# Nonlinear Effects in Plasma Resonance

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The problem of interaction between a plasma and radiation is formulated in a self-consistent fashion. The particular nonlinear effect of interest is associated with the possibility of the excitation of a localized electrostatic mode by an externally impinging radiation. The equation for the localized electrostatic mode is found to be intrinsically nonlinear (or anharmonic), and the explicit expressions for the frequency shift and the amplitude jump of the localized electrostatic mode are calculated.

## I. INTRODUCTION

In the case of plasma resonance induced by radiation, the electric field inside the plasma is, in general, orders of magnitude larger than that of the radiation. The nonlinearity caused by this large electric field associated with the resonance may considerably change the resonance behavior of the plasma, which should, in turn, change the electric field inside the plasma. It is clear that the problem demands a self-consistent treatment between the radiation and the plasma. In other words, the macroscopic electric field of a resonance determines the equilibrium particle distribution, which will then self-consistently determine the macroscopic field itself. It appears that we cannot treat the plasma as having fixed harmonic modes or the electric field as being externally given. These two approximations are too much to ask for in a bounded plasma capable of sustaining *localized modes*. In this paper, we propose to do a "*self-consistent*" treatment of the problem.

The localized macroscopic electric field is treated as an *undetermined* quantity which will be proved to satisfy a nonlinear differential equation. In the derivation, we shall only make use of such general features of the localized field as its being spatially inhomogeneous and larger in magnitude than the externally applied electric field. The distinct result of this approach is that in the localized region the plasma will be shown to be approximately described as a set of *anharmonic* oscillations. The externally applied field will cause the anharmonic oscillations to be forced, and thus we expect the manifestation of such general properties of anharmonic oscillators as frequency shifts and amplitude jumps in our result.

In Sec. II, we shall treat the particle dynamics of an electron in a bounded plasma. For the sake of mathematical rigor, the so-called "stroboscopic method"[1] in nonlinear analysis is used. This study serves as a foundation for Sec. III, where a statistical description of the plasma and the field is derived by using Klimontovich's formulation.[2] For the sake of mathematical simplicity and physical clarity, at the end of Sec. III we resort to a fluid description. With this knowledge, we then particularize to consider a finite plasma that can sustain some localized electrostatic modes, where again for mathematical simplicity we assume the localized region to be smaller than the free-space wavelength of the impinging radiation. At the very end of Sec. III, we arrive at an approximate, nonlinear temporal differential equation for the self-consistent macroscopic electric field of the resonance. This equation is then solved by perturbation techniques in Sec. IV, where the physical significance of the results is explained along with the mathematical developments. In the concluding section, a general discussion is given.

## II. PARTICLE DYNAMICS IN AN INHOMOGENEOUS PLASMA

In describing the dynamics of the electrons in a plasma subject to some externally applied electric field, we find it advantageous to separate the fields acting upon an electron into the "rapidly varying" part and the "slowly varying" part. The "rapidly varying" part is mainly initiated by the externally applied electric field [designated as $\vec{E}_{ex}(x, t)$] and is substantially enhanced by the collective response of the surrounding particles, which act as sources for collective macroscopic fields [designated as $\vec{E}_{mac}(x, t)$] with frequencies close to that of $\vec{E}_{ex}$ and wavelengths larger than or on the order of the Debye length. This part may also include some rapid fluctuations with frequencies on the order of the local plasma frequency and with wavelengths on the order of the local Debye wavelength due to the transition of other electrons in its Debye sphere. These fluctuations will be designated by $\vec{E}_{mic}^{(1)}(x(t))$. The "slowly varying" part is initiated by the random motions of the surrounding electrons. This part consists of the field associated with the time-independent part of the macroscopic field (designated as $\partial\phi/\partial\vec{x}$) and the fields caused by the microscopic fluctuations due to the electrons outside the Debye sphere but inside the "mean-free-path sphere" [designated as $\vec{E}_{mic}^{(2)}(x(t))$].

We assume that an electronic plasma with smeared-out ion background is subject to the external field associated with a radiation

$$\vec{E}_{ex} = \vec{E}_0 \exp[-i(\omega t + \vec{k}\cdot\vec{x})] + \text{Complex Conjugate.} \tag{1}$$

The different length and time scales enter the problem in a *nontrivial* manner. It is worthwhile to enumerate these at the beginning. The different time scales and their associated length scales are assumed to be the following.

(1) The rapidly varying part (the fast part):

$t_f \sim 2\pi/\omega \sim 2\pi/\omega_{pe}$, one period of the fast oscillation ($\omega_{pe}$ is the average electron plasma frequency);

$\lambda_f^{(1)} \sim e|\vec{E}_{mac} + \vec{E}_0|/m_e\omega^2$, the electron excursion[3] in the *total* macroscopic field;

$\lambda_f^{(2)} \sim 2\pi v_{th}/\omega$, the distance travelled by an average electron in $t_f$ ($v_{th} = (\kappa T_e/m_e)^{\frac{1}{2}}$ is the thermal speed of an electron, $T_e$ is the electron temperature);

$\lambda_f^{(3)} \sim 2\pi/k$, the free space wavelength for the external applied field; and

$\lambda_f^{(4)} \sim eE_{mic}^{(1)}/\omega_{pe}^2 m_e \sim (e\langle\phi_{in}\rangle/\kappa T)(v_{th}^2/\omega_{pe}^2\lambda_{De}) \sim \epsilon\lambda_{De}$, the electron excursion due to surrounding electrons in its Debye sphere ($\langle\phi_{in}\rangle$ is the average particle potential; $\lambda_{De}$ is the average electron Debye length; and $\epsilon$ is the plasma parameter $\epsilon \sim \langle\phi_{in}\rangle/\kappa T \sim 1/n_e\lambda_{De}^3$, where $n_e$ is the average electron number density; $\epsilon \ll 1$).

