of pion to N_i.

The limit of the second term of (19) depends upon the equal-time commutator between $A_0(x)$ and $\bar{\psi}(0)$. Gell-Mann[3] proposes an assumption that the time components of the vector and axial-vector currents satisfy $SU(3)\times SU(3)$ commutator algebra upon equal-time commutation. If these currents are expressed in terms of the unrenormalized baryon fields, $A_0(x)$ appears as

$$A_0(x)=a\psi_0^\dagger(x)\gamma_5\psi_0(x)+\cdots, \qquad (22)$$

where the dots stand for the terms which commute at equal time with the unrenormalized nucleon field $\psi_0(x)$. In fact, the current-commutator algebra determines a, for example, as $1/\sqrt{2}$ when $A_0(x)$ refers to the neutral

[5] It is an additional assumption that partial integration is justified. However, this does not appear to be a very serious assumption, according to N. H. Fuchs and M. Sugawara, Phys. Rev. 165, 1839 (1968), for example.

pion and $\psi_0(x)$ is the unrenormalized proton field. Expression (22) implies

$$\int dx\, \delta(x_0)[A_0(x),\bar{\psi}_0(0)] = -a\bar{\psi}_0(0)\gamma_5. \qquad (23)$$

Since the above relation is homogeneous in $\psi_0(0)$, the same relation holds for the renormalized field $\psi(0)$:

$$\int dx\, \delta(x_0)[A_0(x),\bar{\psi}(0)] = -a\bar{\psi}(0)\gamma_5. \qquad (24)$$

We assume in the present paper that the equal-time commutation relation (24) holds, with a determined by current-commutator algebra in the manner explained above. We note that a is the unrenormalized axial-vector coupling constant of the nucleon.

We now let the space components of p approach those of q first and then let p_0 approach q_0. In this approach of $p \to q$, expression (19) tends to

$$\bar{u}(q)\Gamma_i i\gamma_5\left[\frac{(\epsilon_i m_i - \epsilon_j m_j)}{2\epsilon_i m_i}F_{ii}(0) + \frac{\gamma_4}{2q_0} \times (\epsilon_i m_i - \epsilon_j m_j)F_{ii}(0) + (\epsilon_i m_i + \epsilon_j m_j)\frac{a m_\pi^2}{C_\pi}\right]\frac{C_i}{C_j}, \qquad (25)$$

where the first two terms are due to the first term of (19) and γ_4 appears because of the matrix element of $A_4(x)$, which is the only term that contributes to this limit. The last term of (25) is due to the equal-time commutator in (19).

The limit (25) must be equal to the same limit of the left-hand side of Eq. (13), which can be rewritten as

$$\bar{u}(q)\Gamma_i i\gamma_5[\{F_{ij}(-(q-p)^2,-p^2) + (\epsilon_i m_i + \epsilon_j m_j)F_{ij}'(-(q-p)^2,-p^2)\} + i\gamma(p-q)F_{ij}'(-(q-p)^2,-p^2)], \qquad (26)$$

where use was made of the free Dirac equation satisfied by $\bar{u}(q)$. The expressions (18) for the imaginary parts imply that both $F_{ij}(-(q-p)^2,-p^2)$ and $F_{ij}'(-(q-p)^2,-p^2)$ have poles at $p=q$. However, the expressions (18) also imply that the particular combination of these amplitudes that appears inside the curly brackets in (26) is free from the pole in question, since the relevant imaginary parts cancel each other in this combination. Thus the first term of (26) approaches a finite limit as $p \to q$, and so does the second term of (26) because of the factor $p-q$. The limit of this second term can be computed knowing the residue of $F_{ij}'(-(p-q)^2,-p^2)$ at $p=q$ which is contained in the expressions (18) for the imaginary parts. We find in this way that the second term of (26) actually tends to the second term of (25), as it should. Thus we conclude that

$$F_{ij}(-(q-p)^2,-p^2) + (\epsilon_i m_i + \epsilon_j m_j)F_{ij}'(-(q-p)^2,-p^2) \underset{p\to q}{\longrightarrow} \left[\frac{(\epsilon_i m_i - \epsilon_j m_j)}{2\epsilon_i m_i}F_{ii}(0) + (\epsilon_i m_i + \epsilon_j m_j)\frac{a m_\pi^2}{C_\pi}\right]\frac{C_i}{C_j}. \qquad (27)$$

