Coulomb Interference in High-Energy Scattering*

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The relativistic Coulomb interference problem is carefully examined in order to evaluate critically the equations used in analyzing experiments designed to test the forward π-N dispersion relations. We show that with a suitable interpretation the nonrelativistic Bethe formula for the phase difference ϕ between the strong and electromagnetic contributions is valid. However, for high energy near forward scattering there are unknown contributions which can change this phase by $O(\alpha)$ rad; this is to be compared with a magnitude of ϕ of $\sim 2\alpha$ rad. We further show that two previous relativistic calculations of ϕ are incomplete. The effects of calculable radiative corrections such as soft photon emission or vacuum polarization are examined. For the former it is found that although their magnitude is relatively large, they can, in general, be neglected. The reason for this is that by folding an accurate measurement of the total cross section into the data analysis a compensation of errors is induced. No such compensation takes place for the case of vacuum polarization, and its contribution could be important. Finally, we investigate the effects of small admixtures to the strong interactions which are not of the form e^{a+b^2t}. We find that they are unlikely to be of significance here provided they can be assumed to be $\lesssim 20\%$ of the complete amplitude.

INTRODUCTION

In the past few years, there have been several attempts, notably by Lindenbaum and his co-workers,[1] to verify experimentally the forward dispersion relations for π-N scattering. These experiments utilize interference with the presumed known Coulomb amplitude to obtain the real part of the forward π-N scattering amplitude. The imaginary part is obtained from a measurement of the total cross section. Typically, the data are analyzed by assuming that the complete amplitude (strong plus Coulomb) is expressible in the form

$$f_{\text{tot}} = f_N + f_C e^{i\alpha\phi}. \tag{1}$$

Here, f_N is the purely strong interaction amplitude [Fig. 1(a)], f_C the purely Coulombic amplitude [Fig. 1(l)], and ϕ some phase shift induced by the long-range Coulomb interaction. Using a potential model, Bethe,[2] some time ago, derived the following estimate for ϕ:

$$\phi \cong 2 \ln(1.06/|\mathbf{k}_1| b\theta), \tag{2}$$

where k_1 is the c.m. momentum, b the range of the strong-interaction forces, and θ the c.m. scattering angle. Results similar to this have since been rederived within potential theory by several authors.[3,4] However, since the experiments are essentially relativistic it would be satisfying to have a nonpotential relativistic confirmation of this result. The first such attempt appears to be that of Solov'ev,[5] who employed the techniques developed by Yennie, Frautschi, and Suura (YFS)[6] for dealing with the infrared (IR) problem. Solov'ev obtained

$$\phi \cong 2 \ln(2/\theta), \tag{3}$$

a result differing considerably from that of Bethe. The only other published treatment along these lines is apparently the recent one by Rix and Thaler.[3] They derive a result which seems to be qualitatively in agreement with that of Bethe. However, as we shall endeavor to indicate in Sec. II of this paper, the results of both Solov'ev and of Rix and Thaler should be viewed with some suspicion: the former, because of its incomplete treatment of the IR problem, the latter because of its dubious evaluation of a crucial integral. Since a full interpretation of the experiments rests heavily upon a good estimate for the magnitude of ϕ, we have reinvestigated the problem and have concluded that, with a suitable interpretation of the parameter b, the form of Bethe's expression is approximately valid at high energies. However, we do find that there are unknown contributions to ϕ which are $O(1)$. These could be significant in the interpretation of results deduced from the difference cross section (i.e., the difference between positive and negative pion-proton scattering).

The basic reason for the difference between the Solov'ev and the Bethe formulas lies in the different treatments of the IR approximation. In calculating radiative corrections one often ignores the energy-momentum variation of the basic process when it occurs as an intermediate state such as in Figs. 1(b) or 1(c).[6] However, when this variatian is sufficiently rapid it can have a profound effect upon the final results. Indeed, as we show in Sec. II [see Eq. (23)], the inclusion, for example, of the variation of the π-N amplitude with the momentum transferred in the intermediate state precisely accounts for the difference between the Solov'ev and Bethe results. A similar con-

* Work supported in part by the Office of Naval Research and in part by the National Science Foundation.

[1] S. J. Lindenbaum, in *Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1967* (W. H. Freeman and Co., San Francisco, 1967) and references cited therein; K. J. Foley *et al.*, Phys. Rev. Letters **19**, 193 (1967); **19**, 622(E) (1967); and to be published.

[2] H. A. Bethe, Ann. Phys. (N. Y.) **3**, 190 (1958).

[3] J. Rix and R. M. Thaler, Phys. Rev. **152**, 1357 (1966).

[4] M. M. Islam, Phys. Rev. **162**, 1426 (1967). It is also possible to derive this result using the methods discussed in G. B. West, J. Math. Phys. **8**, 942 (1967).

[5] L. D. Solov'ev, Zh. Eksperim. i Teor. Fiz. **49**, 292 (1965) [English transl: Soviet Phys.—JETP **22**, 205 (1966)].

[6] D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N. Y.) **13**, 379 (1961), referred to hereafter as YFS.

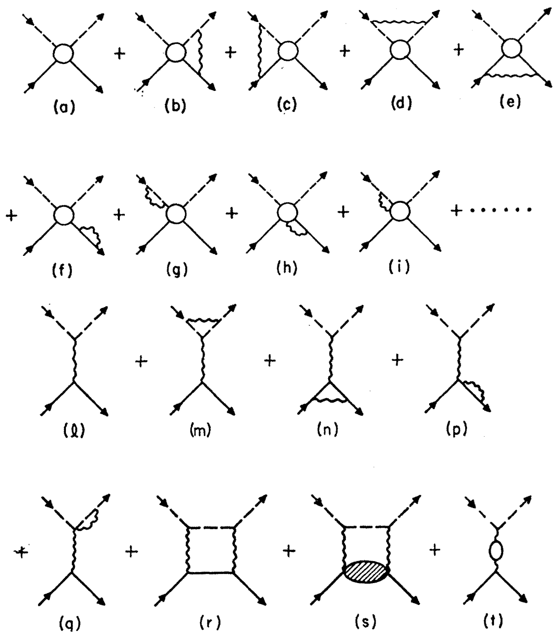

FIG. 1. Examples of graphs which contribute to the expansion of f_{tot}: (a) represents f_N, (b)–(i) contribute to $f_N^{(1)}$, (l) represents f_C, and (m)–(s) contribute to $f_C^{(1)}$ [see Eq. (4)]; (t) is the vacuum polarization contribution and is dominated by the electron-positron pair state. Note that a general electromagnetic form factor can be inserted at any photon vertex (see Sec. II B) and that any intermediate hadronic state can propagate as an excited state (with appropriate quantum numbers) as indicated in (s). Furthermore, examples of crossed graphs have not been explicitly shown here; their inclusion is taken to be understood.

clusion has been reached in a very recent report by Locher.[7] Our treatment of Sec. II goes somewhat beyond his treatment in that we have tried to show just where the many approximations implicit in a result like that of Eq. (2) are located. Our work should therefore be regarded as a general critique of the Bethe formula.

