jectory falls below $J=0$ [$t\approx -0.6$ (BeV/c)^2], the fixed pole should dominate the asymptotic behavior of the difference of the differential cross section for protons and neutrons. Hence we expect

$$\lim_{s\to\infty} s\frac{d\sigma^{(1)}(s,t)}{d\Omega}\bigg|_{t\lesssim -0.6\ (\mathrm{BeV}/c)^2} \to F(t)\,,$$

independent of s, for electroproduction from the nucleons, where $F(t)$ is proportional to the residue of the fixed pole or the coefficient of the δ_{J0} in sense amplitudes.

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Electromagnetic Perturbations on πNN and πNN^* Couplings in the Chew-Low Model: General Features*

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Electromagnetic perturbations on πNN and πNN^* couplings are studied in the N-N^* reciprocal bootstrap model. In the present paper we confine ourselves to rather general features, making the linear-D approximation for simplicity. There are several self-consistent coupling shifts, much as in the analogous $SU(3)$ reciprocal bootstrap. It is shown that, except for even-J exchanges in the t channel, the "driving terms" are orthogonal to these self-consistent coupling shifts. Thus, as in the $SU(3)$ case, no simple predictions can be made for coupling shifts in the linear-D approximation.

I. INTRODUCTION

SYMMETRY-BREAKING perturbations on the Chew-Low model have been much studied,[1–7] in the hope that (i) a unique set of perturbations would be approximately self-consistent; (ii) the set would be "driven" by the electromagnetic, weak, or semistrong interactions, thus allowing the prediction that observed mass and coupling shifts should be in the same ratio as the approximately self-consistent perturbations of the model. A unique set of approximately self-consistent perturbations, resembling the experimental results, was indeed found for electromagnetic and strong mass splittings of the $J=\frac{1}{2}^+$ octet B and the $J=\frac{3}{2}^+$ decuplet Δ,[1–4] and for the parity-violating part of weak decays $B\to B+\Pi$.[5] On the other hand, several different self-consistent perturbations were discovered for the parity-conserving part of the weak decays $B\to B+\Pi$, so no predictions could be made for this amplitude.[6,7] A similar situation was found for strong perturbations on the $BB\Pi$ and $\Delta B\Pi$ couplings,[6,7] except that in this case it was possible to achieve predictions by noting that the B and Δ mass shifts would preferentially "drive" a particular set of the self-consistent coupling shifts.[6]

We wish to report here on general features of an analogous study of electromagnetic coupling shifts in the $SU(2)$ version of the Chew-Low model. The same disease occurs as in $SU(3)$ coupling shifts: There are several different sets of self-consistent coupling shifts. In addition, we find that the one-photon exchange contribution to πN scattering, and contributions such as γN and $\gamma\pi N$ intermediate states in the s and u channels, only "drive" those sets of coupling shifts which are *not* self-consistent. This supports the conclusion of the related $SU(3)$ studies: The perturbed Chew-Low model does not predict any simple pattern of parity-conserving coupling shifts unless the mass shifts impose one.

In the present paper we derive the above-mentioned results in the linear-D approximation, where the mathematics is simple, and discuss how the results are related to general properties of the crossing matrix. In the following paper,[8] an attempt is made to obtain a rough estimate of the coupling shifts in spite of these

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difficulties, by detailed numerical evaluation of the low-energy contributions.

The perturbed Chew-Low bootstrap equations have the general form

$$\delta R_\alpha = X_{\alpha\beta}\delta R_\beta + \delta R_\alpha{}^t, \tag{1.1}$$

where δR_α is a vector whose elements are the residue shifts of all the s-channel states, $X_{\alpha\beta}$ is the u-channel to s-channel crossing matrix, and $\delta R_\alpha{}^t$ is the t-channel contribution. This form is derived in Sec. II.

The self-consistency condition for shifts δR_α is that the eigenvalue of $X_{\alpha\beta}$ equal $+1$. For the *full* crossing matrix, about half of all the eigenvalues fulfill this condition. We are interested, however, only in that part of the crossing matrix which connects the N and N^* states. In Sec. III we find that this "reduced" crossing matrix has fewer $+1$ eigenvalues, but enough "remembrance" of the full crossing matrix survives to yield three different self-consistent solutions.