(2) The slowly varying part (the slow part):

$t_s^{(3)} \sim 1/\epsilon \ln\epsilon\omega_{pe} \gg t_f$, the relaxation time for electrons;

$\lambda_s^{(3)} \sim v_{th}/t_s^{(3)} \gg \lambda_{De}$, the electron mean free path for collisions with other electrons;

$\lambda_s^{(2)} \sim (\partial/\partial\vec{x})\ln|\phi(\vec{x})|$, the characteristic length for the macroscopic static potential $\phi(\vec{x})$, which produces the time-independent part of the macroscopic field;

$t_s^{(2)} \sim \lambda_s^{(2)}/v_{th}$, the transit time for a typical electron to pass through the macroscopic potential;

$t_s^{(1)} \sim 1/\nu_e$, the mean collision time for electrons with neutral particles ($\nu_e$ is the collision frequency); and

$\lambda_s^{(1)} \sim v_{th}/\nu_e$, the mean free path for electron collisions with neutrals.

We assume $t_s^{(3)}, t_s^{(2)}, t_s^{(1)} \gg t_f$, and $\lambda_f^{(3)} \gg \lambda_s^{(2)}, \lambda_s^{(1)}, \lambda_s^{(3)} \gg \lambda_{De} (\sim \lambda_f^{(2)}) \gg \lambda_f^{(1)}, \lambda_f^{(4)}$.

The dynamics of the charged particles inside the plasma are described classically by the set of $N$ Newton's equations ($N$ is the total number of electrons):

$$m_e d\vec{v}_i(t)/dt = -e[\vec{E}_{mac}(\vec{x}_i, t) + \vec{E}_{ex}(t) + \vec{E}_{mic}^{(1)}(\vec{x}_i(t))] + e(\partial/\partial\vec{x}_i)\phi(\vec{x}_i) - e\vec{E}_{mic}^{(2)}(\vec{x}_i(t)), \quad d\vec{x}_i(t)/dt = \vec{v}_i(t). \tag{2}$$

Here $\phi(\vec{x})$ is the macroscopic time-independent potential; $\vec{E}_{mic}^{(1)}(\vec{x}_i(t))$ is the microscopic interparticle force among electrons with characteristic length on the order of $\lambda_{De}$; $\vec{E}_{mic}^{(2)}(\vec{x}_i(t))$ is the microscopic interparticle force among electrons with characteristic length on the order of $\lambda_s^{(3)}$; and $\vec{E}_{mac}(\vec{x}_i, t)$ is the macroscopic collective field with frequency close to $\omega_{pe}$ and wavelength larger than $\lambda_{De}$. Since the fast time scale is at least one or two orders of magnitude faster than the slow time scale, we can make use of the stroboscopic method[1] in nonlinear analysis. We perform an averaging process over one period of the "fast" (macroscopic) fluctuation and obtain an equation for the time evolution of $\vec{v}_i$ and $\vec{x}_i$ on the "slow" time scale only:

$$m_e d\vec{v}_i^{(s)}/dt = \bar{F}(\vec{x}_i^{(s)}) + e(\partial/\partial\vec{x}_i^{(s)})\phi(\vec{x}_i^{(s)}) - e\vec{E}_{mic}^{(2)}(\vec{x}_i^{(s)}), \quad d\vec{x}_i^{(s)}/dt = \vec{v}_i^{(s)}, \tag{3}$$

where $$\bar{F}(\vec{x}_i^{(s)}) = -\frac{e\omega}{2\pi}\int_0^{2\pi/\omega} dt\, \vec{x}_i^{(f)}(\partial/\partial\vec{x}_i^{(s)})\vec{E}_{mac}(\vec{x}_i^{(s)}, t). \tag{4}$$

The superscripts $f$ and $s$ designate the fast and the slow parts, respectively. In getting the above equation, we have used a Taylor series expansion about the velocity minimum in the spatial dependence, and have assumed the electron excursions $\lambda_f^{(1)}$ and $\lambda_f^{(4)}$ to be the smallest length scale of all. Taking the difference of Eqs. (2) and (3) gives a differential equation for the fast fluctuation and it is as follows:

$$m_e d\vec{v}_i^{(f)}/dt = -e[\vec{E}_{mac}(\vec{x}_i^{(s)}, t) + \vec{E}_{ex}(t)] - e\vec{E}_{mic}^{(1)}(\vec{x}_i^{(s)}(t)) - e\vec{x}_i^{(f)}(\partial/\partial\vec{x}_i^{(s)})\vec{E}_{mac} - \tilde{F}(\vec{x}_i^{(s)})$$
$$- e\vec{x}_i^{(f)}(\partial/\partial\vec{x}_i^{(s)})(\partial\vec{x}_i^{(s)})\phi - e\vec{x}_i^{(f)}(\partial/\partial\vec{x}_i^{(s)})\vec{E}_{mic}^{(1)}(\vec{x}_i^{(s)}), \quad d\vec{x}_i^{(f)}/dt = \vec{v}_i^{(f)}. \tag{5}$$

We have retained only the first-order terms in the Taylor expansion, and we can make the following estimation for each term in Eq. (5) (in the light of the various temporal and spatial scales stated at the beginning of this section).

$$|\vec{v}_i^{(f)}| \sim \omega\lambda_f^{(1)} \sim \omega\lambda_f^{(4)} \gg (\lambda_f^{(1)}/\lambda_f^{(2)})(\omega\lambda_f^{(1)})$$
$$\sim (e\phi/\kappa T)(\lambda_{De}/\lambda_s^{(2)})(\omega\lambda_f^{(1)}) \sim (\lambda_{De}/\lambda_s^{(3)})(\omega\lambda_f^{(1)}). \tag{6}$$

By the ordering we have chosen for the various parameters, all the terms in (6) are negligible relative to the first three terms. We thus arrive at an *approximate* equation for the fast variation,and it is as follows:

$$m_e d\vec{v}_i^{(f)}/dt = -e(\vec{E}_{mac} + \vec{E}_{ex} + \vec{E}_{mic}^{(1)}), \quad d\vec{x}_i^{(f)}/dt = \vec{v}_i^{(f)}. \tag{7}$$

Substituting Eq. (7) into Eq. (4), we may perform the integration and write the force $\tilde{F}$ due to the yet *undetermined macroscopic resonance field* $\vec{E}_{mac}$

$$\tilde{F}(\vec{x}_i^{(s)}) = e(\partial/\partial\vec{x}_i^{(s)})\hat{\alpha}\{\vec{E}_{mac}(t')\cdot[\vec{E}_{mac}(t'') + \vec{E}_{ex}(t'')]\} \tag{8}$$

where $\hat{\alpha} = (e\omega/4\pi m_e)\int_0^{2\pi/\omega} dt \int_0^t dt' \int_0^{t'} dt''$.