In analyzing the data, Lindenbaum's group originally used both the Bether and Solov'ev formulas and concluded that their fits apparently favor Solov'ev. (However, a more recent analysis,[1] in which a more complete treatment of the multiple scattering corrections is used, has shown that the Bethe formula is actually favored.) This surprising result has led us to examine the "real" radiative corrections to the experiment; by "real" we mean those electromagnetic corrections, such as soft-photon emission, or vacuum polarization, which can change the *magnitude* of the cross section. The dominant contribution of such effects are calculable and they should in principle be incorporated into the data analysis. Superficially, such corrections are of a magnitude which could strongly affect the results derived from an analysis of the difference cross section. However, a direct numerical evaluation to be reported in Ref. 1 indicates that these corrections are negligible.

[7] M. P. Locher, CERN Report No. 67/859/5-TH 799 (unpublished). (We received this paper after completion of our work.)

The main reason for this lies in the fact that an accurate measurement of the total cross section acts as an anchor point in the fitting procedure. This will be discussed in Sec. III.

Immediately below in Sec. I we give a brief review of some of the relevant basic ideas and results in the theory of radiative corrections. For a complete and detailed discussion of the general theory the reader is referred to YFS.[6]

I. REVIEW OF SOME BASIC IDEAS

A dominant feature of the radiative corrections is the IR divergence phenomenon. The physical origin of this is well known: When charges change their state of motion within a short time interval, the outer regions of their fields cannot adjust instantaneously and some very soft quanta must be radiated. Along with the necessity for real radiation, there is a radiation reaction which reduces the probability for the scattering to take place without radiation. The two features are inseparable experimentally, and the corrections depend sensitively on the detection arrangement, in particular, the energy resolution $\Delta E/E$. A related feature is the long-range Coulomb field, which induces the phase change ϕ. In the following several paragraphs, we review briefly some of the pertinent features of the treatment of this phenomenon.

We assume that the elastic scattering amplitude f_{tot} is expandable as a sum of two series:

$$f_{\text{tot}}=f_N[1+\alpha f_N^{(1)}+\alpha^2 f_N^{(2)}+\cdots]+f_C[1+\alpha f_C^{(1)}+\alpha^2 f_C^{(2)}+\cdots]. \quad (4)$$

The second series is defined to include those terms in which only photons are exchanged between the pion and the nucleon. The remaining terms are defined to be the first series. In general, both the $f_N^{(i)}$ and the $f_C^{(i)}$ are IR-divergent. In Fig. 1 we have shown some simple examples of graphs which contribute to the first-order terms $f_N^{(1)}$ and $f_C^{(1)}$. It should be noted that at each vertex a phenomenological form factor can be inserted. Furthermore, in a diagram like Fig. 1(r) the intermediate state hadrons could propagate as higher spin resonances; such polarization effects, which are expected to be small, will not be discussed in this paper. Of those diagrams shown only (h) and (i) do not suffer the characteristic logarithmic divergence. In each order, one can generally isolate the most divergent terms since these are associated with inner bremsstrahlung processes in which a photon connects external legs [see, e.g., Figs. 1(b)–1(g) and 1(m)–1(s)]. We next use the fact that the IR divergences "exponentiate," i.e., the divergences can be summed to all orders to give exponential factors so that the series (4) can be rearranged into the form

$$f_{\text{tot}}=\exp(\alpha\hat{B}_N)[1+\alpha\hat{f}_N^{(1)}+\alpha^2\hat{f}_N^{(2)}+\cdots]+\exp(\alpha\hat{B}_C)f_C[1+\alpha\hat{f}_C^{(1)}+\alpha^2\hat{f}_C^{(2)}+\cdots]. \quad (5)$$

The infinite (complex) exponents $\hat{B}_N$ and $\hat{B}_C$ are derived from the inner bremsstrahlung graphs contained in $f_N^{(1)}$ and $f_C^{(1)}$. The new amplitudes $\hat{f}_N^{(i)}$ and $\hat{f}_C^{(i)}$ are now *finite* so that, for instance, $\alpha\hat{f}_N^{(1)}$ is truly of order 1/137. It should be noted that the definition of these new amplitudes is not unique, in that they can be altered by redefining $\hat{B}_N$ and $\hat{B}_C$ in the non-IR region. Generally, however, one defines the $\hat{B}$'s in such a way that they contain the bulk of the model-independent radiative corrections. The $\hat{f}_{N,C}^{(i)}$, of course, always depend upon the details of the basic interaction.

We have yet to consider the effect of the real soft bremsstrahlung process. Because of the finite energy resolution of the detection apparatus, such inelastic events are not distinguishable from the elastic ones and one must calculate the (infinite) probability for emission of these very soft photons. The observed cross section is thereby increased by an amount $2\alpha\tilde{B}|f_{\text{tot}}|^2$, say (to first order in α). $\tilde{B}$ is calculated from diagrams in which an external photon line is added to each external leg of Fig. 1. It is a real divergent quantity whose precise form depends critically upon the experimental arrangement. We again invoke the exponentiation theorem in order to sum up, to all orders, these divergent contributions. We are thus able to reexpress the cross section in the form (neglecting real, non-IR photon terms which contribute to relative order $\alpha\Delta E/E$)

$$\begin{aligned} d\sigma/d\Omega \propto |&f_N \exp[\alpha(\text{Re}\hat{B}_N+\tilde{B})] \\ &\times[1+\alpha\hat{f}_N^{(1)}+\alpha^2\hat{f}_N^{(2)}+\cdots] \\ &+f_C \exp[\alpha(\text{Re}B_C+\tilde{B})][1+\alpha f_C^{(1)}+\alpha^2 f_C^{(2)}+\cdots] \\ &\times\exp\{i\alpha[\text{Im}(\hat{B}_C-\hat{B}_N)]\}|^2. \end{aligned} \quad (6)$$

Now, it is a general result that the quantities $(\text{Re}\hat{B}_{N,C}+\tilde{B})$ are finite, i.e., the IR divergences in $\tilde{B}$ are precisely cancelled by the divergences in $\text{Re}\hat{B}_{N,C}$ as anticipated. This may be seen explicitly by noting that the divergence in $\text{Re}\hat{B}_{N,C}$ originates, as is indicated in the Appendix, from the photon pole. That is to say, the divergence in $\text{Re}\hat{B}_{N,C}$ occurs when the virtual bremsstrahlung photon becomes real. The finite remainders $(\text{Re}\hat{B}_{N,C}+\tilde{B})$, which were referred to as the "real" radiative corrections in the Introduction, are explicitly given in Sec. III.

The quantity $\text{Im}(\hat{B}_C-\hat{B}_N)$, which was called ϕ in the Introduction, i.e.,

$$\phi\equiv\text{Im}(\hat{B}_C-\hat{B}_N), \quad (7)$$

is also finite. It receives its contributions only from diagrams in which the virtual photon connects the legs of *different external* particles, e.g., Figs. 1(b), 1(c), and 1(r). In Sec. II we give a detailed account of an approximate calculation of ϕ.

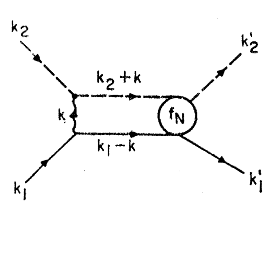

FIG. 2. Graph considered in Sec. II. This gives the dominant "imaginary" contribution to the radiative corrections for near-forward scattering.