Equation (1.1) can be rewritten in the form

$$(\delta_{\alpha\beta} - X_{\alpha\beta})\delta R_\beta = (X_{\alpha\beta} - \delta_{\alpha\beta})\delta R_\beta{}' + \delta R_\alpha{}^t, \tag{1.2}$$

where δR refers to the πNN and πNN^* couplings and $\delta R'$ to all other s-channel states. This is a set of inhomogeneous equations with respect to δR, whose homogeneous part has nontrivial solutions (the eigenvectors associated with unit eigenvalues of the "reduced" part of X which connects the δR). The remaining question is whether the right-hand side of Eq. (1.2) is orthogonal to these eigenvectors, and the answer given in Sec. IV is yes for all components of $\delta R'$ and for contributions to δR^t from $J^P = 1^-, 3^-, \cdots$ exchanges (e.g., one-photon exchange). This result ensures that the solution of Eq. (1.2) is finite (unless we include even-J exchanges in the calculation), but it also means that in more refined calculations, where one removes the linear-D approximation used in deriving (1.1) and the eigenvalues are no longer exactly $+1$, the odd-J exchanges and s- and u-channel terms do not appreciably "drive" the approximately self-consistent δR.

We conclude in Sec. V with a brief discussion of the results of the static model for coupling shifts.

II. DERIVATION OF THE BASIC FORMULA

It is well known[9] that the Chew-Low model with linear D function leads to linear relations of the form

$$R_\alpha{}^s = R_\alpha{}^u + R_\alpha{}^t \tag{2.1}$$

among the residue functions. Relating R^u to R^s by the crossing matrix, one obtains

$$R_\alpha{}^s = X_{\alpha\beta}R_\beta{}^s + R_\alpha{}^t. \tag{2.2}$$

Perturbation of (2.2) gives Eq. (1.1′):

$$\delta R_\alpha{}^s = X_{\alpha\beta}\delta R_\beta{}^s + \delta R_\alpha{}^t. \tag{1.1′}$$

[9] See, for example, J. R. Fulco and D. Y. Wong, Phys. Rev. Letters 15, 274 (1965).

The above argument makes Eq. (1.1) plausible but leaves the status of terms such as one-photon exchange somewhat unclear. In order to specify the content of the equation more precisely, we now rederive it in the S-matrix perturbation formalism of Dashen and Frautschi.[10] We denote the Jth partial-wave amplitude for $\pi N \to \pi N$, with initial and final isospin states $|I_i I_z\rangle$ and $|I_f I_z\rangle$, by $T(I_i I_z \to I_f I_z; J)$. We further denote the unperturbed amplitude by T_0 and the perturbation by $\delta T = T - T_0$. The first-order shift δR in the residue of an s-channel pole with mass M, isospin I_i, and spin J is then given by[10]

$$\delta R(I_i I_z \to I_i I_z; J) = \frac{1}{2\pi i}\int_C \frac{D_{I_i J}{}^2(W')}{(W'-M)^2}[1 - D_{I_i J}{}''(M)(W-M)] \times \delta T(I_i I_z \to I_f I_z; J)dW', \quad I_i = I_f \tag{2.3}$$

$$\delta R(I_i I_z \to I_f I_z; J) = \frac{1}{2\pi i}\int_C \frac{D_{I_i J}(W')D_{I_f J}(W')}{(W'-M)} \times \delta T(I_i I_z \to I_f I_z; J)dW', \quad I_i \neq I_f \tag{2.4}$$

where W is the center-of-mass energy, all $D'(M)$ have been normalized to 1, and the contour C runs clockwise around all singularities *except* the pole at M. In the Chew-Low model with linear D function we specialize to P waves, and take

$$\begin{aligned} D_{11} &= (W-M)\,, \\ D_{13} &= 1\,, \\ D_{31} &= 1\,, \\ D_{33} &= (W-M^*)\,, \end{aligned} \tag{2.5}$$

where now M^* refers to the 3-3 resonance mass and M is reserved for the nucleon mass. With these specializations, Eqs. (2.3) and (2.4) reduce to