The limit of Eq. (15) can be analyzed in exactly the same way. We find from Eq. (15) that

$$F_{ij}(-(q-p)^2,-q^2) + (\epsilon_i m_i + \epsilon_j m_j)F_{ij}'(-(q-p)^2,-q^2) \underset{q\to p}{\longrightarrow} \left[\frac{(\epsilon_j m_j - \epsilon_i m_i)}{2\epsilon_j m_j}F_{jj}(0) + (\epsilon_i m_i + \epsilon_j m_j)\frac{a m_\pi^2}{C_\pi}\right]\frac{C_j}{C_i}. \qquad (28)$$

IV. SELF-CONSISTENCY CONDITIONS

We assume that the off-nucleon-mass-shell decay amplitudes defined by (13) and (15) satisfy the unsubtracted dispersion relations in the respective variables. We furthermore assume that the dispersion integrals in these dispersion relations can be saturated by the contributions from the same set of nucleon resonances.

The latter of the above assumptions is based upon the following observation: As is noted in Sec. II, the intermediate states that contribute to the imaginary parts of these decay amplitudes are necessarily only those states which have the same set of quantum numbers as the nucleon, including spin but not parity. Therefore, when the dispersion integrals are evaluated on the mass shells, these nucleon resonances actually represent the most prominent nearby singularities. Thus it is possible that the dispersion integrals are saturated by these nucleon resonances.

We first consider the unsubtracted dispersion relations satisfied by $F_{ij}(-(q-p)^2,-p^2)$ and $F_{ij}(-(q-p)^2,-q^2)$ evaluated at $-p^2=m_j^2$ and $-q^2=m_i^2$, respectively. If the dispersion integrals are saturated by the contributions from the nucleon resonances which are given by the expressions (18), these dispersion relations reduce to the following self-consistency conditions:

$$\sum_n F_{in}(-(q-p)^2)C_n = 0\,, \quad \sum_n C_n F_{nj}(-(q-p)^2) = 0. \qquad (29)$$

No self-consistency condition follows from F''s, since these amplitudes do not reduce to the physical amplitudes on the mass shell.

The two conditions in (29) are actually the same, since we can always refer to the decays into the neutral pion, and thus $F_{ij}(-(q-p)^2)$ becomes symmetric under the interchange of i and j.

The soft-pion limits given by (27) and (28) enable us to derive additional self-consistency conditions. For this purpose, we define the amplitudes which are given by

$$F_{ij}(-(q-p)^2,-p^2)$$
$$+\left(\frac{m_j^2+p^2}{\epsilon_j m_j-\epsilon_i m_i}\right)F_{ij}'(-(q-p)^2,-p^2) \quad (30)$$

and also by

$$F_{ij}(-(q-p)^2,-q^2)$$
$$+\left(\frac{m_i^2+q^2}{\epsilon_i m_i-\epsilon_j m_j}\right)F_{ij}'(-(q-p)^2,-q^2), \quad (31)$$

both of which are analytic except for the dynamical singularities. The amplitude (30), for example, reduces on the mass shell ($p^2+m_j^2=0$) to $F_{ij}(-(q-p)^2)$ and tends as $p\to q$ to

$$(\epsilon_i m_i+\epsilon_j m_j)\left[\frac{am_\pi^2}{C_\pi}-\frac{F_{ii}(0)}{2\epsilon_i m_i}\right]\frac{C_i}{C_j}, \quad (32)$$

as can be shown from the soft-pion limit (27) and also the imaginary parts given by (18). Thus we can write the once-subtracted dispersion relation for the amplitude (30) with $-(q-p)^2=0$ and the subtraction point chosen at $-p^2=m_i^2$. This subtraction amounts to the suppression of the effect of the polynomial introduced in the amplitude (30). Therefore, when this dispersion relation is evaluated on the mass shell, the dispersion integral can also be saturated by the contributions from the nucleon resonances. In this way, we find

$$\sum\nolimits_n f_{in}C_n=C_i, \quad (33)$$

where

$$f_{ij}=\frac{C_\pi}{am_\pi^2}\frac{F_{ij}(0)}{\epsilon_i m_i+\epsilon_j m_j}=\frac{A_{ij}(0)}{a}, \quad (34)$$

where $A_{ij}(0)$ is given by (21). We note that $f_{ii}(0)$ is the axial-vector coupling-constant renormalization for N_i.

Similarly, we can derive from the amplitude (31)

$$\sum\nolimits_n C_n f_{nj}=C_j, \quad (35)$$

which is, however, the same condition as (33), since f_{ij} defined by (34) is also symmetric under the interchange of i and j.