II. CALCULATION OF THE "IMAGINARY" RADIATIVE CORRECTION

A. General Result

Following our discussion in Sec. II we see that the major contribution to ϕ comes, at least for small values of the c.m. scattering angle, from the IR-divergent terms contained in the graphs of Figs. 1(b), 1(c), and 1(r). The nondivergent terms are assumed to have a polynomial expansion around $t=0$. Such nonlogarithmic contributions are extremely difficult to calculate in general since they depend upon the detailed dynamics. No attempt will be made to calculate them in this paper; we would like to point out, however, that since they can be thought of as truly perturbative in nature their magnitude is presumably $O(\alpha)$.

Let us consider, then, the IR contributions contained in the evaluation of diagrams 1(b) and 1(c); 1(r) will be considered later. Using the standard Feynman rules, one can straightforwardly show that the IR contribution to either 1(b) or 1(c) is

$$I_N=-4\pi i\alpha(4k_1\cdot k_2)\int\frac{d^4k}{(2\pi)^4} \times\frac{f_N(2k_2\cdot k+k^2,\,-2k_1\cdot k+k^2,\,s,\,t')}{(k^2+i\epsilon)[(k_2+k)^2-m^2+i\epsilon][(k_1-k)^2-M^2+i\epsilon]}. \quad (8)$$

The notation is illustrated in Fig. 2: the initial and final nucleon 4-momenta are k_1 and k_1', respectively; those of the pion are k_2 and k_2'. The internal photon 4-momentum is k. The strong interaction is denoted by the circle in the figures and is algebraically represented by $f_N(\Delta m^2,\Delta M^2,s,t)$, where Δm^2 is the amount that the pion is off its mass shell and ΔM^2 is the corresponding quantity for the nucleon. The pion rest mass is denoted by m and that of the nucleon by M. We employ the usual Mandelstam variables

$$s\equiv(k_1+k_2)^2$$

and

$$t\equiv(k_1'-k_1)^2.$$

It should be noted that $t'\equiv(k_1'-k_1+k)^2$ and that the contribution of (8) to αB_N of Eq. (5) is (by definition)

$$I_N/f_N(o,o,s,t). \quad (9)$$

In obtaining Eq. (8) we have kept only the spin-independent convection-current terms. Numerator terms linear in k, such as those associated with the nucleon magnetic moment, have been dropped since they are not IR-divergent. We have, furthermore, ignored the possibility of form factors at the photon vertices. They only serve to complicate the argument at this stage and we have left their discussion to Sec. II B below.

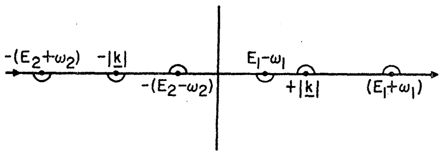

FIG. 3. Complex k^0 plane, showing the positions of the poles in the integral of Eq. (8) and the contour used for evaluation.

Let us examine the explicit pole structure of the k^0 integrand. In the c.m. system of the external particles (where $\mathbf{k}_1+\mathbf{k}_2=\mathbf{0}$), there will be poles at the following values of k^0:

$$\begin{gathered} E_1\pm\omega_1\mp i\epsilon, \\ -E_2\pm\omega_2\mp i\epsilon, \\ \pm|\mathbf{k}|\mp i\epsilon. \end{gathered} \tag{10}$$

Here E_1 and E_2 are the c.m. energies of the nucleon and the pion, respectively, and

$$\omega_{1,2}\equiv+[E_{1,2}^2+(\mathbf{k}^2-2\mathbf{k}_1\cdot\mathbf{k})]^{1/2}. \tag{11}$$

The positions of these poles is illustrated in Fig. 3. Without complete knowledge of the general off-mass-shell analytic behavior of f_N, one clearly cannot evaluate the integrals. However, as we now show, it is possible to obtain a contribution to ϕ which is independent of the analytic properties of f_N; we shall refer to this as the "model-independent" contribution. The k^0 integration contour is closed in the upper half-plane. This picks up poles at the following positions:

$$\begin{aligned} k^0 &= -(E_2+\omega_2) \quad &\text{(a)} \\ &= \quad (E_1-\omega_1) \quad &\text{(b)} \\ &= -|\mathbf{k}|. \quad &\text{(c)} \end{aligned} \tag{12}$$

If we assume that there is no contribution from the contour at infinity and that the singularities in f_N are so "distant" that their contribution may be neglected, then the k^0 integration will give three terms corresponding to the three poles (a), (b), and (c). These terms can be characterized by corresponding denominators now occurring in the integrand in place of the propagators of Eq. (8); they are explicitly exhibited in the Appendix.

We must now perform the $|\mathbf{k}|$ integration ($0<|\mathbf{k}|<\infty$); equivalently, this can be viewed as an integration over ω_1 ($M<\omega_1<\infty$) or over ω_2 ($m<\omega_2<\infty$). Keeping in mind that, at the moment, we are only interested in that part of I_N which is imaginary with respect to f_N, we divide the integration into a principal value (PV) part and a part coming from avoiding the poles (this gives the conventional "$\pi i\times\delta$-function" contribution). These two contributions will be 90° out of phase, provided f_N has no important singularities in the region of integration (so that its phase is not a rapidly varying function). For ϕ we shall only require the pole contributions. It is not difficult to show that the *only* relevant pole is contained in the factor $(E_1-\omega_1+E_2-\omega_2+i\epsilon)$ of the (b) denominator (see the Appendix). The pole is at $\omega_1=E_1$ (this automatically implies $\omega_2=E_2$), and direct evaluation gives, for the model-independent (M.I.) IR contribution, the quantity

$$I_N^{(\mathrm{MI})}=\frac{i\alpha(k_1\cdot k_2)|\mathbf{k}_1|}{2\pi(E_1+E_2)}\int\frac{d\Omega_{\mathbf{p}}}{|\mathbf{k}_1-\mathbf{p}|^2}f_N(o,o,s,t'). \tag{13}$$

We have here introduced the 4-momentum $p\equiv k_1-k$, which represents the intermediate virtual nucleon momentum. The integral in (12) is to be evaluated over the solid angle subtended by the 3-vector $\mathbf{p}$. Note that it is now the *on-mass-shell* strong amplitude which is required.

We can now partially justify some of our assumptions. The contribution $I_N^{(\mathrm{MI})}$ has its origins in the pole at $E_{1,2}=\omega_{1,2}$, corresponding to the region near $k^0=0$ where the poles at $k^0=E_1-\omega_1$ and $-(E_2-\omega_2)$ pinch the contour in the k^0 plane. By considering some simple perturbation diagrams, one can conjecture that the nearest singularity in f_N will not occur until $k^0\sim$ the mass of the first single-particle intermediate state in the t' channel. Furthermore, it has been shown by Feshbach and Yennie[8] that when f_N is expanded about its mass-shell value, terms linear in k^0 vanish. The proof of this statement follows from gauge invariance, in much the same way that the low-energy theorem for bremsstrahlung, as proven by Low,[9] does. Essentially, the off-mass-shell dependence of f_N is partially compensated for by diagrams like 1(h). One can hope, therefore, that in the region where ϕ gets its major contribution, corrections due to the variation of the phase of f_N (i.e., due to its analytic properties) are not large.[10] Nevertheless, this represents another source of our ignorance whose magnitude cannot be estimated without detailed knowledge of the strong interactions.

