Equation of State at Supranuclear Densities and the Existence of a Third Family of Superdense Stars*†

ULRICH H. GERLACH‡§
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
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This paper presents a method for deducing the equation of state of "cold" matter at supranuclear densities from astronomical data. In particular, from the masses and the radii of a sequence of superdense stars composed of degenerate matter, one can determine the equation of state. The relationship between the equation of state and the mass-radius curve is used to construct an equation of state that allows a third family of superdense stars.

INTRODUCTION

REASONS have been advanced for believing that the birth of a neutron star coincides with the occurrence of a supernova explosion.[1] The mass regime within which neutron stars are calculated to be stable is approximately $0.15M_\odot$–$0.7M_\odot$; the exact numbers depend upon an exact knowledge of the equation of state, i.e., the nucleon-nucleon interaction. The central density of the neutron core resulting from a supernova can range—it is calculated—from below nuclear densities, $\sim 2\times 10^{13}$ g/cm^3, to about 20 times nuclear densities, $\sim 6\times 10^{15}$ g/cm^3.[2] At these and higher densities several workers have suggested that hyperons appear.[3]

Thus, because of our ignorance of nuclear interactions at superhigh densities, one cannot exclude the possibility that one or another elementary-particle transformation may strongly influence the compressibility of matter at a certain supranuclear density. In that event the stability of a superdense star at these densities may be affected quite significantly. Significantly enough to alter the properties of the two already predicted families of degenerate stars (white dwarfs and neutron stars)? This is interesting—but not impressive. Significantly enough to give rise to a third family of superdense stars? This effect, though less likely, would be far more dramatic and decisive in what it would tell about the equation of state. Therefore, this paper asks and answers this question: How must the equation of state run to permit a third family of degenerate stars?

In answering this query we are led to formulate and answer a more general question:

Given the mass-radius relation $M=M(R)$ for the entire sequence of stars associated with a given equation of state, $p=p(\rho)$, *find* that equation of state.

The converse problem is well-known and thoroughly studied[4]: Given the equation of state, find the family of equilibrium configurations. Starting with a given value of the central density $\rho=\rho_0$, one integrates the equations of hydrostatic equilibrium[5]

$$\frac{dp^*}{dr}=-\frac{(p^*+\rho^*)(m^*+4\pi r^3 p^*)}{r^2-2m^*r}, \tag{1}$$

$$\frac{dm^*}{dr}=4\pi r^2\rho^* \tag{2}$$

from the center to the point of vanishing pressure. One thus finds the radius $R=R(\rho_0)$ and the mass $M=M(\rho_0)$ and, therefore—repeating the calculation for other values of ρ_0—the desired mass-radius relationship $M=M(R)$. But now reverse the procedure: Given $M=M(R)$, how does one find $p=p(\rho)$?

Consider a "fiducial" configuration with a central density, say $\rho_0=3\times 10^{14}$ g/cm^3, below which the equation of state is known. Now consider a configuration that differs from the "fiducial" configuration by ΔM and ΔR. Since the central density of the "close-by" configuration is

$$\rho_0+\Delta\bar{\rho}(0),$$

this configuration has a central core of radius $r=r_0$ which has the property that all matter contained inside that core is of density larger than ρ_0. The problem now boils down to this: (1) What is the new central density

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‡ Present address: Battelle Memorial Institute, Columbus, Ohio.

§ John B. Putnam, Jr., Memorial Fellow.

[1] See F. Zwicky, Phys. Rev. **55**, 726 (1939); A. G. W. Cameron, Astrophys. J. **129**, 676 (1959); **130**, 884 (1959); H. Y. Chiu, Ann. Phys. (N. Y.) **26**, 364 (1964); in *Quasi-Stellar Sources and Gravitational Collapse*, edited by I. Robinson *et al.* (University of Chicago Press, Chicago, 1965), Chap. 33; S. A. Colgate and R. H. White, Astrophys. J. **143**, 626 (1966); W. D. Arnett, in *High Energy Astrophysics*, edited by C. DeWitt *et al.* (Gordon and Breach, Science Publishers, Inc., New York, 1967), Vol. III, p. 113.

[2] B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitational Theory and Gravitational Collapse* (University of Chicago Press, Chicago, 1964), Chap. 6, p. 46.