In terms of f_{ij} defined by (34), the self-consistency condition (29) with $(q-p)^2=0$ can be put in the form

$$\epsilon_i m_i C_i+\sum\nolimits_n \epsilon_n m_n f_{in}C_n=0, \quad (36)$$

where use was made of the condition (33). The number of conditions implied by each set of self-consistency conditions (33) and (36) is exactly the number of nucleon resonances, including the nucleon.

If we multiply the condition (36) by C_i and sum over i, we obtain, using again the condition (33),

$$\sum\nolimits_i \epsilon_i m_i C_i^2=0. \quad (37)$$

We note that this last condition can never be satisfied unless the nucleon resonances with opposite parities both appear, as is actually seen in experiments.

V. COMPARISON WITH EXPERIMENTS

We show in this section that the self-consistency conditions are actually consistent with experiments.

We summarize in Table I the experimental data on the nucleon resonances listed under the notation of the present paper. We assume in the present comparison those figures due to Lovelace,[6] because the self-consistency conditions (33) and (36) seem to favor them over the figures due to Rosenfeld,[7] as is remarked in conclusion (3) of Sec. VI.

The data in Table I determine some of the decay amplitudes. When the decay into the neutral pion is considered, then a, which is defined by (24), is $1/\sqrt{2}$ and the corresponding decay width is $\frac{1}{3}$ of the elastic decay width Γ_{el} in Table I. Thus, ignoring the off-pion-mass-shell correction, the decay widths quoted in Table I determine the three decay amplitudes

$$f_{12}=0.374,\ f_{13}=0.155,\ f_{14}=0.288. \quad (38)$$

The positive sign assumed in (38) normalizes the sign of all the states of the nucleon resonances. Table I shows that N_4 has an inelastic width of 64 MeV. Thus the decay rates for $N_4\to N_2+\pi$ and $N_4\to N_3+\pi$ must individually be smaller than 64 MeV, which yields

$$|f_{24}|<0.703,\ |f_{34}|<3.26. \quad (39)$$

There is no limitation on f_{23}, since the transition $N_3\to N_2+\pi$ is kinematically forbidden, though allowed according to Rosenfeld's figures [see the remarks in conclusion (3) of Sec. VI]. The axial-vector coupling-constant renormalization f_{11} is known to be

$$f_{11}=1.18. \quad (40)$$

All other parameters are totally unknown; these are the other coupling constants f_{22}, f_{33}, and f_{44} and the renormalization constants C_2, C_3, and C_4 (C_1 being unity by definition).

TABLE I. Data on nucleon resonances reported by Lovelace[a] and by Rosenfeld[b] (those of the latter are shown in parentheses).

Notation	J^P	Mass (MeV)	Γ_{tot} (MeV)	Γ_{el} (MeV)
N_1	$\frac{1}{2}^+$	940	...	...
N_2	$\frac{1}{2}^+$ (P_{11})	1466 (~1400)	211 (~200)	139 (~140)
N_3	$\frac{1}{2}^-$ (S_{11})	1548 (1570)	116 (130)	38 (39)
N_4	$\frac{1}{2}^-$ (S_{11})	1709 (1700)	300 (240)	236 (240)

[a] Reference 6.
[b] Reference 7.

[6] C. Lovelace, in *Proceedings of the Heidelberg International Conference on Elementary Particles, Heidelberg, 1967*, edited by H. Filthuth (John Wiley & Sons, Inc., New York, 1968).

[7] A. H. Rosenfeld, in *Proceedings of the Heidelberg International Conference on Elementary Particles, Heidelberg, 1967*, edited by H. Filthuth (John Wiley & Sons, Inc., New York, 1968).

There are altogether eight conditions in the self-consistency conditions (33) and (36). Therefore, if all the figures in (38) and (40) are taken as inputs and also one of the bounds in (39) is imposed, then all the remaining eight parameters are determined within certain bounds by the conditions (33) and (36). This solution of the self-consistency conditions can be obtained as follows. First, the conditions (33) and (36) with $i=1$ are combined with the condition (37). The only unknowns in these three conditions are C_2, C_3, and C_4, which can then be determined as

$$C_2=-2.02,\ C_3=1.83,\ C_4=1.02,$$
or
$$C_2=-1.98,\ C_3=-0.054,\ C_4=1.98. \tag{41}$$