We digress momentarily to point out that, had we simply taken the δ-function contributions directly from the propagators in Eq. (8) without proper factorization into individual pole terms, we would have obtained precisely *one-half* of the above result. This was the procedure used by Rix and Thaler. Effectively, this procedure omits important PV contributions.[6] Since the final result of Rix and Thaler agrees with ours, we can only conclude that they have made compensating errors. Locher, on the other hand, essentially obtains our Eq. (13) by directly calculating the imaginary parts of the relevant diagrams from unitarity. It is well known that this gives a factor of 2 over the procedure in which only the δ-function part of the propagator is retained. It should be noted, however, that what is required is

[8] H. Feshbach and D. R. Yennie, Nucl. Phys. **37**, 150 (1962).

[9] F. E. Low, Phys. Rev. **110**, 974 (1958).

[10] As a check on our procedure the k^0 integration can be evaluated in precisely the same way that we evaluated the $|\mathbf{k}|$ integration, namely, by dividing it into PV parts and pole contributions. These are again assumed to be 90° out of phase. There are now 6 pole terms; however, from the $|\mathbf{k}|$ integration one can show that only two contribute: $k^0=E_1-\omega_1+i\epsilon$, $k^0=-E_2+\omega_2-i\epsilon$, and that these pinch the contour at $k^0=0$. We obtain Eq. (13).

not the absolute imaginary parts of these graphs but rather the imaginary parts relative to the on-mass-shell strong-interaction amplitude. Our discussion above clearly indicates that there are some very strong assumptions involved in equating these.

In order to complete the angular integrations in Eq. (12), one must have an explicit form for the on-mass-shell f_N. Generally this is parametrized as a function of its c.m. scattering angle, in this case

$$\theta' \equiv \cos^{-1}(\hat{p}\cdot\hat{k}_1');$$

the corresponding t value is

$$\begin{aligned} t' &= (k_1'-k_1+k)^2 \\ &= -(\mathbf{k}_1'-\mathbf{p})^2 \\ &= -2\mathbf{k}_1^2(1-\cos\theta'). \end{aligned} \tag{14}$$

For the over-all process we can define the corresponding variables:

$$\theta \equiv \cos^{-1}(\mathbf{k}_1\cdot\mathbf{k}_1')$$

and

$$t \equiv -2\mathbf{k}_1^2(1-\cos\theta). \tag{15}$$

Introducing an effective Coulomb coupling constant

$$\eta \equiv \frac{\alpha(k_1\cdot k_2)}{s^{1/2}|\mathbf{k}_1|} = \frac{\alpha[s-(m^2+M^2)]}{[s-(m+M)^2]^{1/2}[s-(M-m)^2]^{1/2}}, \tag{16}$$

we can rewrite $I_N^{(\mathrm{MI})}$ in the more convenient form

$$\begin{aligned} I_N^{(\mathrm{MI})} &= \frac{i}{4\pi}\eta \int d\Omega_{\mathrm{p}} \frac{f_N(s,t')}{(1-\cos\Theta)} \\ &= -\tfrac{1}{2}\eta f_N(s,t)\left\{\frac{1}{2\pi}\int \frac{d\Omega_{\mathrm{p}}}{(1-\cos\Theta)} \right. \\ &\quad \left. -\frac{1}{2\pi}\int \frac{d\Omega_{\mathrm{p}}[1-f_N(s,t')/f_N(s,t)]}{(1-\cos\Theta)}\right\}, \end{aligned} \tag{17}$$

where

$$\cos\Theta \equiv \hat{p}\cdot\hat{k}_1 = \cos\theta'\cos\theta + \sin\theta'\sin\theta\cos\phi'$$

and ϕ' is the azimuthal angle of $\mathbf{p}$ with respect to $\mathbf{k}'$ as z axis. The denominators occurring in the integrands of Eq. (17) can vanish at $\Theta=0$ (i.e., when $\theta=\theta'$). The ensuing divergence is, of course, the explicit manifestation of the IR divergence. It is generally parametrized by giving the photon a small mass λ, in which case the denominators become

$$[(1+\lambda^2/2\mathbf{k}_1^2)-\cos\Theta].$$

The limiting procedure $\lambda \to 0$ is always assumed to be understood.

The first integral of Eq. (17) is straightforward and we obtain

$$2\ln(\lambda/2|\mathbf{k}_1|). \tag{18}$$

To evaluate the second integral, first perform the ϕ' integration to obtain

$$\int_{-1}^{1} d\cos\theta' \frac{[1-f_N(s,t')/f_N(s,t)]}{|\cos\theta'-\cos\theta|} = \int_{-4\mathbf{k}_1^2}^{0} \frac{dt'}{|t'-t|}\left[1-\frac{f_N(s,t')}{f_N(s,t)}\right]. \tag{19}$$

We have assumed that $f_N(s,t)$ itself does not have a singularity at $t=0$. Using (9) we can finally write the imaginary part of the IR contribution as

$$\mathrm{Im}\hat{B}_N \simeq \left(\frac{\eta}{2\alpha}\right)\left\{2\ln\frac{\lambda}{2|\mathbf{k}_1|} + \int_{-4\mathbf{k}_1^2}^{0} \frac{dt'}{|t'-t|}\left[1-\frac{f_N(s,t')}{f_N(s,t)}\right]\right\}. \tag{20}$$

The complete "imaginary" IR contribution will actually be twice this since we must include contributions from both diagrams 1(b) and 1(c).

A precisely analogous procedure can be applied to the photonic series in order to isolate the imaginary phase contribution arising from the IR-divergent diagram 1(r). We will again obtain Eq. (20), but with f_N replaced by the known Coulomb amplitude

$$f_C(s,t') \propto (t'-\lambda^2)^{-1}. \tag{21}$$

Again, since we are only concerned with the IR contribution, we have kept only the convection terms. Carrying out the integrations, one obtains the well-known Coulomb phase factor[11]

$$\mathrm{Im}\hat{B}_C = -\left(\frac{2\eta}{\alpha}\right)\ln\left(\frac{2|\mathbf{k}_1|\sin\frac{1}{2}\theta}{\lambda}\right). \tag{22}$$

Notice that, in this case, there is no extra factor of 2 since there is only one diagram, namely 1(r), contributing.

From Eqs. (20) and (22) we can read off the quantity of physical interest:

$$\begin{aligned} \alpha\phi &\equiv \alpha\,\mathrm{Im}(\hat{B}_C-\hat{B}_N) \\ &= -2\eta\ln\sin\tfrac{1}{2}\theta + \eta\int_{-4\mathbf{k}_1^2}^{0} \frac{dt'}{|t'-t|}\left[1-\frac{f_N(s,t')}{f_N(s,t)}\right]. \end{aligned} \tag{23}$$

Equation (23) immediately indicates the added approximation involved in Solov'ev's work, for we obtain his result, Eq. (3), by neglecting the second term (recall that $\theta \to 0$ and that for high energies $\eta \to 1$). Neglecting this second term is equivalent to neglecting the t dependence of f_N.