[3] V. A. Ambartsumyan and G. S. Saakyan, Astron. Zh. **37**, 193 (1960) [English transl.: Soviet Astron.—AS **4**, 187 (1960)]. See also S. Tsuruta and A. G. W. Cameron, in *High Energy Astrophysics*, edited by C. DeWitt *et al.* (Gordon and Breach, Science Publishers, Inc., New York, 1967), Vol. III, p. 163 for a good bird's eye view of various equations of state.

[4] Reference 2, Chap. 6, and references cited therein.

[5] The units used here are geometrical units expressed in centimeters. $m^*=mG/c^2$ in cm; $\rho^*=\rho G/c^2$ in cm^{-2}; $p^*=pG/c^4$ in cm^{-2}; here m, ρ, and p are the mass, mass-energy density, and pressure in cgs units, respectively. We shall drop the asterisk in the future.

and (2) what is the stiffness,

$$\beta^2 = dp/d\rho\,,$$

in the central core?

The solution to the problem is facilitated by considering the difference between the two configurations $\Delta m(r)$ and $\Delta\rho(r)$. These "changes" are determined by variation equations of the equations of hydrostatics, Eqs. (1) and (2),

$$\frac{d\Delta\rho}{dr} = A(r)\Delta\rho + B\Delta m\,, \tag{3}$$

$$\frac{d\Delta m}{dr} = 4\pi r^2 \Delta\rho\,. \tag{4}$$

Here $A(r)$ and $B(r)$ are well-determined functions of $p(r)$, $\rho(r)$, and $m(r)$. Two steps are necessary:

(1) Relate the changes at the surface of the star $\{\Delta M(R),\ \Delta\rho(R) = -\Delta R d\rho/dr|_{r=R}\}$ to changes at the surface of the central core $r=r_0$, $\{\Delta m(r_0), \Delta\rho(r_0)\}$.

(2) Determine $\Delta m(r)$ and $\Delta\rho(r)$ *inside* the central core $r=r_0$ by matching *smoothly* onto the outside "changes" *that* unique solution which is nonsingular at the origin.

Since the solution inside the central core depends upon the increment in the central density and upon the stiffness $dp/d\rho$ inside $r=r_0$, it is clear that we now have equations that relate the observable changes between two stars to the equation of state at the very center of these two stars.

Let us make the above arguments exact by obtaining the actual equations.

TRANSFER MATRIX

In order to relate the "changes" at the surface of a star (close to the "fiducial" configuration) to those at the surface of the central core $r=r_0$, solve Eqs. (3) and (4). The equations can be written in the form of a matrix equation

$$\frac{d}{dr}\psi(r) = H(r)\psi(r)\,, \tag{5}$$

where $\psi(r)$ is the vector

$$\psi(r) = \begin{pmatrix} \Delta\rho(r) \\ \Delta m(r)/m(r) \end{pmatrix}$$

and $H(r)$ is the matrix

$$H(r) = \begin{pmatrix} A(r) & B(r) \\ 4\pi r^2/m & -4\pi r^2\rho/m \end{pmatrix}.$$

Give the initial value of ψ at $r=r_0$,

$$\psi(r_0)\,.$$

Then the solution of the "Schrödinger" equation (5) at $r=r_0+\Delta r$ is

$$\psi(r+\Delta r) = [1 + H(r_0)\Delta r]\psi(r_0)\,.$$

Consequently, by iteration, the solution for arbitrary r is

$$\psi(r) = \lim_{\Delta r\to 0;\, n\to\infty} \prod_{i=0}^{n} [1 + H(r_i)\Delta r]\psi(r_0)\,.$$

Here the product consisting of an infinite number of factors is a *product integral*,[6,7] which we call the transfer matrix[8]

$$T_{r_0}{}^{r} = \lim_{\Delta r\to 0;\, n\to\infty} \prod_{i=0}^{n} [1 + H(r_i)\Delta r]\,.$$

It relates "changes" at the surface of the star to "changes" outside or at the surface of the central core $r=r_0$,

$$\begin{pmatrix} \Delta\rho(r_0) \\ \Delta m(r_0)/m(r_0) \end{pmatrix} = T_R{}^{r_0} \begin{bmatrix} -\left.\dfrac{d\rho}{dr}\right|_{r=R} & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \Delta R \\ \Delta M/M \end{pmatrix}. \tag{6}$$

Now we shall determine $\Delta m(r)$ and $\Delta\rho(r)$ inside the central core $r=r_0$ by performing the above-mentioned matching process.