$$\delta R_\alpha = \frac{1}{2\pi i}\int_C \delta T_\alpha(W')dW', \tag{2.6}$$

where α stands for $(I_i I_z \to I_f I_z; J)$. We see that mass-shift terms such as $[R/(W'-M'-\delta M) - R/(W'-M')]$ in δT do not contribute to (2.6), but hadron-coupling shift terms such as $\delta R'/(W'-M')$ do contribute, as well as the new cuts associated with one-photon exchange, $\gamma\pi$ exchange, and so forth. Since the integral is taken clockwise, contributions from terms $\delta T_\alpha \sim \delta R_\alpha{}'/(W_s{}'-M')$ on the right cut pick up a minus sign $(-\delta R_\alpha{}')$. Left-cut terms can be rewritten

$$\delta T_\alpha(W_s) = X_{\alpha\beta}\delta T_\beta(W_u)\,, \tag{2.7}$$

where $X_{\alpha\beta}$ is the static crossing matrix. With static

[10] R. Dashen and S. Frautschi, Phys. Rev. 137, B1318 (1965).

kinematics, contributions from left-cut terms

$$\delta T_\beta(W_u)\sim X_{\alpha\beta}\frac{\delta R_\beta'}{W_u-M'}=X_{\alpha\beta}\frac{\delta R_\beta'}{-W_s+2M-M'}$$

retain the plus sign ($X_{\alpha\beta}\delta R_\beta'$) when the clockwise integral is performed. Thus we may write (2.6) in the form

$$\delta R_\alpha=-\int_M^\infty \delta R_\alpha'(W)dW+\int_M^\infty X_{\alpha\beta}\delta R_\beta(W)dW+\delta R_\alpha^t, \quad (2.8)$$

where the first (primed) integral includes all s-channel terms except the N or N^* coupling, the second integral includes all u-channel terms including the γN intermediate state, and the t-channel contributions including one-photon exchange are lumped into δR_α^t. This is the exact specification of our basic equation (1.1).

Before proceeding further, we would like to make two comments about Eq. (2.8):

(i) The equation, and therefore all the discussion in this paper, is valid only in the static model. Specifically, its derivation depended on the use of linear D functions and static crossing relations in which the full infinite-dimensional crossing matrix is replaced by the finite, energy-independent P-wave crossing matrix $X_{\alpha\beta}$.

(ii) Carrying the separation of N and N^* coupling shifts from the other contributions a step further, we can write

$$\delta R_\alpha=X_{\alpha\beta}\delta R_\beta-\int_M^\infty [\delta R_\alpha'(W)-X_{\alpha\beta}\delta R_\beta'(W)]dW+\delta R_\alpha^t, \quad (2.9)$$

where δR_α refers to N and N^* couplings and the primed integral to all other s- and u-channel terms. The remainder of this paper will deal with the equation in this form; the first term on the right being discussed in Sec. III, the integral and δR_α^t in Sec. IV. Note that Eq. (2.9) is a special case of the Dashen-Frautschi relation[10]

$$\delta R_\alpha=A_{\alpha\beta}\delta R_\beta+D_\alpha, \quad (2.10)$$

with the "A matrix" becoming simply the crossing matrix in our model, and the "driving term" supplied by the remaining s-, u-, and t-channel contributions in (2.9). Thus the search for unit eigenvalues of A becomes a search for unit eigenvalues of the crossing matrix, and the explicit formulation of (2.9) makes some study of the driving terms possible.

III. EIGENVALUES OF THE TRUNCATED CROSSING MATRIX

The full crossing matrix, connecting the complete set of all πN P-waves in the Chew-Low model, satisfies

$$X^2=1, \quad (3.1)$$

and therefore all its eigenvalues are ± 1.[11]

In the study of perturbations on the Chew-Low model, however, only part of the complete set of all πN P waves is used, because we begin with particles in the 1-1 and 3-3 states, but not the 1-3 and 3-1 states. As a result, only part of the full crossing matrix comes into play, and the eigenvalues of this part of the crossing matrix are *not* all ± 1. In the present section, we wish to spell out the relationship between the full crossing matrix and the truncated crossing matrix used in the perturbation studies.