[11] W. Rolnick, Phys. Rev. 148, 1539 (1966). It is interesting to note that there seems to be no simple way of reproducing the complete nonrelativistic Coulomb phase which contains the extra term $2i\eta\ln\gamma$, where γ is Euler's constant.

B. Bethe's Formula and the Inclusion of Form Factors

A result which is similar to Bethe's can be obtained by parametrizing f_N in the following standard way[12]:

$$f_N(s,t)=e^{a+b^2t}. \tag{24}$$

The experiments of Foley *et al.*[1] show that this is an excellent parametrization in the small-t region [i.e., $-t<0.5$ $(\text{BeV}/c)^2$]. Using this in (23) one can express the integral in the form

$$E(b^2t)+E(b^2t+4b^2\mathbf{k}_1^2), \tag{25}$$

where we have defined

$$E(z)=\int_0^z \frac{dx}{x}(1-e^{-x}).$$

We are, of course, interested mostly in the small θ and large $\mathbf{k}_1^2$ region, where one can approximate $\alpha\phi$ by [13]

$$\alpha\phi\simeq-\eta\left[2\ln(|\mathbf{k}_1|b\theta)+\gamma+O(\mathbf{k}_1^2b^2\theta^2)+O\left(\frac{e^{-4\mathbf{k}_1^2/b^2}}{\mathbf{k}_1^2/b^2}\right)\right]. \tag{26}$$

Equation (26) is reminiscent of Bethe's formula, Eq. (2).

Before discussing Eq. (26) let us first consider the effects of including form factors at the photon vertices. This can be accomplished by making the replacement

$$\frac{f_N(s,t')}{f_N(s,t)}\to\frac{f_N(s,t')}{f_N(s,t)}F_\pi(k^2)F_{\text{ch}}(k^2) \tag{27}$$

in Eq. (17). Here $F_\pi(k^2)$ is the pion form factor and $F_{\text{ch}}(k^2)$ the proton charge form factor. A slightly different replacement holds for the Coulomb amplitude:

$$\frac{f_C(s,t')}{f_C(s,t)}\to\frac{f_C(s,t')}{f_C(s,t)}\frac{F_\pi(k^2)F_{\text{ch}}(k^2)}{F_\pi(t)F_{\text{ch}}(t)}. \tag{28}$$

For small t we can, of course, take $F_\pi(t)F_{\text{ch}}(t)\sim1$. It should be noted that in introducing these form factors, one is implicitly assuming, just as was assumed for f_N, that the k^0 singularities in F_π and F_{ch} are far away from the origin.

Since $k^2=-2\mathbf{k}_1^2(1-\cos\Theta)$ the integration over ϕ', although evaluable analytically, is no longer straightforward. For simplicity we shall, therefore, immediately invoke the small-t limit, where

$$k^2\simeq-2\mathbf{k}_1^2(1-\cos\Theta')=t'$$

is independent of ϕ'. We can, therefore, make the replacement (27) directly into Eq. (20); similarly for the Coulomb term. Since f_N falls off significantly faster as a function of t than either of the electromagnetic form factors, we should obtain a reasonable estimate for ϕ by using the analytically tractable parametrizations.[14]

$$F_\pi\sim\exp(r_\pi^2t),\quad F_{\text{ch}}\sim\exp(r_p^2t). \tag{29}$$

The integrations are now straightforward[15] and one reproduces Eq. (26), but with b replaced by the effective radius

$$b'\equiv(b^2+r_\pi^2+r_p^2)^{1/2}. \tag{30}$$

We conclude, therefore, that the form of Bethe's result is valid but that the parameter b' must be regarded as a model-dependent parameter. However, it should be noted that in using Eq. (2) one would apparently be ignoring the constant contribution γ $(=0.5772\cdots)$. That this contribution is non-negligible can be seen as follows: A typical experimental value of $t(\sim\mathbf{k}_1^2\theta^2)$ is 0.01 $(\text{GeV}/c)^2$ and from the experiment $b^2\sim5(\text{GeV}/c)^{-2}$; from electron scattering[14] $r_\pi^2\sim r_p^2\sim2(\text{GeV}/c)^{-2}$. Hence, $b'\sim3.0(\text{GeV}/c)^{-1}$ and we find that

$$2\ln(|\mathbf{k}_1|b'\theta)\sim2.5.$$

We thus see that γ represents about 25% of the complete ϕ. Of course, the precise value of such a contribution cannot be taken seriously since it was derived using special analytic forms of the form factor. The effect of varying these analytic forms has been investigated by Locher.[7] He limited his study to the simple case in which $f_N(t)$ and $F_\pi(k^2)$ are taken to be constants and $F_{\text{ch}}(k^2)$ allowed to vary. He concluded that such a variation could easily induce a change in ϕ of the order of γ. However, in the more realistic case the comparatively fast falloff of f_N with t reduces such an effect and hence ignorance of the precise form of these form factors will presumably not be an important source of error.

III. REAL RADIATIVE CORRECTIONS

In this section we shall be concerned with those radiative corrections, such as soft-photon emission or vacuum polarization [see, for example, Fig. 1(t)], which can alter the magnitude of the scattering amplitudes. There are no important vacuum polarization contributions to the quantities $(\hat{B}+\text{Re}\hat{B})$ and we shall consider them first. They can be calculated in much the same way as $\text{Im}\hat{B}$ was calculated in the previous section. However, there is one simplifying feature here. When calculating $\text{Im}\hat{B}$ we had to perform an integration over the intermediate-state scattering angles θ' and ϕ'.

[12] Occasionally a third term ct^2 is added to the exponent with $c\sim2.4(\text{BeV}/c)^{-4}$. Neglecting this term has essentially no effect on our conclusions.

[13] *Handbook of Mathematical Functions*, edited by M. Abromowitz and I. A. Stegun (National Bureau of Standards, Washington, D. C., 1964), 5.1.39 and 5.1.52.

[14] Such a parametrization of the nucleon form factors has actually been tried [see S. D. Drell, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967), p. 85].

[15] Note that the form factors have negligible effect upon the photon terms since Coulomb scattering in the forward direction is dominated by large impact parameters.

Because f_N falls off quickly with t', this integration resulted in an expression which differed considerably from the naive formula which ignores any t' dependence of f_N. Of course, it was this fact that originally motivated us to consider the problem. Now, in the present case, the major contributions involve integrations over the corresponding intermediate-state scattering angles in the t channel,[16] i.e., for instance, over $s'\equiv(k_1-k_2-k)^2$. The simplifying feature is that f_N is essentially independent of s' so that the integral corresponding to the t' integration in Eq. (23) can here be neglected. This corresponds to the "usual" IR approximation and implies that $\text{Re}\hat{B}_N\simeq\text{Re}\hat{B}_C$.