MATCHING CONDITIONS

Inside any star the pressure must be continuous, otherwise the star would not be in equilibrium. Furthermore, the mass must be continuous, unless mass shells of infinite density are allowed. Consequently, across the boundary $r=r_0$, the following equalities must hold

$$\Delta p(r_0^+) = \Delta p(r_0^-)\,, \tag{7}$$

$$\Delta m(r_0^+) = \Delta m(r_0^-)\,. \tag{8}$$

The explicit expressions for the solutions inside and outside the core are obtained by means of Taylor series expansions around the origin. We can write the first few terms of the "fiducial" configuration, and obtain the following simplified versions of Eqs. (3) and (4):

$$\frac{d^2\Delta m}{dr^2} - \left(\frac{2}{r} + br\right)\frac{d\Delta m}{dr} + (d + er^2)\Delta m = 0\,, \tag{9}$$

$$\Delta\rho(r) = \frac{1}{4\pi r^2}\frac{d\Delta m}{dr}\,, \tag{10}$$

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[6] F. R. Gantmacher, *Application of the Theory of Matrices* (Interscience Publishers, Inc., New York, 1959), Chap. IV.
[7] G. Rasch, J. Reine u. Angew. Math. **171**, 65 (1934).
[8] L. A. Pipes, J. Franklin Inst. **283**, 357 (1967).

where

$$b=-(d/d\rho)(k/\beta^2)+\tfrac{4}{3}\pi(p_0+\rho_0)/\beta^2,$$
$$d=4\pi(p_0+\rho_0)/\beta^2,$$
$$e=(4\pi/\beta^2)[8\pi(p_0+\rho_0)(p_0+\tfrac{2}{3}\rho_0)+(d\beta^2/d\rho_0)\times(p_0+\rho_0)k/2\beta^4-\tfrac{1}{2}(1+1/\beta^2)k],$$
$$k=4\pi(p_0+\rho_0)(p_0+\tfrac{1}{3}\rho_0),$$
$$\beta^2=dp/d\rho,$$

and the subscript zero refers to the "fiducial" configuration at $r=0$. The general solution to these equations (to second order) is

$$\Delta m(r)=E(1+\tfrac{1}{2}dr^2)+Fr^3\left[1-r^2\frac{3}{10}\frac{d}{d\rho}\left(\frac{k}{\beta^2}\right)\right],\quad (11)$$

$$\Delta\rho(r)=E\frac{d}{4\pi r}+\frac{3F}{4\pi}\left[1-\tfrac{1}{2}r^2\frac{d}{d\rho}\left(\frac{k}{\beta^2}\right)\right],\quad (12)$$

$$\Delta p(r)=E\frac{\beta^2 d}{4\pi r}+\frac{3F\beta^2}{4\pi}\left(1-\tfrac{1}{2}r^2\frac{dk}{dp}\right).\quad (13)$$

The constants are determined by the fact that this solution must coincide at $r=r_0$ with the solution, Eqs. (6). The series expansions Eqs. (11)–(13) are accurate as long as the relative changes $\Delta m(r)/m(r)$, $\Delta\rho(r)/\rho(r)$, and $\Delta p(r)/p(r)$ are much smaller than unity. This is true for $r\geq r_0$ if we let $\Delta M/M$ and $\Delta R/R$ be small enough.

It is clear that in general $E\neq 0$. Thus *inside* the sphere $r=r_0$ the relative changes $\Delta m(r)/m(r)$, etc., are not small compared to unity, and consequently Eqs. (11)–(13) are invalid in that central region.

Our aim is to calculate the actual difference in the central pressures and the central densities of the two "close-by" configurations. Let us denote these quantities by $\Delta\bar{p}(0)$ and $\Delta\bar{\rho}(0)$, respectively. Thus the square of the speed of sound at the density $\rho_0+\Delta\bar{\rho}(0)$ is given by

$$\frac{\Delta\bar{p}(0)}{\Delta\bar{\rho}(0)}\equiv\bar{\beta}^2.$$

Since it is futile to calculate $\Delta\bar{p}(0)$ and $\Delta\bar{\rho}(0)$ from Eq. (6) or Eqs. (11)–(13) directly, let us write down the expansion for the pressure, density, and mass difference which *is* valid inside the sphere $r=r_0$. This expansion is

$$\Delta\bar{m}(r)=\tfrac{4}{3}\pi\Delta\bar{\rho}(0)r^3,\quad (14)$$

$$\Delta\bar{\rho}(r)=\Delta\bar{\rho}(0)\left[1-\tfrac{1}{2}r^2\frac{d}{d\rho}\left(\frac{k}{\beta^2}\right)\right],\quad (15)$$

$$\Delta\bar{p}(r)=\bar{\beta}^2\Delta\bar{\rho}(0)\left[1-\tfrac{1}{2}r^2\frac{dk}{dp}\right].\quad (16)$$