In obtaining the above result, we have made the assumption that $\vec{E}_{mac}$ is an electrostatic mode, namely

$$(\partial/\partial\vec{x}_i^{(s)}) \times \vec{E}_{mac}(\vec{x}_i^{(s)}, t) = 0.$$

If we further assume the macroscopic resonant field to be dominant inside a localized region in the bounded plasma, we can neglect $|\vec{E}_{ex}|$ compared with $|\vec{E}_{mac}|$ and rewrite Eq. (8) as

$$\tilde{F}(\vec{x}_i^{(s)}) = -(\partial/\partial\vec{x}_i^{(s)})[(e^2/4\omega^2 m_e)|\vec{E}_{mac}(\vec{x}_i^{(s)})|^2]. \tag{9}$$

This is in agreement with previous results.[4] It is immediately realized that the average force obtained above can be combined with the macroscopic potential $\phi(\vec{x}_i^{(s)})$ to form an "effective macroscopic potential." We thus define

$$\phi_{eff}(\vec{x}) \equiv \phi(\vec{x}) + \hat{\alpha}\{[\vec{E}_{mac}(\vec{x}_i(t'')) + \vec{E}_{ex}(t'')]\cdot\vec{E}_{mac}(\vec{x}_i t')\} = \phi(\vec{x}) - (e/4\omega^2 m_e)|\vec{E}_{mac}(\vec{x})|^2$$
$$(\text{for } |\vec{E}_{mac}| > |\vec{E}_{ex}|). \tag{10}$$

A few words about the physical significance of this result should be offered. In a bounded plasma resonance, the electrostatic modes excited are, in general, some *local modes*,[5] and the particle dynamics should be strongly influenced by the existence of these modes as indicated by Eq. (8) or (9). On the other hand, in the case of the interaction between the plasma and some propagating modes, the macroscopic electric field inside the plasma is on the same order as the externally applied electric field, and thus the average force given by Eq. (8) or (9) should not be important. This problem has been treated in many recent papers.[6–8]

Combining Eqs. (7) and (3), we can write the set of Newton's equations in the following *approximate* form:

$$m_e d\vec{v}_i/dt = -e(\vec{E}_{mac} + \vec{E}_{ex}) + e\frac{\partial}{\partial\vec{x}}\phi_{eff} - e\vec{E}_{mic}', \quad d\vec{x}_i/dt = \vec{v}_i \quad (\text{where } \vec{E}_{mic} = \vec{E}_{mic}^{(1)} + \vec{E}_{mic}^{(2)}). \tag{11}$$

These are single-electron orbit equations, which can be used as a basis for deriving statistical description.

## III. GOVERNING EQUATIONS FOR RESONANCE IN A BOUNDED PLASMA

In order to get a complete description of the field and the plasma, we shall proceed in the Klimontovich formulation.[2] For the particle dynamics, Klimontovich's equation is derived from the set of Newton's equations given in Eq. (11); for the field quantity, Poisson's equation is used.

$$\partial N_e/\partial t+\vec{v}\cdot\partial N_e/\partial\vec{x}+[-(e/m_e)(\vec{E}_{mac}+\vec{E}_{ex})+(e/m_e)(\partial/\partial\vec{x})\phi_{eff}]\cdot\partial N_e/\partial\vec{v}=(e/m_e)\vec{E}_{mic}\cdot\partial N_e/\partial\vec{v}. \tag{12}$$

$$(\partial/\partial\vec{x})\cdot(\vec{E}_{mac}-\partial\phi_{eff}/\partial\vec{x}+\vec{E}_{mic})=-4\pi e\int N_e d\vec{v}. \tag{13}$$

Here $N_e$ is recognized as the particle probability density (normalized to the system volume) for the electrons and is defined as

$$N_e=(1/n_e)\sum_{i=1}^{N}\delta(\vec{x}-\vec{x}_i(t))\delta(\vec{v}-\vec{v}_i(t)),$$

where $n_e$ is the average electron spatial density and $N$ the total number of electrons. It should be noted that the Klimontovich equation given in Eq. (12) is a *complete microscopic* description and is totally equivalent to the set of Newton's equations given in Eq. (11).

The lack of detailed information about the microstates at any moment *compels* us to introduce a statistical description. This is conveniently done by taking the ensemble average of Eqs. (12) and (13) with respect to the assumed initial *N-particle* distribution function $f_N(\vec{x}_i(t=0),\ \vec{v}_i(t=0))$. The ensemble average of $N_e$ is recognized as the one-particle distribution function $f$, while the ensemble average of $\vec{E}_{mic}$ is found to be zero. Therefore, the average equations are found to be

$$\partial f/\partial t+\vec{v}\cdot\partial f/\partial\vec{x}+[-(e/m_e)(\vec{E}_{mac}+\vec{E}_{ex})+(e/m_e)\partial\phi_{eff}/\partial\vec{x}]\cdot\partial f/\partial\vec{v}=(\partial/\partial\vec{v})\cdot\langle(e/m_e)\vec{E}_{mic}\delta N_e\rangle \tag{14}$$

$$(\partial/\partial\vec{x})\cdot(\vec{E}_{mac}-\partial\phi_{eff}/\partial\vec{x})=-4\pi e\int f d\vec{v}, \tag{15}$$

where the angular brackets $\langle\ \rangle$ indicate the ensemble average

$$f\equiv\langle N_e\rangle=\int\prod_{i=1}^{N}d\vec{x}_i d\vec{v}_i f_N N_e,\quad \delta N_e\equiv N_e-f.$$