There are ten different πN scattering reactions, and two P states ($J=\frac{1}{2}$ and $\frac{3}{2}$) for each, so the *full* P-wave crossing matrix operates on a 20-dimensional space. It is convenient to choose as a basis the 20 linear combinations

$$\begin{aligned}
&\delta R(\pi^+p\to\pi^+p;\ J=\tfrac{1}{2}^+)+\delta R(\pi^-p\to\pi^-p;\ J=\tfrac{1}{2}^+),\\
&\delta R(\pi^+p\to\pi^+p;\ \tfrac{1}{2}^+)-\delta R(\pi^-p\to\pi^-p;\ \tfrac{1}{2}^+),\\
&\delta R(\pi^+n\to\pi^+n;\ \tfrac{1}{2}^+)+\delta R(\pi^-n\to\pi^-n;\ \tfrac{1}{2}^+),\\
&\delta R(\pi^+n\to\pi^+n;\ \tfrac{1}{2}^+)-\delta R(\pi^-n\to\pi^-n;\ \tfrac{1}{2}^+),\\
&\delta R(\pi^0p\to\pi^0\mathrm{p};\ \tfrac{1}{2}^+),\\
&\delta R(\pi^0n\to\pi^0n;\ \tfrac{1}{2}^+),\\
&\delta R(\pi^+n\to\pi^0p;\ \tfrac{1}{2}^+)+\delta R(\pi^0n\to\pi^-p;\ \tfrac{1}{2}^+),\\
&\delta R(\pi^+n\to\pi^0p;\ \tfrac{1}{2}^+)-\delta R(\pi^0n\to\pi^-p;\ \tfrac{1}{2}^+),\\
&\delta R(\pi^-p\to\pi^0n;\ \tfrac{1}{2}^+)+\delta R(\pi^0p\to\pi^+n;\ \tfrac{1}{2}^+),\\
&\delta R(\pi^-p\to\pi^0n;\ \tfrac{1}{2}^+)-\delta R(\pi^0p\to\pi^+n;\ \tfrac{1}{2}^+),\\
&\delta R(\pi^+p\to\pi^+p;\ \tfrac{3}{2}^+)+\delta R(\pi^-p\to\pi^-p;\ \tfrac{3}{2}^+),\\
&\delta R(\pi^+p\to\pi^+p;\ \tfrac{3}{2}^+)-\delta R(\pi^-p\to\pi^-p;\ \tfrac{3}{2}^+),\\
&\qquad\text{etc., for } J=\tfrac{3}{2}^+.
\end{aligned}$$

In this basis, each linear combination of charge states crosses to itself, and the crossing matrix has the simple form

$$X=\begin{pmatrix}-\frac{1}{3}C & \frac{4}{3}C\\ \frac{2}{3}C & \frac{1}{3}C\end{pmatrix}, \quad (3.2)$$

where the numbers $-\frac{1}{3}$, $\frac{4}{3}$, $\frac{2}{3}$, and $\frac{1}{3}$ refer to static spin crossing coefficients relating the $J=\frac{1}{2}^+$ and $J=\frac{3}{2}^+$ states of the s channel to those of the u channel, and

$$C=\begin{bmatrix}1 &&&&&&&&&\\ &-1&&&&&&&0&\\ &&1&&&&&&&\\ &&&-1&&&&&&\\ &&&&1&&&&&\\ &&&&&1&&&&\\ &&&&&&-1&&&\\ &&&&&&&1&&\\ &&&&&&&&-1&\\ &0&&&&&&&&1\end{bmatrix} \quad (3.3)$$

[11] See, for example, R. C. Hwa and S. H. Patil, Phys. Rev. 138, B933 (1965).

gives the crossing between s- and u-channel charge states. Since the spin crossing matrix

$$\begin{pmatrix} -\frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

has eigenvalues ± 1, one easily sees that the full crossing matrix has ten $+1$ eigenvalues and ten -1 eigenvalues.