Like $\bar{\beta}^2$, the derivatives in the square brackets are meant to be evaluated at $\rho_0+\Delta\bar{\rho}(0)$. Although these derivatives are not in general the same as those in Eqs. (11)–(13), we will not denote them differently because they will not enter into the final result.[9] The bars over $\Delta\bar{m}(r)$, etc., indicate that these quantities are correct expansions for $r\leq r_0$ [and also for $r>r_0$, provided, of course, that $\Delta\bar{\rho}(r)/\rho(r)$, etc., are small compared to unity]. See Fig. 1.

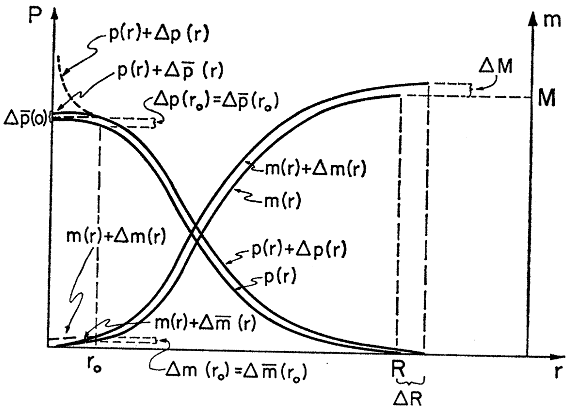

FIG. 1. Two "close-by" equilibrium configurations. The mass coordinate is the right-hand ordinate, the pressure coordinate is the left-hand ordinate. The "fiducial" configuration is $\{m(r),p(r)\}$; its total radius and mass are R and M. The difference between the "close-by" configuration and the "fiducial" configuration is given by $\{\Delta m(r),\Delta p(r)\}$ for $r>r_0$. This difference is the solution to the variation equations $d\Delta\rho/dr=A\Delta\rho+B\Delta m$, and $d\Delta m/dr=C\Delta\rho$ together with the boundary condition $\Delta m(R)=\Delta M$, $\Delta p(R)=-\Delta R dp(R)/dr$. Inside the sphere $r=r_0$ the variation equations are useless; their solution becomes singular at the origin, and the solution is only valid as long as it describes "small changes" $\Delta m(r)$, etc. The reason for the singularity is that in general the square of the velocity of sound, $v^2=dp/d\rho$, at the center of the "close-by" configuration is different from $dp/d\rho$ at $r=r_0$ of this configuration where the pressure is $p(r_0)+\Delta p(r_0)=p(0)$. A singular solution resulting from a "bad" choice of the equation of state inside $r=r_0$ is shown by the dotted curves that represent $\Delta m(r)$, $\Delta p(r)$ for $r<r_0$. A unique "good" choice of the equation of state results in a unique nonsingular solution $\{\Delta\bar{m}(r),\Delta\bar{p}(r)\}$ that can be joined smoothly onto $\{\Delta m(r),\Delta p(r)\}$ at $r=r_0$.

At the boundary $r=r_0$, Eqs. (14) and (16) have to be matched to Eqs. (11) and (13):

$$\Delta p(r_0)=\Delta\bar{p}(r_0),$$
$$\Delta m(r_0)=\Delta\bar{m}(r_0).$$

Solving these equations for E and F yields

$$E=\tfrac{4}{3}\pi\Delta\bar{\rho}(0)r_0^3(1-\bar{\beta}^2/\beta^2),$$
$$F=\tfrac{4}{3}\pi\Delta\bar{\rho}(0)\bar{\beta}^2/\beta^2.$$

[9] The expansions in Eqs. (15) and (16) [and also in Eqs. (11), (12), and (13)] are only valid as long as the second-order terms in $\frac{1}{2}r^2$ are negligible compared to the first-order terms. Consequently, the *second* variation equations of the equations of hydrostatics would have to be invoked in order to describe the "changes" accurately to second order. The only reason the second-order terms in $\frac{1}{2}r^2$ have been included in the above equations is to see how discontinuities in the equation of state affect the expansion. If the equation of state is discontinuous ("change of phase") see footnote 10.