Then we use a relation satisfied by f_{24} and f_{34} which can be obtained from the conditions (33) and (36) with $i=4$ by eliminating f_{44}. When the second set in (41) is assumed, the above relation for f_{24} and f_{34}, when $|f_{34}|<3.26$ is imposed, yields $-0.962<f_{24}<-0.953$, in violation of the first inequality in (39). In fact, the above figures of f_{24} correspond to the decay width for $N_4\to N_2+\pi$ of at least 120 MeV, which is certainly too large to be consistent with experiments. Thus we discard the second set in (41). For the first set in (41), the same bounds for f_{34} yield $-0.276>f_{24}>-0.576$, which is certainly consistent with experiments. Thus the self-consistency conditions (33) and (36) are consistent with experiments.

We summarize below the acceptable solution of the self-consistency conditions (33) and (36):

$$\begin{aligned}\text{inputs:}\ & f_{11}=1.18,\ f_{12}=0.374,\ f_{13}=0.155,\\ & f_{14}=0.288,\ +3.26>f_{34}>-3.26;\\ \text{outputs:}\ & C_2=-2.02,\ C_3=1.83,\ C_4=1.02,\\ & f_{24}=-0.426\pm0.150,\quad f_{23}=-0.867\mp0.087,\\ & f_{22}=0.192\mp0.004,\quad f_{33}=-0.05\mp1.91,\\ & f_{44}=-0.13\mp5.56,\end{aligned} \tag{42}$$

where the upper and lower signs correspond to the upper and lower bounds for f_{34}, respectively, and all these uncertainties are linearly proportional to that of f_{34}.

VI. CONCLUSIONS

(1) The analysis in the previous section shows that the self-consistency conditions (33) and (36) are actually consistent with all the experiments available on the four nucleons which are summarized in Table I. Moreover, the self-consistency conditions determine all the unknown resonance parameters as summarized in (42), which allows us to make the following observations.

(2) The bounds on f_{44} in the solution (42) correspond to $F_{44}(0)$, the renormalized coupling constant of pion to N_4, which is nearly 10 times the renormalized pion-nucleon coupling constant $F_{11}(0)$. It seems quite reasonable to assume that $F_{44}(0)$ is at most of the order of $F_{11}(0)$ in magnitude. Then all the uncertainties in (42) would have to be reduced at least by a factor of 5 or so. In this case, f_{24} assumes approximately its central value that appears in (42) and $|f_{34}|$ is substantially smaller than 3.26. These figures of f_{24} and f_{34} correspond to the following decay widths for $N_4\to N_2+\pi$ and $N_4\to N_3+\pi$:

$$\begin{aligned}&\Gamma(N_4\to N_2+\pi)\sim24\ \text{MeV},\\ &\Gamma(N_4\to N_3+\pi)\lesssim2.6\ \text{MeV}.\end{aligned} \tag{43}$$

In particular, the first of the above two expressions seems to provide the best way of testing further the self-consistency conditions (33) and (36) by experiment. Incidentally, the bounds on f_{24} that appear in (42) correspond to the decay widths

$$10\ \text{MeV}<\Gamma(N_4\to N_2+\pi)<45\ \text{MeV}. \tag{44}$$

(3) According to Rosenfeld's figures in Table I, the mass difference m_3-m_2 is 168 MeV or $1.2m_\pi$. If m_3-m_2 is actually as big as $1.2m_\pi$, the figure of f_{23} in (42) implies that the decay $N_3\to N_2+\pi$ would occur with the width of 27 MeV, in comparison with 38 MeV for the experimental decay width of $N_3\to N_1+\pi$. Since the decay $N_3\to N_2+\pi$ should appear as $N_3\to N_1+2\pi$, it appears that such a decay width for $N_3\to N_2+\pi$ is too large to be consistent with experiment. We remark that the above change in masses does not appreciably affect the value of f_{23} which is determined by the self-consistency conditions (33) and (36). It is, in fact, very difficult to make f_{23} much smaller than unity in the solutions of the self-consistency conditions (33) and (36) when the inputs are given approximately by those which appear in (42). Thus the self-consistency conditions (33) and (36) seem to favor Lovelace's figures over Rosenfeld's in Table I. We note that the main difference between them is the mass of N_2 or $N(1400)$. In other words, the self-consistency conditions (33) and (36) are in favor of increasing its mass considerably over 1400 MeV.