In our model of perturbations we set some of the δR equal to zero, for two reasons. The first reason is time-reversal invariance, which provides the four conditions

$$\delta R(\pi^+n \to \pi^0 p;\, J) - \delta R(\pi^0 p \to \pi^+ n;\, J) = 0\,,$$
$$\delta R(\pi^0 n \to \pi^- p;\, J) - \delta R(\pi^- p \to \pi^0 n;\, J) = 0\,,$$
$$J = \tfrac{1}{2}^+,\, \tfrac{3}{2}^+. \qquad (3.4)$$

To see the effect of removing these four linear combinations, we choose a new basis in which the first eight linear combinations of residues satisfy time-reversal invariance and the next two do not:

$$\delta R(\pi^+ p \to \pi^+ p;\, \tfrac{1}{2}^+) + \delta R(\pi^- p \to \pi^- p;\, \tfrac{1}{2}^+)\,,$$
$$\delta R(\pi^+ p \to \pi^+ p;\, \tfrac{1}{2}^+) - \delta R(\pi^- p \to \pi^- p;\, \tfrac{1}{2}^+)\,,$$
$$\delta R(\pi^+ n \to \pi^+ n;\, \tfrac{1}{2}^+) + \delta R(\pi^- n \to \pi^- n;\, \tfrac{1}{2}^+)\,,$$
$$\delta R(\pi^+ n \to \pi^+ n;\, \tfrac{1}{2}^+) - \delta R(\pi^- n \to \pi^- n;\, \tfrac{1}{2}^+)\,,$$
$$\delta R(\pi^0 p \to \pi^0 p;\, \tfrac{1}{2}^+)\,,$$
$$\delta R(\pi^0 n \to \pi^0 n;\, \tfrac{1}{2}^+)\,,$$
$$[\delta R(\pi^+ n \to \pi^0 p;\, \tfrac{1}{2}^+) + \delta R(\pi^0 p \to \pi^+ n;\, \tfrac{1}{2}^+)] + [\delta R(\pi^0 n \to \pi^- p;\, \tfrac{1}{2}^+) + \delta R(\pi^- p \to \pi^0 n;\, \tfrac{1}{2}^+)]\,,$$
$$[\delta R(\pi^+ n \to \pi^0 p;\, \tfrac{1}{2}^+) + \delta R(\pi^0 p \to \pi^+ n;\, \tfrac{1}{2}^+)] - [\delta R(\pi^0 n \to \pi^- p;\, \tfrac{1}{2}^+) + \delta R(\pi^- p \to \pi^0 n;\, \tfrac{1}{2}^+)]\,,$$
$$[\delta R(\pi^+ n \to \pi^0 p;\, \tfrac{1}{2}^+) - \delta R(\pi^0 p \to \pi^+ n;\, \tfrac{1}{2}^+)] + [\delta R(\pi^0 n \to \pi^- p;\, \tfrac{1}{2}^+) - \delta R(\pi^- p \to \pi^0 n;\, \tfrac{1}{2}^+)]\,,$$
$$[\delta R(\pi^+ n \to \pi^0 p;\, \tfrac{1}{2}^+) - \delta R(\pi^0 p \to \pi^+ n;\, \tfrac{1}{2}^+)] - [\delta R(\pi^0 n \to \pi^- p;\, \tfrac{1}{2}^+) - \delta R(\pi^- p \to \pi^0 n;\, \tfrac{1}{2}^+)]\,,$$

and, similarly, for $J = \tfrac{3}{2}^+$. The crossing matrix in this new basis turns out to be just the same as before [Eqs. (3.2) and (3.3)]. Thus one easily sees that the combinations which are noninvariant under time reversal cross only among themselves, with two $+1$ eigenvalues and two -1 eigenvalues. Dropping these combinations from consideration, one is left with a 16×16 crossing matrix which has eight $+1$ eigenvalues and eight -1 eigenvalues.