Substituting these expressions into Eqs. (11) and (12) yields to lowest order[10]

$$\Delta\rho(r_0)=\Delta\bar{\rho}(0)\bar{\beta}^2/\beta^2, \tag{17}$$

$$\Delta m(r_0)=\tfrac{4}{3}\pi\Delta\bar{\rho}(0)r_0^3. \tag{18}$$

These two equations relate "changes" at $r=0$ to "changes" at the surface of the "small" central core $r=r_0$. Substituting Eqs. (17) and (18) into Eq. (6), relates "changes" at the surface of the star to "changes" at the surface of the central core $r=r_0$, yields the desired relationships between the surface and the center of the star[11]:

$$\frac{1}{\beta^2}\begin{pmatrix}\Delta\bar{\rho}(0)\bar{\beta}^2\\ \Delta\bar{\rho}(0)\beta^2/\rho_0\end{pmatrix}=T_R{}^{r_0}\begin{pmatrix}-\Delta R dp/dr|_{r=R}\\ \Delta M/M\end{pmatrix}. \tag{19}$$

This relationship is very useful. It allows one to calculate directly the internal properties of a star from externally observable characteristics, the mass and the radius. In particular, it allows one to calculate the equation of state of cold matter[12] (catalyzed to the end point of thermonuclear reactions) from the mass-radius relationship of a sequence of stars built out of "cold" matter ("cold": actual temperature low compared to Fermi temperature).

As a preliminary step towards determining an equation of state that allows a third family of stable equilibria (tertiary stars), we have to get first a qualitative idea of how the mass-radius relationship is affected by the equation of state. The idea is to construct along some M-R curve a vector field (see Fig. 2) that tells us how the curve would change if the equation of state were changed. To do this use Eqs. (19). Calculate what the increments ΔR and ΔM would be if the following two conditions are satisfied: (1) The central density is increased by $\Delta\bar{\rho}(0)$, and (2) in the central region occupied by matter at supranuclear densities the velocity of sound propagation is $\bar{\beta}^2$ instead of β^2. Various choices for $\bar{\beta}^2$ result in a collection of vectors emanating from the point representing the configuration with central density $\rho_0=3\times10^{14}$ g/cm^3. A set of such collections of vectors drawn at strategic places along a particular curve, such as the one in Fig. 2, gives a fairly good indication of the qualitative effects of the equation of state on the M-R relationship. The vector fields drawn in Fig. 2 are actual results from computer calculations. One observes the following interesting phenomenon: All the vectors have the tendency to avoid one side of the curve. More precisely, focus attention on some particular configuration, say, the one having central density $\rho_0=3\times10^{14}$ g/cm^3 and a sound velocity at the center $\beta\sim(0.015)^{1/2}=0.125$. Go from this configuration to configurations with central density $\rho=\rho_0+\Delta\bar{\rho}(0)$ and with equations of state having the slopes $\bar{\beta}^2=1$, 10^{-3}, 10^{-4}, 0 in the range of densities from ρ_0 to $\rho_0+\Delta\bar{\rho}(0)$. What is the position of these configurations relative to the "fiducial" configuration? They lie somewhere in the direction of the four vectors indicated in Fig. 2.