The calculation of $(\tilde{B}+\text{Re}\hat{B})$ under these conditions has already been carried out by Meister and Yennie.[17] Reading off from their Eq. (2.20), we find

$$\tilde{B}+\text{Re}\hat{B}\simeq\frac{1}{\pi}\left[\frac{(2m^2-t)}{(t^2-4m^2t)^{1/2}}\times\ln\left\{\frac{2m^2-t+(t^2-4tm^2)^{1/2}}{2m^2}\right\}-1\right]\ln\left(\frac{\Delta E}{E}\right), \quad (31)$$

where $\Delta E/E$ is the energy resolution of the detection apparatus. Again, only the convection current contributions have been retained. Terms representing radiative corrections to the proton lines have been dropped since for small momentum transfers these are heavily damped by the comparatively heavy proton mass. Also dropped are logarithmic terms which do not depend on the resolution and terms which are only linear in the resolution; since these do not contain the $\ln(\Delta E/E)$ factor they can be safely ignored.

As stated in the Introduction, this correction has only a negligible effect upon the results when folded into the data. In order to understand this we present below a hypothetical data analysis. A further motivation for such an analysis is to investigate the effect of introducing a small admixture into the strong interactions whose behavior *cannot* be parametrized by a form like that of Eq. (24). It is to be noted that in the analysis of Ref. 1 the spin-flip contribution to f_N was assumed to be negligible; it was argued[18] that since such a contribution has a linear dependence on t near $t=0$, the bulk of its effect will, in any case, be automatically absorbed into the phenomenological parameter b. Now, one result of our hypothetical analysis is that such an argument is valid only if this linear dependence holds over the whole range of t. If, as is the case with radiative corrections, the analytic form of the admixture deviates from linearity over this range, then its effect cannot be cleanly absorbed into b and an over-all correction remains. We now show this explicitly: We introduce the quantities

$$\alpha_\pm\equiv\text{Re}f_N^\pm/\text{Im}f_N^\pm. \quad (32)$$

If for the moment we neglect real radiative corrections, we can express the complete differential cross section as

$$d\sigma^\pm/dt=|f_C|^2\mp2f_C\,\text{Im}f_N^\pm[\alpha_\pm\cos\alpha\phi\pm\sin\alpha\phi]+(1+\alpha_\pm^2)(\text{Im}f_N^\pm)^2, \quad (33)$$

where $\pm$ refers to $\pi^\pm-p$ scattering. At $t=0$, the optical theorem relates $\text{Im}f_N$ to the measured total cross section

$$\text{Im}f_N^\pm(s,o)=\sigma_{\text{tot}}^\pm/4\pi. \quad (34)$$

A parametrization like that of Eq. (24) is used for $\text{Im}f_N(s,t)$ so that, for instance, one can write

$$|f_N|^2=e^{2a+2b^2t}, \quad (35a)$$

where

$$e^{2a}=(1+\alpha_\pm^2)\sigma_{\text{tot}}^{\pm2}/16\pi^2. \quad (35b)$$

The parameters $\alpha_\pm$ and b are treated as free and independent of t in analyzing the data. In this way one can deduce $\alpha_\pm$ as a function of the incident energy and this can be directly compared with the dispersion relations predictions. The actual comparison was done for the sum and difference quantities $(\alpha_-\pm\alpha_+)$, and, as it will become evident, real radiative corrections can assume importance only in the difference quantity $(\alpha_--\alpha_+)$.

The values of t covered by the experiment were $0<-t<0.2(\text{GeV}/c)^2$; this is to be compared with $m^2\sim0.02\text{ GeV}^2$. In order to investigate the effects of the real radiative corrections it will prove convenient to consider two distinct regions: (i) the interference region $-t\ll m^2$ and (ii) outside the interference region $t\gg m^2$. From Eq. (31) we can write

$$B+\text{Re}B\simeq\frac{1}{\pi}\ln\left(\frac{\Delta E}{E}\right)\times\begin{cases}(-t/3m^2), & -t\ll m^2\\ \ln(-t/m^2)-1, & -t\gg m^2.\end{cases} \quad (36)$$

Outside the Coulomb interference region we can approximate $|f_{\text{tot}}|^2$ by $|f_N|^2$ and thus deduce the parameters of Eq. (35a). However, the actual measured cross section is not $|f_N|^2$ but rather

$$(d\sigma/dt)_{\text{expt}}\simeq|f_N|^2e^{-\Delta_1}\equiv|\tilde{f}_N|^2, \quad (37)$$

where

$$\Delta_1\equiv(2\alpha/\pi)\ln(\Delta E/E)[\ln(-t/m^2)-1].$$

In this region then, it is actually $|\tilde{f}_N|^2$ which is being fitted to the form (35a) and *not* $|f_N|^2$. Clearly, such a fit differs from the "true" one in both shape (value of b^2) and normalization (value of a). Since Δ_1 is effectively constant in this region one might expect the major error to occur in the normalization (e^{2a}). This conclusion, however, is correct, only if a is assumed to be a free parameter. In fact, of course, a is essentially completely known since σ_{tot} is accurately measured and $\alpha_\pm^2$ is a very small quantity which can be sufficiently well determined. Furthermore, radiative corrections to

[16] There is, of course, a contribution from the real parts of the graphs considered in the previous section which depends upon the variation of f_N with t'. Since the effects of (32) are small, we do not expect this added effect [which is not included in (32)] to change our conclusions.

[17] N. Meister and D. R. Yennie, Phys. Rev. **130**, 1210 (1963).

[18] S. J. Lindenbaum (private communications).

$\alpha_\pm^2$ will clearly not significantly affect the value of e^{2a}. In this region then the fits will be almost totally insensitive to the value of $\alpha_\pm$ and the major effect of the radiative corrections will be to alter the value of b. Hence, instead of (37) we should write

$$(d\sigma/dt)_{\text{expt}} \equiv |\tilde{f}_N|^2 \simeq |f_N|^2 e^{-\Delta_1 t/t_1} = e^{2a+2b^2 t} e^{-\Delta_1 t/t_1}, \quad (38)$$

where Δ_1 is to be evaluated at some value of t which is characteristic of the region beyond interference [$t_1 \sim 0.1(\text{GeV}/c)^2$, say]. The form of Eq. (38) ensures the correct normalization condition

$$|f_N|^2(t=0) = |f_N|^2(t=0) = e^{2a}.$$

It should be noted that because there is sufficient error in the data a good fit can still be obtained by making this effective change in the slope; this is particularly true since in the actual data analysis an extra term ct^2 is added to the exponent in Eq. (35a).[12]