The second consideration which further reduces the number of δR's in our model is that some of the residues of the N and N^* poles [namely, $\delta R(I=\tfrac{3}{2} \to I=\tfrac{3}{2};\, J=\tfrac{1}{2}^+)$ and $\delta R(I=\tfrac{1}{2} \to I=\tfrac{1}{2};\, J=\tfrac{3}{2}^+)$] correspond to $SU(2)$ symmetry breaking at both vertices simultaneously and are thus negligible, of order e^4 (these residues would appear in an order e^2 calculation such as ours if there were low-mass physical particles with $I=\tfrac{1}{2}$, $J=\tfrac{3}{2}^+$ and $I=\tfrac{3}{2}$, $J=\tfrac{1}{2}^+$). Removing the linear combinations representing these six states from the list of δR's, we are left with ten δR's, i.e., with a 10×10 crossing matrix.

Since the 10×10 crossing matrix arises from truncating a 16×16 matrix in this manner, its eigenvalues are not all ± 1. Specifically, Thornber[8] finds that it has three $+1$ eigenvalues and two -1 eigenvalues, with the rest at the intermediate values $5/9$, $1/3$, $-1/9$, $-7/9$, $-7/9$. These values are consistent with the following general statements.

(i) Each δR removed in the truncation process can be expressed as a linear combination of eigenvectors of the crossing matrix. By suitable choices of the degenerate eigenvectors, the first δR can always be taken as a linear combination of at most one eigenvector corresponding to eigenvalue $+1$, and one eigenvector corresponding to eigenvalue -1. Thus after the crossing matrix has been reduced from 16×16 to 15×15 by removal of the first δR, there are still at least seven $+1$ eigenvalues and seven -1 eigenvalues. Similarly, each successive δR removed can always be taken as a linear combination of at most one of the remaining eigenvectors with eigenvalue $+1$, and one remaining eigenvector with eigenvalue -1, plus the mixed eigenvectors with nonunit eigenvalues produced by previous truncations. By this general reasoning, at least two $(=8-6)$ eigenvalues $+1$ and two eigenvalues -1 must remain after the crossing matrix is truncated from 16×16 to 10×10. Actually, Thornber[8] finds three eigenvalues at $+1$ and two at -1, which means that the six removed δR had components along only five independent directions in the space of eigenvectors with eigenvalue $+1$.

(ii) It can also be shown that if all eigenvalues of the full crossing matrix were ± 1, the eigenvalues λ_i of the truncated crossing matrix must satisfy $-1 \leq \lambda_i \leq 1$. Again, this is consistent with Thornber's results.

The analogous arguments for the truncated $SU(3)$ version of the static model, with B and Δ poles, do not require any ± 1 eigenvalues but do of course predict $-1 \leq \lambda_i \leq 1$. Detailed calculations[6,7,12] have shown that in fact all eigenvalues satisfy $-1 < \lambda_i < 1$ in the $SU(3)$ static model. Some of the eigenvalues did lie *near* $+1$, however. From the present point of view, this represents a near-survival of a few of the $+1$ eigenvalues in the full untruncated crossing matrix

IV. CONTRIBUTIONS FROM OTHER TERMS

Aside from the crossed channel N and N^* poles, many other terms contribute to δR_α. In the linear-D approximation with static crossing relations we have, according to Eq. (2.9),

$$\delta R_\alpha - X_{\alpha\beta}\delta R_\beta = -\int_M^\infty [\delta R_\alpha'(W') - X_{\alpha\beta}\delta R_\beta'(W')] dW' + \delta R_\alpha{}^t, \qquad (2.9)$$

[12] R. Dashen, Y. Dothan, S. Frautschi, and D. Sharp, Phys. Rev. **151**, 1127 (1966).

where the $\delta R_\alpha'$ are the s-channel contributions, the $X_{\alpha\beta}\delta R_\beta'$ are the u-channel contributions, and the t-channel effects have been lumped together in δR_α^t.

We may think of (2.9) as an inhomogeneous equation in the N and N^* residues δR_α, which has homogeneous solutions corresponding to the unit eigenvalues of the truncated X,

$$X_{\alpha\beta}R_\beta = R_\alpha. \tag{4.1}$$

There are two possibilities:

(i) The homogeneous solutions are orthogonal to the right-hand side of (2.9). In this case they are finite, but our equations do not determine their size or require them to be large.