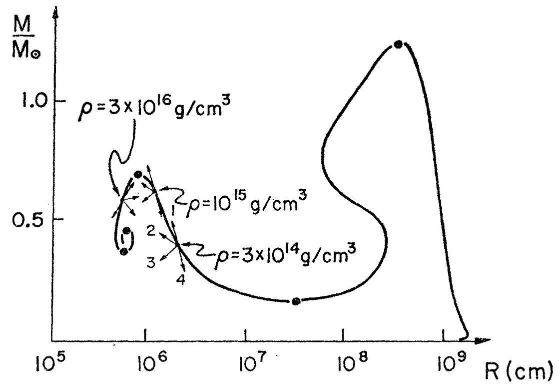

FIG. 2. The mass-radius relationship for a sequence of stars that are made of matter obeying the Harrison-Wheeler equation of state. The heavy dots indicate the critical configurations which signal a change of stability of a particular pulsation mode. Focus attention on one of the configurations from which a group of arrows is emanating, say, the configuration with central density $\rho_0=3\times10^{14}$ g/cm^3. Consider other configurations with higher central densities and with a $dp/d\rho$ at the center corresponding to the HW equation of state. These configurations are points along the curve through the configuration labeled by $\rho_0=3\times10^{14}$ g/cm^3. Instead of considering the HW equation of state, consider now for $\rho>\rho_0$ an equation of state characterized (case 1) by $\beta_1^2=1$ (case 2: $\beta_2^2=10^{-3}$; case 3: $\beta_3^2=10^{-4}$; case 4: $\beta_4=0$). Question: How does the M-R curve for such an equation of state continue from the configuration with central density 3×10^{14} g/cm^3? Answer: The direction of the M-R curve for four different equations of state is indicated by the four arrows emanating from this configuration.

An Equation of State for a Third Family of Stable Equilibria?

Let us determine what equation of state would allow a third stable sequence. The aim will be to specify the

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[10] Observe that in order for Eq. (17) to hold, the sound velocity should not change too violently [specifically, $|d\ln\beta^2/d\ln\rho|\ll(\rho_0r_0^2/\beta^2)^{-1}$] either inside or just outside the surface $r=r_0$. The reason is that in solving for E and F, terms such as $r_0^2\rho_0/\beta^2$ and $r_0^2(d/d\rho)(k/\beta^2)$ [$\approx\frac{4}{3}\pi(\rho_0r_0^2/\beta^2)(2-d\ln\beta^2/d\ln\rho)$] were neglected because $\rho_0r_0^2/\beta^2$ can be made arbitrarily small. However, this requirement does not exclude the possibility of having discontinuous changes in $dp/d\rho$ ("change of phase"). When there is a jump in $dp/d\rho$, one can still calculate $\Delta\bar{\rho}(0)$ and β^2 in the "central region" if one lets the discontinuity occur at the surface $r=r_0$. In practice this means that if one looks at two configurations that differ by ΔR and ΔM, then there may be quite a large jump in $\beta^2=dp/d\rho$ at ρ_0 but not inside the range $\rho_0<\rho<\rho_0+\Delta\rho(0)$. Therefore, the above analysis leaves room for discontinuous equations of state.

[11] The transfer matrix $T_R{}^{r_0}$ can be calculated from a knowledge of the "fiducial" configuration [$m(r)$ and $p(r)$] and from the fact that the equation of state is known for all densities below ρ_0, the density that characterizes the "fiducial" configuration. This is done by means of a high-speed computer.

For Eq. (19) to hold, r_0 has to satisfy two conditions: (1) $r_0^2\rho_0/\beta^2\ll1$, and (2) r_0 must not be so close to the origin that the singular contribution in $T_R{}^{r_0}$ becomes too large, i.e., that $\Delta\rho(r)/\rho(r)$, $\Delta m(r)/m(r)\ll1$ in Eq. (6) is violated. With these two provisos it is clear that $T_R{}^{r_0}$ is independent of r_0 to lowest order.

Furthermore, one may note that in practical calculations it may be necessary to let the pressure p be the independent variable, instead of the radius r. This necessity arises from the fact that at the surface of the star $d\ln(dp/dr)/d\ln r$ is usually very large. For details see pp. 63 and 64 in Gerlach, Ref. 14.

[12] See Ref. 2, Chap. 9, p. 83.

equation so that (1) after the Landau-Oppenheimer-Volkoff (LOV) maximum the $\{M(\rho),R(\rho)\}$ curve represents an unstable family that is terminated by a mass minimum; (2) after the mass minimum the *M-R* curve represents a stable family that is finally terminated by a mass maximum. In short, the *M-R* curve is to turn clockwise (with increasing central density) at the mass minimum that follows the LOV maximum.

To specify an equation of state with these properties, consider a blow-up of the *M-R* relationship near the LOV maximum. Referring to Fig. 3, the maximum has coordinates (R_1,M_1), and the central density for this critical configuration is $\rho_{0(1)}$. Let us expand the mass as a function of R at its critical value M_1:

$$M(R)=M_1+\frac{1}{2}\frac{d^2M}{dR_1^2}(R-R_1)^2,$$

$$\frac{dM}{dR}=\frac{d^2M}{dR_1^2}(R-R_1).$$

For the sake of concreteness let the equation of state that characterizes the *M-R* curve be the Harrison-Wheeler (HW) equation of state.[13] The square of the velocity of sound in the central region of stars close to the critical configuration (M_1,R_1) is $\beta^2=dp/d\rho$.