The fluctuating part satisfies the following equations in the plasma limit $(\epsilon=1/n_e\lambda_{De}{}^3\ll 1)$:

$$\partial\delta N_e/\partial t+\vec{v}\cdot\partial N_e/\partial\vec{x}+[-(e/m_e)\vec{E}_{mac}+\vec{E}_{ex})+(e/m_e)\partial\phi_{eff}/\partial\vec{x}]\cdot\partial\delta N_e/\partial\vec{v}=(e/m_e)\vec{E}_{mic}\cdot\partial f/\partial\vec{v} \tag{16}$$

$$(\partial/\partial\vec{x})\cdot\vec{E}_{mic}=-4\pi e\int\delta N_e d\vec{v}. \tag{17}$$

The right-hand side of Eq. (14) can be evaluated by using Eqs. (16) and (17).[9]

Two clearly distinct cases can be discussed. *First*, if the external field is so dominantly large that one can neglect the right-hand sides of Eqs. (14) and (16), one can then absorb the information of Eq. (16) into Eq. (14). One can therefore consider Eq. (14) without the right-hand side as the *basic equation.* This might be considered as a justification for the use of the Vlasov equation in such studies as those by Aliev and Silin[6] and Jackson.[8] However, in this section we shall dwell upon only the *second* case, namely, when the average *electron excursion* $\lambda_f{}^{(1)}$ is small compared with Debye length. This case was studied *partially* by DuBois and Goldman[7] and can also be partially covered by the work of Jackson. The word "partially" was used, because the above-mentioned works have considered only the case $|\vec{E}_{ex}|>|\vec{E}_{mac}|$. What we shall treat here is the case where some *localized electrostatic mode* is excited and the plasma becomes *nontransparent* $(|\vec{E}_{mac}|>|\vec{E}_{ex}|)$. In this case, the effective macroscopic potential $\phi_{eff}$ in Eqs. (14) and (16) is given by Eq. (9). By the assumption that the electron excursion is smaller than the Debye length, we can neglect the third term on the left-hand side of Eq. (16) and thus calculate the right-hand side of Eq. (14).[10] The result is a kinetic equation with the Balescu-Lenard collision term:

$$\partial f/\partial t+\vec{v}\cdot\partial f/\partial\vec{x}+[-(e/m_e)(\vec{E}_{mac}+\vec{E}_{ex})+(e/m_e)\partial\phi_{eff}/\partial\vec{x}]\cdot\partial f/\partial\vec{v}=(\partial/\partial\vec{v})\cdot[\vec{A}(\vec{v})f+\vec{B}(\vec{v})\cdot\partial f/\partial\vec{v}]. \tag{18}$$

Equations (15) and (18) thus constitute the *basic equations*.

The first feature of Eq. (18) is that the zeroth-order distribution function $f^{(0)}$ must be locally a Maxwell distribution which satisfies the following equation

$$\vec{v}\cdot\partial f^{(0)}/\partial\vec{x}+(e/m_e)(\partial\phi_{\text{eff}}/\partial\vec{x})\cdot(\partial f^{(0)}/\partial\vec{v})=(\partial/\partial\vec{v})\cdot[\vec{A}(\vec{v})f^{(0)}+\vec{\vec{B}}(\vec{v})\cdot\partial f^{(0)}/\partial\vec{v}]. \tag{19}$$

Changing the independent variables to an energy variable, we can rewrite Eq. (19) in the following form

$$\{\vec{v}\cdot[\partial\mathcal{E}_e(\vec{x}\cdot\vec{v})/\partial\vec{x}]+(e/m_e)(\partial\phi_{\text{eff}}/\partial\vec{x})\cdot[\partial\mathcal{E}_e(\vec{x}\cdot\vec{v})/\partial\vec{v}]\}\partial f^{(0)}/\partial\epsilon_e=(\partial/\partial\vec{v})\cdot[\vec{A}(\vec{v})f^{(0)}+\vec{\vec{B}}(\vec{v})\cdot\partial f^{(0)}/\partial\vec{v}], \tag{20}$$

where $\mathcal{E}_e(\vec{x},\vec{v})=\frac{1}{2}m_e|\vec{v}|^2-e\phi_{\text{eff}}$.

The left-hand side of Eq. (20) vanishes and shows that $f^{(0)}$ is a function of the defined *energy variable only*, while the right-hand side of this equation vanishes for the following general condition

$$f^{(0)}=\exp[-\tfrac{1}{2}C_1(\vec{x})|\vec{v}|^2+C_2(\vec{x})\cdot\vec{v}+C_3(\vec{x})],\quad\text{where}\quad C_1(\vec{x})>0.$$

Comparing both sides of Eq. (20), we obtain as the most general zeroth-order one-particle distribution function

$$f^{(0)}(\vec{x},\vec{v})=\text{const}\times\exp\{[-\tfrac{1}{2}m_e|\vec{v}|^2+e\phi_{\text{eff}}(\vec{x})]/\kappa T_e\}, \tag{21}$$

where $T_e$ is the electron temperature and may be assumed to be constant.

In order to find the high-frequency behavior of Eqs. (15) and (18), we shall perturb around $f^{(0)}$ and find the first-order equations to be

$$\partial f^{(1)}/\partial t+\vec{v}\cdot\partial f^{(1)}/\partial\vec{x}+(e/m_e)(\partial\phi_{\text{eff}}/\partial\vec{x})\cdot\partial f^{(1)}/\partial\vec{v}-(e/m_e)(\vec{E}_{\text{mac}}+\vec{E}_{\text{ex}})\cdot\partial f^{(0)}/\partial\vec{v}=-\nu(\vec{v})f^{(1)} \tag{22}$$

$$(\partial/\partial\vec{x})\cdot\vec{E}_{\text{mac}}=-4\pi e\int f^{(1)}d\vec{v}, \tag{23}$$

where $\nu(\vec{v})$ is introduced as a *phenomenological* collision frequency to take care of collisions with particles of other species. Notice that the ordering is not merely according to magnitude, but may be regarded as a *frequency ordering* as well. In other words, the zeroth order corresponds to the zero-frequency solution of Eqs. (15) and (18), while the first order corresponds to nonzero-frequency part of the same equations.