We now examine the interference region. In contradistinction to the above the fits here must clearly be sensitive to $\alpha_\pm$. We therefore suppose that $\alpha_\pm$ is effectively determined by solving Eq. (33) in this region. Without regard to radiative corrections we would thus determine the following incorrect value:

$$\tilde{\alpha}_\pm \simeq \mp \frac{(d\sigma/dt)_{\text{expt}} - |f_C|^2 - |\tilde{f}_N^\pm|^2}{2 f_C \,\text{Im} \tilde{f}_N^\pm}, \quad (39)$$

where $\tilde{f}_N$ is now to be thought of as the extrapolation of the fit (38) "determined" above back into the interference region. (For reasons of simplicity we have here ignored the small interference term in the numerator, i.e., we have set ϕ equal to zero.) The *correct* value for $\alpha_\pm$ is obtained from (39) by (i) allowing for radiative corrections in the experimental quantity $(d\sigma/dt)_{\text{expt}}$ in this region and (ii) using the correct strong amplitude f_N in place of the incorrectly parametrized quantity $\tilde{f}_N$:

$$\alpha_\pm \simeq \mp \frac{(d\sigma/dt)_{\text{expt}} e^{\Delta_2 t} - |f_C|^2 - |f_N|^2}{2 f_C \,\text{Im} f_N}, \quad (40)$$

where

$$\Delta_2 \equiv (2\alpha/3\pi m^2) \ln(\Delta E/E).$$

From Eqs. (39) and (40) we can immediately calculate the correction to $\alpha_\pm$ due to the real radiative corrections:

$$\alpha_\pm - \tilde{\alpha}_\pm \simeq \mp \frac{[(d\sigma/dt)_{\text{expt}}(\Delta_2 - \Delta) + (|f_C|^2 - |f_N|^2)\Delta]t}{2 f_C \,\text{Im} f_N} \equiv \mp \delta. \quad (41)$$

We have here introduced the quantity

$$\Delta \equiv \tfrac{1}{2}|\Delta_1|/t_1. \quad (42)$$

The corresponding error in the difference amplitude is

$$[(\alpha_- - \alpha_+) - (\tilde{\alpha}_- - \tilde{\alpha}_+)] \simeq 2\delta. \quad (43)$$

[Notice that the error in $(\alpha_- + \alpha_+)$ cancels within these approximations.]

In order to get a rough estimate of how large δ is expected to be, we go to the value of t in the interference region (call it t_0), where

$$|f_N|^2 \sim |f_C|^2 \sim \tfrac{1}{2}(d\sigma/dt)_{\text{expt}}.$$

At this point we can simplify Eq. (41) to give

$$\delta \sim (\Delta_2 - \Delta)t. \quad (44)$$

From Ref. 1 we see that $t_0 \sim -0.003(\text{GeV}/c)^2$ at 24 GeV and that a reasonable choice for t_1 is $\sim -0.1(\text{GeV}/c)^2$. We then find $\delta \sim 0.001$. As a crude check on this estimate we can evaluate δ near the "edge" of the interference region, where

$$|f_N|^2 \sim 5|f_C|^2 \sim (d\sigma/dt)_{\text{expt}}, \quad \text{say}. \quad (45)$$

Although the form of δ is quite different (the cross section is varying rapidly in this region) the previous estimate is reproduced. This estimate agrees with the exact numerical evaluation given in Ref. 1. Notice that if the analysis is not "anchored" by the measurement of σ_{tot} the correction would have been significant since Δ would then be replaced by $\Delta t_1/t_0 \sim 30\Delta$.

Now, as stated above, an important extension of the above analysis is to an examination of the effects of small admixtures of the strong interactions which cannot be expressed in the form of Eq. (35a) over the whole range of t. For instance, suppose there were a term in the strong interactions that had the form

$$\Delta f_N \sim 0, \quad t \sim t_0$$
$$\sim \beta_1, \quad t \sim t_1 \gg t_0 \quad (46)$$

where β has some logarithmic dependence on t (so that it could be considered essentially constant outside the interference region). Because the parametrization would still be correctly normalized such a term would behave in a manner very similar to Δ_1 above. The error thus induced in $(\alpha_- = \alpha_+)$ would, therefore, be

$$\sim (\beta_1/f_N) 2t_0/t_1 \sim (1/15)(\beta_1/f_N). \quad (47)$$

If we demand that such an error be no larger than other errors discussed in this paper (i.e., ~ 0.04), we must have

$$\beta_1/f_N \lesssim 0.6. \quad (48)$$

As another example, suppose the admixture were of the form

$$\Delta f_N \sim \beta_2, \quad t \lesssim t_0$$
$$\sim 0, \quad t \sim t_1. \quad (49)$$

The error thus induced in $(\alpha_- - \alpha_+)$ is then

$$2\delta \sim 2\beta_2/f_N.$$

If such a term has a linear t dependence

$$\beta \sim \epsilon t/m_\pi^2,$$

then for $\delta < 0.02$ we require

$$\epsilon/f_N \lesssim 0.15. \qquad (50)$$

If, on the other hand, β_2 were taken to be a constant (a, say) so that there was a correction to the measurement of total cross section, then we would require

$$a/f_N \lesssim 0.02. \qquad (51)$$

The above results, (48) and (50), are quite encouraging, for they indicate that there can be quite large nonlinear admixtures to f_N without grossly affecting the results of the experiment. In particular, (51) indicates that there would have to be an error of 1% or so in the normalization (essentially σ_{tot}) before its effect shows up within the systematic errors.

We shall complete this section with a brief discussion of the corrections due to vacuum polarization[19] [Fig. 1(t), for example]. The insertion of virtual pairs into the photon propagator has the effect of increasing the Coulomb amplitude from f_C to $f_C(1+\Delta_v)$, say. The most important insertion is that of an electron-positron pair; a standard calculation yields for this contribution (in the limit $-t \gg m_e^2$, where m_e is the electron mass)

$$\Delta_v^{(e)} = (\alpha/3\pi)\ln(-t/4m_e^2). \qquad (52)$$

For higher-mass pairs, such as muons, Δ_v is linear in t in the interference region and has a magnitude which is down by more than a factor of 10 from the electron pair contribution. Similarly, the insertion of a vector meson, such as the ρ, into the photon propagator is expected to have a negligible effect since its contribution to Δ_v is given by

$$\Delta_v^{(\rho)} \simeq (\alpha/\gamma_\rho^2)(-t/m_\rho^2), \qquad (53)$$

where M_ρ is the mass of the ρ and γ_ρ is the γ-ρ coupling constant [$\gamma_\rho^2 \simeq 4\pi \times 2.5$]. We find

$$\Delta_v^{(\rho)} \simeq 10^{-6}$$

to be compared to $\Delta_v^{(e)} \simeq 5 \times 10^{-3}$.

We can use Eq. (39) to estimate the change in $\alpha_\pm$ due to an effective change in f_C:

$$\Delta\alpha_\pm \simeq \pm \Delta f_C/\text{Im} f_N^\pm - \alpha_\pm \Delta f_C/f_C$$
$$\simeq \pm [f_C/\text{Im} f_N^\pm]\Delta_v - \alpha_\pm \Delta_v. \qquad (54)$$

The last term in Eq. (54) is not important. The first term, however, is a rapidly varying function of t, and we can estimate its order of magnitude by evaluating it at $t=t_0$, where $\text{Im} f_N \sim f_C$. We then obtain

$$\Delta\alpha_\pm \simeq \pm \Delta_v$$

or

$$\Delta(\alpha_- - \alpha_+) \simeq -2\Delta_v^{(e)} \simeq -0.01.$$

This again is of the order of magnitude of other corrections, although its sign is such as to worsen the agreement between theory and experiment. Note, finally, that the error $\Delta(\alpha_- + \alpha_+)$ is essentially zero.