(4) The values of the renormalization constants in (42) enable us to evaluate an upper bound on the wave-function renormalization constant Z for the nucleon. By definition,[8]

$$1/Z=\textstyle\sum_i C_i^2+\text{a positive number}. \tag{45}$$

Thus the solution (42) implies

$$Z\lesssim0.106, \tag{46}$$

which is quite a low upper bound.

(5) We remark that f_{22} is nearly completely determined in the solution (42) and also that f_{22} is quite small compared with f_{11}. This means that N_2 interacts with pions very much more weakly than N_1 (the nucleon) does with pion. This may be understood by observ-

[8] See, for example, S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Brothers, New York, 1962), Sec. 17.

ing that axial-vector coupling is totally induced for N_2 and therefore f_{22} is expected to be of the order of, say, $f_{11}-1$, which is, in fact, the case in the solution (42). Such an observation would then imply that not only f_{22} but also f_{33} and f_{44} are both small compared with unity. We note that the solution (42) implies that, if f_{33} is small compared with unity, then f_{44} must also be small compared with unity, and vice versa. Thus it appears that the self-consistency conditions (33) and (36) suggest the very interesting conjecture that all f_{22}, f_{33}, and f_{44} are actually small compared with f_{11}.

(6) Concerning the use of the self-consistency conditions (33) and (36) for the purpose of bootstrapping the resonances, it is interesting to note that the condition (37) can never be satisfied unless the nucleon resonances with opposite parities both appear. This is exactly what is actually seen in experiments. Thus the self-consistency conditions are certainly useful in this respect. However, we also remark that as the number of the resonances increases, the total number of parameters that enter the self-consistency conditions increases faster than the total number of conditions contained in the self-consistency conditions. Therefore such self-consistency conditions are no longer very useful, when the number of the resonances is high, unless we have a great many experimental data.

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Third Sector of the Lee Model*

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We study the third sector of the Lee model. In the present work the model is augmented by a third static source U in addition to N and V, with the coupling $U\leftrightarrow V+\theta$. In the third sector the processes $U+\theta\leftrightarrow V+\theta+\theta\leftrightarrow N+\theta+\theta+\theta$ occur, providing a model enriched with a two-particle channel. Using the methods of dispersion theory, the dynamics are reduced to the solution of a Fredholm integral equation in one variable. A variational principle is given for the equation which yields the elastic scattering amplitude. Diagonalization of the second-sector connected S matrix plays an important part in the analysis. Finally, we discuss the relevance of the results to static models with crossing—specifically, to a three-meson solution of the charged-scalar static theory.

I. INTRODUCTION

THE Lee model has been extensively studied in the first and second sectors, but up to the present little work has been done on higher sectors.[1] However, higher sectors have the interesting feature that intermediate states containing many particles are present. In particular, the third sector has four-particle intermediate states, and hence it may provide hints as to how to incorporate four-particle unitarity in more interesting static models, namely those with crossing. The second sector of the Lee model served just this purpose with three-particle unitarity in the case of the charged-scalar theory.[2,3] Because we have models with crossing in mind, we study the Lee model by means of dispersion theory. Off-energy-shell methods are simpler in the case at hand, but they do not permit the inclusion of crossing, whereas dispersion methods do. It is also with more complicated models in mind that we add an elastic channel to the third sector. This is easily accomplished by adding a static source U to the Lee model with the coupling $U\leftrightarrow V+\theta$.[4] This coupling, together with the standard coupling $V\leftrightarrow N+\theta$, causes the states $U+\theta$, $V+\theta+\theta$, and $N+\theta+\theta+\theta$ to communicate in the third sector of the model. The usual Lee model, without the channel $U+\theta$, is recovered from our results by setting the $UV\theta$ coupling λ equal to zero.

In general approach, our work follows the classic paper of Amado on the second sector of the Lee model, which involves the states $V+\theta$ and $N+\theta+\theta$.[5] Amado found the $V\theta$ elastic amplitude by a scheme of contractions which avoids integrations over three-particle intermediate states in the dynamical equations. In spite of this, his elastic amplitude, being exact, naturally satisfies two- and three-particle unitarity equations. In

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[1] It is obvious that integral equations can be written for any sector which sum the Wigner-Brillouin perturbation series. Recently, D. I. Fivel [University of Maryland Report (unpublished)] has given a method, based on a dynamical algebra, for deriving equations in any sector.

[2] J. B. Bronzan, J. Math. Phys. 7, 1351 (1966).

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[4] J. B. Bronzan, Phys. Rev. 139, B751 (1965).

[5] R. D. Amado, Phys. Rev. 122, 696 (1961).