Now let us consider a simple physical problem where we have a slab of plasma subject to an external applied electric field $\vec{E}_{\text{ex}}$. Some localized electrostatic modes are excited by $\vec{E}_{\text{ex}}$, and we shall be interested in the effect of increasing $|\vec{E}_{\text{ex}}|$ on the behavior of the modes. Mathematically, the localized modes should be represented by some temporal and spatial partial differential equation. We can derive such an equation from Eqs. (22) and (23). Taking the time derivative of Eq. (23) and substituting Eq. (22) into it, we obtain

$$(\partial/\partial\vec{x})\cdot(\partial\vec{E}_{\text{mac}}/\partial t-4\pi e\int\vec{v}f^{(1)}d\vec{v})=4\pi e\int f^{(1)}\nu(\vec{v})d\vec{v}. \tag{24}$$

Note that if the number of electrons is assumed to be conserved, we have to let the right-hand side of Eq. (24) vanish. Taking the first moment of Eq. (22), we obtain

$$(\partial/\partial t)(\int e\vec{v}f^{(1)}d\vec{v})+(\partial/\partial\vec{x})\cdot(\int e\vec{v}\vec{v}f^{(1)}d\vec{v})+(e^2/m_e)(\partial\phi_{\text{eff}}/\partial\vec{x})\cdot\int(\partial f^{(1)}/\partial\vec{v})\vec{v}d\vec{v}-(e^2/m_e)(\vec{E}_{\text{mac}}+\vec{E}_{\text{ex}})\cdot$$
$$\int(\partial f^{(0)}/\partial\vec{v})\vec{v}d\vec{v}=-\int e\nu(\vec{v})\vec{v}f^{(1)}d\vec{v}. \tag{25}$$

These equations can be simplified by rewriting them in terms of the following defined fluid quantities and making the following approximations:

$$n^{(0)}\equiv\int f^{(0)}d\vec{v},\quad n^{(1)}\equiv\int f^{(1)}d\vec{v},\quad\vec{j}\equiv\int e\vec{v}f^{(1)}d\vec{v},\quad\vec{\vec{P}}\equiv\int m_e\vec{v}\vec{v}f^{(1)}d\vec{v}\simeq\gamma n\kappa T_e\vec{\vec{I}}$$

$$\phi_{\text{eff}}=(\kappa T_e/e)(\partial\ln n^{(0)}/\partial\vec{x})\quad[\text{from Eq. (21) and definition of } n^{(0)}]$$

$$\int(\partial f^{(1)}/\partial\vec{v})\vec{v}d\vec{v}\simeq n\vec{\vec{I}},\quad|\vec{E}_{\text{mac}}|>|\vec{E}_{\text{ex}}|. \tag{26}$$

Substituting Eqs. (26) and (23) into Eq. (25), we obtain

$$(\partial/\partial t+\langle\nu\rangle)\vec{j}+(\gamma\kappa T_e/4\pi m_e)(\partial/\partial\vec{x})(\partial/\partial\vec{x})\cdot\vec{E}_{mac}-(\kappa T_e/4\pi m_e)(\partial\ln n^{(0)}/\partial\vec{x})(\partial/\partial\vec{x})\cdot\vec{E}_{mac}$$
$$-(n^{(0)}e^2/m_e)\vec{E}_{mac}=0, \quad (27)$$

where $\langle\nu\rangle\vec{j}=\int e\nu(\vec{v})\vec{v}f^{(1)}d\vec{v}$.

For mathematical convenience, let us assume that the localized region for the excited mode is much smaller than the wavelength of the exciting field. Thus, we can write Eq. (24) in the following form

$$\partial\vec{E}_{mac}/\partial t+4\pi\vec{j}=\partial\vec{E}_{ex}/\partial t. \quad (28)$$

Combining Eqs. (27) and (28), we get the wave equation for the localized modes:

$$[\partial^2/\partial t^2+\langle\nu\rangle\partial/\partial t-(\gamma\kappa T_e/m_e)(\partial/\partial\vec{x})(\partial/\partial\vec{x})+(\kappa T_e/m_e)(\partial\ln n^{(0)}/\partial\vec{x})(\partial/\partial\vec{x})+(4\pi n^{(0)}e^2/m_e)]\vec{E}_{mac}$$
$$=(\partial/\partial t)(\partial/\partial t+\langle\nu\rangle)\vec{E}_{ex}. \quad (29)$$

The above equation is nonlinear, because $n^{(0)}$ depends upon the time average value of $E_{mac}$ through $\phi_{eff}$ as given by Eq. (10). Substituting Eq. (21) into the first expression of Eq. (26), we get a normalizing condition for $n^{(0)}$

$$n^{(0)}(\vec{x}, T_e, \Lambda)\int d\vec{x}\exp\{[e\phi(x)/\kappa T_e]-\tfrac{1}{4}\Lambda(|\vec{E}_{mac}|^2/|\vec{E}_0|^2)\}=N, \quad (30)$$

where the absolute-value symbol designates the maximum amplitude of the enclosed quantity, and

$$\Lambda=e^2|\vec{E}_0|^2/\omega^2 m_e\kappa T_e$$

$$n_0(\vec{x}, T_e)\equiv n^{(0)}(\vec{x}, T_e, \Lambda=0)=N\exp[e\phi(\vec{x})/\kappa T_e]/\int d\vec{x}\exp[e\phi(\vec{x})/\kappa T_e].$$

$\Lambda$ is the same parameter as defined by Goldman[7] and is a measure of the ratio of the order energy to the thermal energy, while $n_0(\vec{x}, T)$ is the density profile without the localized resonance. If we further assume $\Lambda$ to be a small parameter (which is a consequence of the smallness assumption of $\lambda f^{(1)}$, we can write approximately

$$n^{(0)}(\vec{x}, T, \Lambda)=n_0(\vec{x}, T)\{1+\tfrac{1}{4}\alpha\Lambda[|\vec{E}_{mac}(\vec{x})|^2/|\vec{E}_0|^2]\} \quad (31)$$

where $\alpha|\vec{E}_{mac}|^2/|\vec{E}_0|^2=\int(|\vec{E}_{mac}|^2/|\vec{E}_0|^2)\exp[e\phi(\vec{x})/T_e]d\vec{x}/\int\exp[e\phi(\vec{x})/T_e]d\vec{x}$.