Consider a sequence of equations of state $p=p_{(n)}(\rho)$ such that the equation of state coincides with the HW equation of state for $\rho<\rho_{0(n)}$, but for $\rho>\rho_{0(n)}$ the square of the sound velocity is larger than $\beta^2=dp/d\rho$, say $\bar{\beta}^2>\beta^2$. The result of calculating the *M-R* curves for three such equations of state is shown in Fig. 3. Here the "change vectors" tangent to these *M-R* curves emanate from the three points corresponding to the three "fiducial" densities $\rho_{0(1)}$, $\rho_{0(2)}$, $\rho_{0(3)}$. Let these "change vectors" have the slopes $\gamma(R_1)$, $\gamma(R_2)$, $\gamma(R_3)$. As shown in Fig. 3 (based on computer calculations), the "change vectors" have the property that

$$\gamma(R)-\frac{dM(R)}{dR}\approx\text{const}<0,$$

i.e., the difference between the slopes of these vectors and the tangents to the *M-R* curve of the HW equation of state changes little for all three cases.

Because of this constancy of the difference of the slopes of the "change vector" and the "fiducial curve" $M(R)$, a central density along $M(R)$ will be reached where

$$\gamma(R_3)=0 \quad (\text{at } \rho=\rho_{0(3)},\ R=R_3).$$

Therefore, there exists a central density such that

$$\gamma(R_2)<0, \quad \text{say} \quad \gamma(R_2)=\tfrac{1}{2}\gamma(R_1).$$

Among the members of the sequence of equations of state considered above, select the one that has the "fiducial" density $\rho_0=\rho_{0(2)}$. The new resulting sequence of equilibrium configurations will follow an *M-R* curve shown by the dashed line in Fig. 3. The fact that this new curve now starts off with a slope $dM/dR=\gamma(R_2)<0$ implies that it characterizes a *stable* sequence of equilibrium configurations. Because of the general relativistic effects, the stable sequence will reach an end at the next mass maximum and will start winding itself up into a spiral (if there are no more sudden changes in $dp/d\rho$) as the central density increases without limit.[14] Consequently, we may conclude that a necessary and sufficient condition for the existence of a third family is that (1) the particle interactions at supranuclear densities are just "right" to produce a discontinuity in the "stiffness," and (2) a formation process exists.

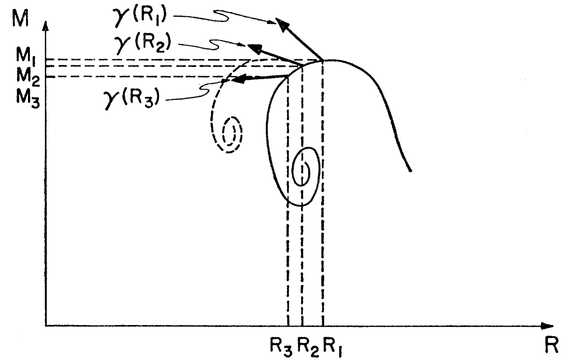

FIG. 3. A blow-up of the LOV maximum. The central densities $\rho_{0(1)}$, $\rho_{0(2)}$, and $\rho_{0(3)}$ correspond to the configurations (R_1,M_1), (R_2,M_2), and (R_3,M_3). The solid curve up to the density $\rho_{0(2)}$ and the dashed curve from thereon is produced by an equation of state that allows a third family of stable equilibria.

CONCLUSION

A question that immediately comes to one's mind is: *Has* nature provided us with a universal equation of state $p=p(\rho)$ that allows the existence of superdense tertiary stars (a third family of stable equilibrium configurations)? The answer is not yet known. However, consider the requirements: (1) The speed of sound (or $dp/d\rho$) must increase abruptly as the density is increased through a certain "zone of transition."[15] (2) This increase must occur at a density slightly above

[13] See Ref. 2, Chap. 10, p. 108.

[14] See Ref. 2, Chap. 5, p. 30; B. K. Harrison, Phys. Rev. **137**, B1644 (1965); N. A. Dmitriev and S. A. Holin, Vopr. Kosmogonni, Akad. Nauk SSSR **9**, 254 (1963); U. H. Gerlach, Ph.D. thesis, Princeton University, 1967 (unpublished), available from University Microfilms, Inc., Ann Arbor, Mich.