(ii) The homogeneous solutions are *not* orthogonal to the right-hand side of (2.9). In this case the solutions of (2.9) diverge, but if the eigenvalues are perturbed slightly away from unity (e.g., by adding a nonlinear term to D), the corresponding eigenvectors are determined by the equation to be large ("enhanced") and finite [of the form $\delta R_\alpha = (1-X)_{\alpha\beta}^{-1}$ (right-hand side)$_\beta$].

We now proceed to show that in fact all terms on the right-hand side of (2.9) are orthogonal to the homogeneous solutions, so that possibility (i) is what occurs, unless t-channel exchanges with $J^P=0^+, 2^+, \cdots$ are included. For each s-channel term, together with its u-channel partner, the proof is immediate. Each s-channel contribution crosses (in the static model with linear D function) in the same way as the N and N^* residues did. Thus each $\delta R'$ contributes

$$\delta R_\alpha' - X_{\alpha\beta}\delta R_\beta' = (\delta R_\alpha' - \delta R_\alpha') = 0 \tag{4.2}$$

to an eigenvector satisfying (4.1). Note that the full untruncated X matrix may occur in some of these s- and u-channel terms, but the additional terms this implies are orthogonal to the eigenvectors satisfying (4.1).

Turning to t-channel exchanges, we first note that the t channel contributes equally to $J=\frac{1}{2}^+$ and $J=\frac{3}{2}^+$ couplings in the static limit. To see this, recall that the $J=l\pm\frac{1}{2}$ amplitude, $f_{l^\pm}=e^{i\delta_{l\pm}}\sin\delta_{l^\pm}/q$, is related to partial-wave projections on the invariant amplitudes A and B by[13]

$$f_{l^\pm} = (1/32\pi W^2)\{[(W+M)^2-\mu^2][A_l+(W-M)B_l] + [(W-M)^2-\mu^2][-A_{l\pm1}+(W+M)B_{l\pm1}]\}. \tag{4.3}$$

In the static limit $[\mu/M\to 0,\ (W-M)/M\to 0]$, the coefficient of $A_{l\pm1}$ is down by a factor $[(W-M)^2-\mu^2]/4M^2$ compared to the coefficient of A_l. Explicit evaluation shows that the t-channel contributions to $A_{l\pm1}$ (unlike the s- or u-channel contributions) cannot provide compensating factors of M/μ or poles at $W=M$, and therefore they can be neglected. Similarly, the coefficient of $B_{l\pm1}$ relative to B_l in the static limit is $[(W-M)^2-\mu^2]/2M(W-M)$, and t-channel contributions to $B_{l\pm1}$ do not provide compensating factors, so the B_l term dominates $B_{l\pm1}$ except in a circle about $W=M$ whose radius shrinks to zero in the static limit. Since the t-channel contributions to $B_{l\pm1}$ inside this circle are finite (again unlike the s- and u-channel case), the t-channel part of $B_{l\pm1}$ can be neglected altogether as the circle shrinks to zero radius. Thus, to the leading order, only the terms A_l and B_l which contribute the *same* to $J=l\pm\frac{1}{2}$ survive.

The pattern of t-channel contributions to different *charge* states depends on whether the t-channel exchange couples to a symmetric or antisymmetric $\pi\pi$ charge state. Biswas, Patil, and Saxena[14] have shown that exchanges coupling to symmetric $\pi\pi$ charge states are eigenvectors of C_{su} (the untruncated s- to u-channel charge crossing matrix) with eigenvalue $+1$, and exchanges coupling to antisymmetric $\pi\pi$ charge states are eigenvectors of C_{su} with eigenvalue -1. Since the pions satisfy generalized Bose statistics, these results also imply that even- (odd-) J exchanges contribute only to couplings which are eigenvectors of C_{su} with eigenvalue $+1$ (-1). Furthermore, since the t channel contributes equally to $J=\frac{1}{2}^+$ and $J=\frac{3}{2}^+$ couplings, and equal $\frac{1}{2}^+$ and $\frac{3}{2}^+$ couplings constitute an eigenvector of the s- to u-channel *spin* crossing matrix

$$S_{su} = \begin{pmatrix} -\frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \tag{4.4}$$

with eigenvalue $+1$, we see that the above statements apply to the combined charge and spin crossing matrix X as well as to C. An immediate consequence is that vector exchanges (γ, ρ, ω, ϕ), being odd in J, contribute nothing to coupling shifts which are eigenvectors of X with eigenvalue $+1$, as can be verified by direct calculation. This conclusion survives the truncation of the N-N^* crossing matrix, of course, because contributions which are orthogonal to all eigenvectors of X with eigenvalue $+1$ *before* truncation must remain orthogonal to the surviving eigenvectors with eigenvalue $+1$ *after* truncation.