In explaining the meaning of $\alpha$, let us note that $|\vec{E}_{mac}|$ is assumed to be smaller than or of the order as $|\vec{E}_0|$ except in the small localized region where the resonance occurs. This region contributes dominantly in the numerator of the expression for $\alpha$, and therefore we can roughly state that $\alpha$ is a *measure* of the size of the local region and is given as the ratio of the volume of the localized resonance region to the total volume of the plasma interacting with the radiation.

In the next section, we shall study the nonlinearity of Eq. (29) using the *multiple-time-scale* method,[11] which is essentially a small-parameter perturbation analysis.[12] But first, we want to transform Eq. (29) into a more convenient form. When the external applied field $|\vec{E}_{ex}|$ is sufficiently small, we can replace the spatial part of the operator in Eq. (29) by its *eigenvalue* $\omega_0^2$, which should be of the order of $\omega_{pe}^2$. Therefore, the *approximate time-evolution equation* for the localized mode yields the following equation:

$$(\partial^2/\partial t^2+\langle\nu\rangle\partial/\partial t+\omega_0^2)\vec{E}_{mac}=(\partial/\partial t)(\partial/\partial t+\langle\nu\rangle)\vec{E}_{ex}-\alpha\Lambda\omega_{pe}^2(|\vec{E}_{mac}|/|\vec{E}_0|)^2\vec{E}_{mac}, \quad (32)$$

where $\omega_{pe}^2=4\pi n_0 e^2/m_e$.

Keeping in mind the assumed external field in Eq. (32) and shifting the time scale to $t+\tan^{-1}(\langle\nu\rangle/\omega)$ we can write Eq. (32) into the following normalized dimensionless form

$$(\partial^2/\partial\tau^2+\delta\partial/\partial\tau+\Omega^2)\mathcal{E}=-\cos\tau-\epsilon|\mathcal{E}|^2\mathcal{E}, \quad (33)$$

where the absolute-value sign designates the maximum amplitude of $\epsilon$ and

$$\tau=\omega t,\quad \delta=\langle\nu\rangle/\omega,\quad \mathcal{E}=E_{\mathrm{mac}}/2E_0,\quad \epsilon=\alpha\Lambda\omega_{pe}{}^2/w^2,\quad \Omega^2=w_0{}^2/w^2.$$

This is the equation we shall study in the next section. A few remarks should be made at this point. First, it should be noted that Eq. (33) is an *approximate equation* designed for the *localized region* where the electrostatic mode is excited; if any detailed information is needed we should always refer back to Eqs. (22) and (23). For this localized region, the spatial dependence determines the value of $\omega_0$, which is assumed to have only a weak dependence on external power and is mainly determined by average plasma characteristics (e.g., $\omega_{pe}T$). Secondly, Eq. (33) has the form of Duffing's equation[14] and would be identical with it if there were no absolute-value sign. Because of this slight difference, we shall devote a whole section to solving the equation.

## IV. THE FREQUENCY SHIFT AND THE AMPLITUDE JUMP FOR A LOCALIZED ELECTROSTATIC MODE

Equation (33) has two linearly superposed solutions: the homogeneous solution and the particular solution. If the magnitude of $\Omega$ differs much from 1, the homogeneous part will be damped because of $\langle\nu\rangle$. The only solution that has physical meaning is the particular solution which has the frequency of the forced term, $-\omega$. But when the magnitude of $\Omega$ is close to 1, the two solutions will mingle together and give us such interesting effects as the frequency shift and the amplitude jumps.

The mathematical method we choose here is the so-called "multiple-time-scale" method. That is, we consider the actual time to be imbedded in a multiple of fictitious time scales in order to avoid secular behavior in a perturbation analysis. Let

$$\mathcal{E}=\mathcal{E}_0(\tau,\epsilon\tau,\dots)+\epsilon\,\mathcal{E}_1(\tau,\epsilon\tau,\dots) \tag{34}$$

in Eq. (33) and obtain in zeroth order

$$\left(\frac{\partial^2}{\partial\tau^2}+\delta\frac{\partial}{\partial\tau}+\Omega^2\right)\mathcal{E}_0=-\cos\tau, \tag{35}$$

and in the first order

$$\left(\frac{\partial^2}{\partial\tau^2}+\delta\frac{\partial}{\partial\tau}+\Omega^2\right)\mathcal{E}_1$$

$$=-2\frac{\partial^2\mathcal{E}_0}{\partial\tau\partial\epsilon\tau}-\delta\frac{\partial\mathcal{E}_0}{\partial\epsilon\tau}-\epsilon|\mathcal{E}_0|^2\mathcal{E}_0. \tag{36}$$

Since we are looking at the region where $\Omega$ is close to 1, we are free to choose

$$\mathcal{E}_0(\tau,\epsilon\tau)=\mathcal{A}_0(\epsilon\tau)e^{i\tau}+\text{complex conjugate}. \tag{37}$$