IV. CRITIQUE AND CONCLUSIONS

In this section we would like to summarize some of our results and to comment briefly upon their accuracy.

In Sec. II we calculated ϕ, the imaginary part of the phase difference between the strong and purely electromagnetic interactions and showed that a relativistic treatment reproduces Bethe's potential theory result,[2] Eq. (2), provided the parameter b is suitably interpreted [see Eq. (30), for example]. In so doing we demonstrated that the Solov'ev formula,[5] Eq. (3), is incomplete in that it does not take into account the t variation of the strong interactions [see Eq. (23)]. We also showed that a previous relativistic confirmation of Bethe's formula due to Rix and Thaler[3] is misleading because the Feynman integrals were not properly evaluated.

The question that naturally arises is: How accurate is the Bethe formula? This is not an easy question to answer. In deriving Eq. (20) we neglected all off-mass-shell effects and essentially all analytic properties of the strong amplitude. We tried to argue that this is justified because ϕ is dominated by the pion and proton poles of Fig. 2. We made no attempt to consider polarization effects; note that these could be enhanced by hadronic resonances. Without calculating them directly, none of these effects can be easily estimated. There seems no reason, therefore, not to believe that their contribution is $O(\alpha)$. [Notice that this is to be compared with $2\alpha \ln(|\mathbf{k}_1| b'\theta) \sim 2\alpha$.] Further errors must be associated with the interpretation of b'. Furthermore, in carrying out the integrals over $f_N(s,t)$, Eq. (23), we have assumed that the exponential fit, Eq. (24), holds out to large values of t. Finally, it should be noted that throughout we have tacitly assumed that there is a unique way of separating the strong from the electromagnetic interactions. From the very way in which the phenomonological functions $F_\pi(k^2)$, $F_{\text{ch}}(k^2)$, and $f_N(s,t)$ enter, it is not clear that there is an unambiguous way of doing this. With these various points in mind we feel, therefore, that the Bethe formula is good at best only up to a variation of a factor of ~ 2.5 in the parameter b'. To see the significance of this, we note that since $\cos\phi \sim 1$ we can straightforwardly deduce from Eq. (33) the error in $\alpha\pm$ induced by an error in ϕ[20]:

$$\Delta\alpha_\pm \simeq \alpha\Delta\phi \simeq 2/137 \sim 0.015.$$

[19] We are indebted to L. S. Brown and J. Stuart Godfrey of Yale University for drawing our attention to this important contribution. They also have considered some of the contributions considered in this paper but have paid somewhat more attention to the details of the calculations than to the experimental consequences which we have tried to stress.

[20] *Note added in proof.* This estimate should not be interpreted as being equivalent to one standard deviation. Estimating the theoretical uncertainty is a very subjective matter, and we know of no objective criterion for placing it on the same footing as an experimental error. We feel this estimate is conservative but would not be surprised if the actual correction turned out to be larger.

Hence, $\Delta(\alpha_- - \alpha_+) \sim 0.03$, which could be significant since $(\alpha_- - \alpha_+)$ itself is of this order of magnitude.

In Sec. III we examined the question of real radiative corrections and showed that because of the s independence of f_N one could use the Meister-Yennie[17] corrections originally derived for electron-proton scattering. Without completely reanalyzing the data, we estimated these corrections by proposing a hypothetical analysis. We assumed that outside the Coulomb interference region, the complete scattering amplitude f_{tot} can be approximated by f_N. In this region, the radiative corrections are effectively constant (actually logarithmic). When this f_N is extrapolated back into the interference region (where the radiative corrections are linear in t and are very small in magnitude) and used to calculate $\alpha_\pm$ [from Eq. (39)], an error will be made. This error is very small for the combination $(\alpha_+ + \alpha_-)$ but can be comparatively large for the difference $(\alpha_- - \alpha_+)$. Because f_N is "partially" normalized at $t=0$ to the total cross section σ_{tot}, this source of error is greatly reduced. Our estimate of its magnitude agrees with the numerical evaluation given in Ref. 1. We extended this analysis to a consideration of small admixtures of f_N which cannot be parametrized with a form like that of Eq. (24). We find that they would have to be quite large ($\sim 15\%$ of f_N) before they significantly perturb the results of the experiment. A similar conclusion holds for errors in the total cross-section measurement. Provided that they are $\lesssim 2\%$, they too will not appreciably affect the results. Finally, we examined the effects of vacuum polarization and found that they could change $(\alpha_- - \alpha_+)$ by possibly 1% in a direction which tends to worsen the agreement between theory and experiment.

In summary, then, we can say that, at the present level of experimental accuracy, the theoretical uncertainties lie within the quoted error brackets. However, if the experimental accuracy were to improve appreciably, then such uncertainties would certainly become important and considerations of the radiative corrections beyond those discussed in this paper would clearly become necessary. Of course, it may be that detailed model calculations or some new physical insight into the problem would show that the corrections to our calculations are actually significantly less than $O(\alpha)$; within our approach we are unable to make such a claim.

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APPENDIX

The denominators resulting from the k^0 integration in Eq. (8) are

(a) from the pion pole at $k^0 = -(E_2+\omega_2)+i\epsilon'$:

$$-(E_2+\omega_2+|\mathbf{k}|-i\epsilon)^{-1}(E_2+\omega_2-|\mathbf{k}|-i(\epsilon-\epsilon'))^{-1} \times 2(\omega_2-i\epsilon)^{-1}(E_2+\omega_2+E_1+\omega_1-i\epsilon)^{-1} \times(E_2+\omega_2+E_1-\omega_1+i(\epsilon-\epsilon'))^{-1};$$

(b) from the proton pole at $k^0 = E_1-\omega_1+i\epsilon'$:

$$-(E_1-\omega_1-|\mathbf{k}|+i\epsilon)^{-1}(E_1-\omega_1+|\mathbf{k}|+i(\epsilon-\epsilon'))^{-1} \times(E_1-\omega_1+E_2-\omega_2+i\epsilon)^{-1} \times(E_1-\omega_1+E_2+\omega_2+i(\epsilon-\epsilon'))^{-1}2(\omega_1-i\epsilon)^{-1};$$

(c) from the photon pole at $k^0 = -|k|+i\epsilon'$:

$$-2(|\mathbf{k}|-i\epsilon)^{-1}(E_2-\omega_2-|\mathbf{k}|+i\epsilon)^{-1} \times(E_2+\omega_2-|\mathbf{k}|-i(\epsilon-\epsilon'))^{-1}(E_1+\omega_1+|\mathbf{k}|-i\epsilon)^{-1} \times(E_1-\omega_1+|\mathbf{k}|+i(\epsilon-\epsilon'))^{-1}.$$

The photon pole is to be considered at $k^0 = -\sqrt{(\mathbf{k}^2+\lambda^2)}$, in which case there is no direct pole contribution from this term when carrying out the $|\mathbf{k}|$ integration. Hence, (c) does not contribute to ϕ. By writing down the Feynman amplitude for soft-photon emission, one can verify that its divergence is precisely of the form of that of the PV contribution from (c). A detailed calculation shows that divergences from all such photon pole terms exactly cancel those from soft-photon emissions—see YFS.