It is also interesting to note that in the static limit, as pointed out by Chew,[15] *all* t-channel contributions to the residue of a pole at $W=M$ vanish. Thus in a model where the N^* pole is set at the same W as the nucleon pole, none of the t-channel exchanges would contribute to coupling shifts. This behavior arises as follows. The partial-wave amplitude simplifies to

$$f_{l^\pm} = (1/8\pi)[A_l+(W-M)B_l+(q_s^2/4M^2) \times(-A_{l\pm1}+2MB_{l\pm1})] \tag{4.5}$$

in the static limit. The residue of a pole at $W=M$ is

$$R_{l^\pm} = \lim_{W\to M} [(W-M)f_{l^\pm}]; \tag{4.6}$$

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for poles at $W=M$ the B_l term does not contribute and

$$R_{l^\pm}=\lim_{W\to M}\left[\frac{(W-M)}{8\pi}\left(A_l-\frac{q_s^2}{4M^2}A_{l\pm1}+\frac{q_s^2}{2M}B_{l\pm1}\right)\right]$$

$$=\lim_{W\to M}\frac{1}{2\pi i}\int_C dW'\frac{(W'-M)}{(W'-M)}\frac{1}{8\pi}\times\left(A_l-\frac{q_s'^2}{4M^2}A_{l\pm1}+\frac{q_s'^2}{2M}B_{l\pm1}\right)$$

$$=\frac{1}{2\pi i}\int_C\frac{dW'}{8\pi}\times\left(A_l(W')-\frac{q_s'^2}{4M^2}A_{l\pm1}(W')+\frac{q_s'^2}{2M}B_{l\pm1}(W')\right). \quad (4.7)$$

The discontinuities due to t-channel exchanges include cuts from

(i) $s=(M-\mu)^2$ to $(M+\mu)^2$ (exchanges of mass zero to 2μ);

(ii) $S=M^2$ to $s=-M^2$ along the circle $|s|=M^2$;

(iii) $S=-M^2$ to $s=0$ and $-\infty$.

Explicit evaluation of these discontinuities shows that they cancel systematically [the part from $s=(M-\mu)^2$ to $s=M^2$ cancels the $s=M^2$ to $s=(M+\mu)^2$ piece; the two halves of the circle cancel, and the pieces from $s=-M^2$ to 0 and from $-M^2$ to $-\infty$ cancel]. Thus the t channel contributes nothing to residues at $W=M$. For poles at $W=M^*\neq M$, one must include the term $(W-M)B_l$, and the factor $(W-M)$ in this term spoils the cancellation, so the t channel *does* contribute to residues in this case.

V. DISCUSSION

We have found several self-consistent coupling shifts in the N-N^* static model. Within the context of the linear-D approximation, we find no particular reason why one of these self-consistent coupling shifts should be dominant—i.e., no simple predictions emerge.

The most closely related previous study was the analogous investigation of perturbations on the $SU(3)$ static model.[6,7] A similar conclusion was reached; there were several approximately self-consistent coupling shifts. By the introduction of curved D functions, the effect of mass shifts on coupling shifts was investigated and shown to be an important effect which might dominate. The present study supports this conclusion, inasmuch as we have shown that a number of other effects fail to "drive" the self-consistent shifts.

The one case in which a *single* self-consistent set of coupling shifts has been found in perturbed static models is the parity-violating weak couplings of the B-Δ reciprocal bootstrap.[5] Experimentally, this self-consistent set fits the observed effects very well. Theoretically, however, it has not been established that the "driving terms" actually drive this particular set of couplings. The results of the present paper do not bear directly on this matter, since we have assumed C and P conservation, so the question of the driving terms in the parity-violating weak interaction remains open.

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