Substituting this into Eq. (36) and removing the secular terms, we obtain the following equations for $\mathcal{A}_0$ in the time scale $\epsilon\tau$:

$$2i(\partial\mathcal{A}_0/\partial\epsilon\tau)+\delta(\partial\mathcal{A}_0/\partial\epsilon\tau)+4|\mathcal{A}_0|^2\mathcal{A}_0=0,$$

$$2i(\partial\mathcal{A}_0{}^*/\partial\epsilon\tau)-\delta(\partial\mathcal{A}_0{}^*/\partial\epsilon\tau)-4|\mathcal{A}_0|^2\mathcal{A}_0{}^*=0. \tag{38}$$

The above equations imply $\partial|\mathcal{A}_0|^2/\partial\epsilon\tau=0$, which allows only the following form for $\mathcal{A}_0$

$$\mathcal{A}_0=Ae^{i\theta(\epsilon\tau)}. \tag{39}$$

Substituting Eq. (39) into Eq. (38), we obtain

$$\theta(\epsilon\tau)=-[4|A|^2/(2-i\delta)]\epsilon\tau. \tag{40}$$

Referring to Eq. (37), we can write the *most general form* for $\mathcal{E}_0(\tau,\epsilon\tau)$ without causing the occurrence of secular terms in the perturbation analysis, namely

$$\mathcal{E}_0(\tau,\epsilon\tau)=A\exp\{i[1-\epsilon4|A|^2/(2-i\delta)]\tau\}+\text{c.c.} \tag{41}$$

Note that the above equation indicates both a frequency shift and a damping due to the nonlinear term in Eq. (33). The amplitude $A$ has to satisfy Eq. (35). After substitution and simplification, we obtain

$$A\left[\Omega^2-\left(1-\epsilon\frac{4|A|^2}{2-i\delta}\right)^2\right]=\tfrac{1}{2}\left(\cos\epsilon\frac{4|A|^2}{2-i\delta}\tau\right),$$

$$i\delta A\left(1-\epsilon\frac{4|A|^2}{2-i\delta}\right)=-\tfrac{1}{2}\left(\sin\epsilon\frac{4|A|^2}{2-i\delta}\tau\right). \tag{42}$$

The above equations determine the amplitude $A$. From this point on we shall assume $\delta\ll1$ (with the usual laboratory plasma in mind). In this case, we get from Eq. (42) the following algebraic relation

$$4A^2\{[\Omega^2-(1-2\epsilon|A|^2)^2]^2+\delta^2(1-2\epsilon|A|^2)^2\}=1. \tag{43}$$

The above equation can be rewritten in terms of the physical quantities by referring to Eq. (33).

$$E_{\mathrm{mac}}{}^2\{[\omega_0{}^2-(1-\epsilon E_{\mathrm{mac}}{}^2/8E_0{}^2)^2\omega^2]^2$$

$$+\nu^2\omega^2(1-\epsilon E_{\mathrm{mac}}{}^2/8E_0{}^2)^2\}=4\omega^4E_0{}^2. \tag{44}$$

We can simplify this relation further by taking the ordering $\epsilon E_{\mathrm{mac}}{}^2/E_0{}^2\sim\nu\ll\omega$, and rewriting Eq. (44) in terms of $\Delta\omega\equiv\omega-\omega_0$ ($|\Delta\omega|\ll|\omega|$),

$$[\Delta\omega-\tfrac{1}{8}\epsilon\omega(E_{\mathrm{mac}}/E_0)^2]^2+(\tfrac{1}{2}\nu)^2=$$

$$=\omega^2(E_{\mathrm{mac}}/E_0)^2. \tag{45}$$

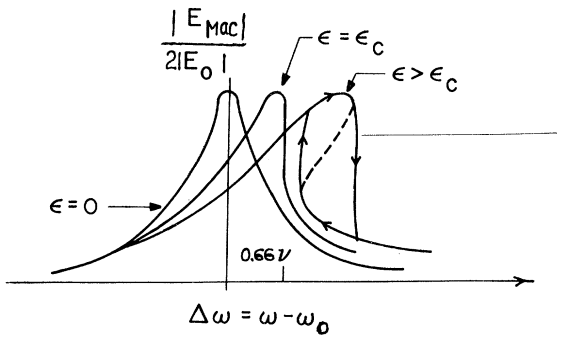


FIG. 1. Schematic diagram shows the results of Eq. (45) for various $\epsilon$'s. Note that $\Delta\omega|_{\text{shift}}=0.66\,\nu$ for $\epsilon=\epsilon_C=1.1\nu^3/\omega^3$. The dotted line indicates the unstable solution, and the jumps occur in the arrowed direction.

Now we are at a stage to interpret the *physical meaning* of Eq. (44) or (45). We shall itemize the results and show them in the sketched diagram.

(1) $E_{\text{mac}}$ will reach its maximum value ($E_{\text{max}}=2\omega E_0/\nu$) not at $\Delta\omega=0$, but at

$$\Delta\omega|_{\text{shift}}=\tfrac{1}{8}\epsilon\omega(E_{\text{max}}/E_0)^2=\tfrac{1}{2}\epsilon(\omega/\nu)^2\omega$$

$$\sim\alpha\omega(eE/m\nu)^2/V_{\text{th}}{}^2\,. \tag{46}$$

(2) When the *frequency shift* given in Eq. (41) becomes sufficiently large because of the increasing value of $\epsilon$, the forced oscillation and the natural oscillation cannot have the same energy. This will cause the resonance to be diminished and the localized mode will *disappear*. Mathematically, this is shown as an amplitude jump condition for Eq. (45). This is given by $dE_{\text{mac}}/d\Delta\omega\to\infty$ or $d\Delta\omega/dE_{\text{mac}}\to 0$. After differentiating Eq. (45) and simplifying, the condition for a jump is found to be

$$\left[\Delta\omega=\tfrac{1}{8}\epsilon\omega\left(\frac{E_{\text{mac}}}{E_0}\right)^2\right]\left[\Delta\omega-\tfrac{3}{8}\epsilon\omega\left(\frac{E_{\text{mac}}}{E_0}\right)^2\right]$$

$$+\tfrac{1}{4}\nu^2=0. \tag{47}$$

The above equation has two roots, with the difference between two jumps given by

$$\Delta\omega|_{\text{between jumps}}=\left[\left(\frac{\epsilon\omega E_{\text{mac}}{}^2}{4E_0}\right)^2-\nu^2\right]^{\frac{1}{2}}. \tag{48}$$