[15] In order to produce an *M-R* curve with a sharp kink, such as the one labeled by $\gamma(R_2)$ in Fig. 3, it is obviously necessary to have a discontinuity in $dp/d\rho$. In that event the "zone of transition" would be merely a point. However, it is more likely that nature is best described by a $dp/d\rho$ curve that is not really discontinuous, but rather by a curve that has a "zone of transition" that extends over a finite density range (which should be small compared to $\Delta\rho/\rho_0=10^{-2}$, the range of densities over which tertiary stars exist). In that event, the stable and unstable sequence of superdense stars meet smoothly on the *M-R* curve at R_2.

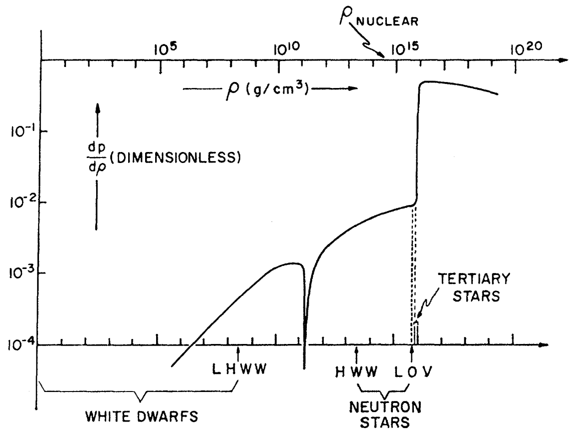

FIG. 4. An equation of state that allows tertiary stars (a third family of stable equilibria). The density of mass-energy in g/cm³ is plotted to the right. The vertical coordinate is the square of the speed of sound, $dp/d\rho$, in units of c^2. This quantity may not exceed unity. The sharp spike in the curve occurs at the density $\rho = 3\times10^{12}$ g/cm³ where a phase transition ("neutron drip") occurs. LHWW, HWW, and LOV (L stands for Landau, HWW stands for Harrison-Wakano-Wheeler, LOV stands for Landau-Opperheimer-Volkoff) indicate the central densities at which the stability of equilibrium configurations changes.

To allow the existence of tertiary stars the compressibility should increase abruptly by a factor of 50–100 at a density slightly above ρ_{LOV}, which labels the critical LOV configuration. With an equation of state as indicated above, there exists an *unstable* branch of stars between the density marking the LOV critical mass and the density at which the compressibility has reached a high value (about 75 times the compressibility at ρ_{LOV}). The third family of stable equilibria follows the unstable one. Both families exist over a density regime $\Delta\rho/\rho_0 \sim 1/100$. The mass range of tertiary stars is $\Delta M/M \sim 10^{-5}$, whereas the range of radii is $\Delta R/R \sim 10^{-4}$. The features of an equation of state required to make the existence of tertiary stars possible are: (1) The square of the speed of sound must increase abruptly by a factor of $\sim$75; (2) the density at which this increase occurs must be just *above* the density that characterizes the LOV maximum.

the central density ($\sim 4\times10^{15}$ g/cm³ for the HW equation of state) that characterizes the Landau-Oppenheimer-Volkoff mass maximum. (3) The speed of sound must be less than that of light. The value of $dp/d\rho$ just below the "zone of transition" must be low enough so that an increase by a factor $\sim$75 is actually possible without violating this condition. The range of densities within which $d^2p/d\rho^2$ must become large must be above the density that characterizes the critical LOV configuration. Consequently, the densities at which $d^2p/d\rho^2$ becomes large ("zone of transition") is not arbitrary; it depends upon the equation of state at lower densities. Thus, it is clear that the behavior of the equation of state at low densities is important because the low-density behavior influences the central density for which the LOV maximum occurs. All these requirements seem to be fairly stringent, and it would be interesting to find out exactly what kind of physical mechanism could produce such an equation of state. Figure 4 presents one conceivable equation of state that allows tertiary stars.

Quite apart from the existence of a third family of equilibrium configurations, nuclear physics stands to gain from measurements on neutron stars, the second family of equilibrium configurations. *From future measurements on the mass-radius relation for neutron stars, plus the mathematical "inversion procedure" summarized in Eqs. (19), one can hope to determine the equation of state of nuclear matter up to the densities of the order of 20 times those encountered in atomic nuclei.*

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