Equation (48) indicates that there is a critical value of $\epsilon$ (designated by $\epsilon_{\text{cr}}$) where the jump starts. It is given by setting $\Delta\omega|_{\text{between jumps}}=0$, and the corresponding $\Delta\omega$ is given by Eq. (47) as

$$\Delta\omega|_{\text{cr}}=\tfrac{1}{4}\sqrt{7}\,\nu. \tag{49}$$

Substituting these into Eq. (45) and recalculating $\epsilon_{\text{cr}}$, we obtain

$$\epsilon_{\text{cr}}=\tfrac{1}{4}(15-4\sqrt{7})(\nu/\omega)^3. \tag{50}$$

Referring to Eqs. (31), (33), and (50), we obtain a critical $E_0$ for the jump to occur:

$$E_0|_{\text{cr}}=1.37(m_e\omega\kappa T_e\nu^3/\alpha e^2\omega_{pe}{}^2)^{\frac{1}{2}}\,. \tag{51}$$

In the light of Eq. (47), we shall make a distinction among three cases, depending upon the magnitude of $E_0$. For $E_0<E_0|_{\text{cr}}$, no amplitude jump is possible, and there is merely a frequency shift in the maximum, as stated in item (1); for $E_0=E_0|_{\text{cr}}$ the jump just starts; for $E_0>E_0|_{\text{cr}}$ there will be a hysteresis loop in the frequency of the linear mode, $\omega_0$, as the localized electrostatic mode is excited and suppressed.

## V. DISCUSSION

We have considered the interaction between an externally applied electric field and a nontransparent bounded plasma. The parameter of this problem is found to be the same as before: $\Lambda=e^2|E_0|^2/\omega^2 m_e\kappa T_e$. However, our work here should be considered as complementary to that of Refs. (6), (7), and (8) rather than a reformulation, because the previous calculations, in contrast to our work, have all made the assumption that the plasma is transparent to the externally applied electric field. The fact that we did not consider the particle dynamics of the ions prevented us from discussing various parametric excitations. But if the effects we have considered are important, the problem of the parametric excitations has to be re-examined. We have justified a *different set of basic equations*, given by Eqs. (22) and (23), for the nonlinear effect we looked at. This nonlinear effect was treated by Gurevich and Pitaevskii,[4] but our result in Sec. IV shows that the *critical E field* needed for this nonlinear effect to occur is *many orders of magnitude smaller than* the one they obtained. By putting typical laboratory plasma numbers into Eqs. (51), we find that $E_0|_{\text{cr}}$ is in the order of 1 V/cm. Thus, the author and his collaborators performed an experiment to test this effect, and found it to be in excellent qualitative as well as reasonable quantitative agreement.[15] Further work along this line is likely to be beneficial. For example, Eqs. (49) and (51) suggest a new possible diagnostic technique; and the frequency-sensitive effect in the case of $E_0>E_0|_{\text{cr}}$ suggests one way of controlling the localized electrostatic modes.

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# Convergence of the Two-Component Plasma Correlation Function

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It has been suggested that the divergence of the two-component plasma correlation function at small interparticle distances may be removed by taking quantum corrections to the classical result in a certain manner. It is shown here that this approach is not possible in general. A completely quantum-mechanical treatment is given instead, and an explicit convergent expression for the radial distribution function at small $r$ is obtained. Also discussed is the fact that there does not seem to exist any simple interpolation formula that bridges the classical and the quantum-mechanical results for $r$ in the region of thermal de Broglie wavelengths.

## I. INTRODUCTION

It is well known that the classical correlation function of a two-component plasma has a singularity at small interparticle distances. As we shall see below, the radial distribution function consists of two terms, one representing the contribution from the bound states and the other from the continuum states. Classically both terms are divergent at small $r$. While the contribution from the continuum states diverges more strongly at lower energies, the most serious difficulty comes from the bound states.

A suggestion has been made by Lamb[1] that if one takes the quantum-mechanical correction, one can obtain a radial-distribution function which appears to be finite everywhere. The method he used was developed by Goldberger and Adams.[2] It was essentially a generalization of Wigner's method[3] of power-series expansion in $\hbar$. The question of whether this method can be applied to resolve the divergence difficulty in this problem needs more careful consideration. In fact, as is well known,[3] the expansion in a power series in $\hbar$ is valid only when the behavior of the system is nearly correctly given by the classical theory. In the present case of a two-component plasma with Coulomb interaction, this expansion is actually a power series in the spatial derivatives of the interaction potential as well as in $\hbar$, and is clearly inapplicable as the interparticle distance $r$ approaches zero. The result obtained by Lamb may be valid for $r$ much greater than the thermal de Broglie wavelength $\lambda = (\hbar^2/2mkT)^{1/2}$, in which case the classical theory is a good approximation anyway. However, his result cannot be used to discuss the divergence difficulty at $r<\lambda$ where the expansion breaks down.

On the other hand, Trubnikov and Elesin[4] have calculated the radial distribution function quantum-mechanically. However, they neglected the bound states completely and calculated the continuum contribution by making a large $ka_0$ or high-energy expansion (Born approximation). We would like to point out that their results cannot be justified unless the fundamental divergence difficulty arising from the bound states is first resolved.

It is the purpose of this paper to analyze the behavior of the radial distribution function at distances $r<\lambda$ by including contributions from all the bound and continuum states. It will be shown explicitly by a completely quantum-mechanical treatment that there is no divergence as $r \to 0$.

In Sec. II we review briefly the expansion method and point out its inapplicability to the present problem. Section III is devoted to the calculation of the radial distribution function at $r<\lambda$. Discussions of these results are presented in Sec. IV.