Motion of a Time-Dependent Harmonic Oscillator, and of a Charged Particle in a Class of Time-Dependent, Axially Symmetric Electromagnetic Fields*

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A class of explicitly time-dependent invariants for time-dependent harmonic oscillators is used to construct simple and elegant representations of the general solution of the equations of motion. Then the results for the oscillator are used to derive representations of the general solution of the equations of motion for a charged particle moving classically in the axially symmetric electromagnetic field consisting of an arbitrarily time-dependent, uniform magnetic field, the associated induced electric field, and the electric field due to an arbitrarily time-dependent, uniform charge distribution.

Recently a class of explicitly time-dependent invariants was reported for a time-dependent harmonic oscillator.[1,2] In this paper we use these invariants to construct a corresponding class of simple and elegant representations of the general solution of the equations of motion for the oscillator. Then we use the results for the oscillator to derive representations of the general solution of the equations of motion for a charged particle moving classically in the axially symmetric electromagnetic field consisting of an arbitrarily time-dependent, uniform magnetic field, the associated induced electric field, and the electric field due to an arbitrarily time-dependent, uniform charge distribution. This is possible because the dynamical variables of the charged-particle system are simply related to those of the oscillator.

TIME-DEPENDENT HARMONIC OSCILLATOR

We define a time-dependent harmonic oscillator to be a system described by the equation

$$\epsilon^2\ddot{q}+\Omega^2(t)q=0\,, \tag{1}$$

where $\Omega(t)$ is an arbitrary piecewise continuous function of time, ϵ is a constant parameter, and time differentiation is denoted by a dot. It has been shown[1,2] that the quantity

$$I=\tfrac{1}{2}[\rho^{-2}q^2+\epsilon^2(\rho\dot{q}-q\dot{\rho})^2] \tag{2}$$

is an exact invariant of Eq. (1) as long as $\rho(t)$ is *any* particular solution of

$$\epsilon^2\ddot{\rho}+\Omega^2(t)\rho-\rho^{-3}=0\,. \tag{3}$$

The quantities q and Ω may be complex, and this fact will be important in our treatment of charged-particle motion.

Equations (1) and (2) can be simplified significantly by replacing the variables q and t by variables Q and τ defined by

$$Q=q/\rho \quad\text{and}\quad \tau=\frac{1}{\epsilon}\int^t \rho^{-2}(t')dt'. \tag{4}$$

The inverse powers of ρ in these definitions do not lead to difficulty because, as a result of the ρ^{-3} term in Eq. (3), ρ is bounded away from zero. It is easily verified that the expression for I in terms of Q and τ is

$$I=\tfrac{1}{2}[Q^2+(dQ/d\tau)^2] \tag{2'}$$

and that the differential equation for Q as a function of τ is

$$d^2Q/d\tau^2+Q=0\,. \tag{5}$$

The general solution of Eq. (5) is

$$Q=Ce^{i\tau}+De^{-i\tau}\,, \tag{6}$$

where C and D are arbitrary complex constants related to I by

$$I=2CD\,. \tag{7}$$

Equation (6) provides an elegant representation of the general solution of Eq. (1) for *each* ρ that satisfies Eq. (3).

CHARGED PARTICLE

We consider a particle of mass M and charge e moving classically in an axially symmetric electromagnetic field defined by the potentials

$$\mathbf{A}=\tfrac{1}{2}B(t)\mathbf{k}\times\mathbf{r}$$

and

$$\varphi=\tfrac{1}{2}(e/Mc^2)\eta(t)r^2=\tfrac{1}{2}(e/Mc^2)\eta(t)(x^2+y^2)\,, \tag{8}$$

where $\mathbf{r}$ is the position vector, $\mathbf{k}$ is a unit vector along the symmetry axis, r is perpendicular distance from the symmetry axis, x and y are Cartesian coordinates perpendicular to the symmetry axis, $B(t)$ and $\eta(t)$ are arbitrary piecewise continuous functions of time, and c is the speed of light. The potential φ corresponds to an axially symmetric, time-dependent uniform charge density equal to $-(1/2\pi)(e/Mc^2)\eta(t)$. The

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[1] H. R. Lewis, Jr., Phys. Rev. Letters **18**, 510 (1967); **18**, 636 (E) (1967).

[2] H. R. Lewis, Jr., J. Math. Phys. (to be published).

electric and magnetic fields are

$$\mathbf{E}=-\nabla\varphi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}=-(e/Mc^2)\eta(t)(x\mathbf{i}+y\mathbf{j})-\frac{1}{2c}\frac{\partial\mathbf{B}}{\partial t}\mathbf{k}\times\mathbf{r} \tag{9}$$

and

$$\mathbf{B}=\nabla\times\mathbf{A}=B(t)\mathbf{k},$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the positive x and y directions, respectively, and $\mathbf{k}=\mathbf{i}\times\mathbf{j}$. Since the axial motion of a particle in these fields is trivial, we shall ignore it and only treat the motion perpendicular to the symmetry axis. The equations of motion for the particle are

$$\begin{aligned}\epsilon\ddot{x}&=-(1/\epsilon)\eta(t)x+\tfrac{1}{2}\dot{B}(t)y+B(t)\dot{y},\\ \epsilon\ddot{y}&=-(1/\epsilon)\eta(t)y-\tfrac{1}{2}\dot{B}(t)x-B(t)\dot{x},\end{aligned} \tag{10}$$

where

$$\epsilon=Mc/e. \tag{11}$$

The equations of motion can be written simply in terms of a complex variable q defined by

$$r\exp(i\theta)=x+iy=q\exp\left[-\frac{i}{2\epsilon}\int^t B(t')dt'\right]. \tag{12}$$

The quantities r and θ are the usual cylindrical coordinates of the particle. The variable q satisfies Eq. (1) with

$$\Omega^2(t)=\tfrac{1}{4}B^2(t)+\eta(t). \tag{13}$$

The function $\Omega^2(t)$ may be negative. Since q satisfies the time-dependent harmonic-oscillator equation, we can transcribe all of the results for the oscillator into results for the particle. The remainder of the discussion consists primarily in the exploitation of this fact. As before, we define variables Q and τ by Eqs. (4). However, for convenience, we now restrict the function $\rho(t)$ to be *any positive, real* solution of Eq. (3).

The variables r and θ that correspond to the general solution of Eqs. (10) are given by

$$R\equiv r/\rho=|Q|$$

and

$$\theta=\arg Q-\frac{1}{2\epsilon}\int^t B(t')dt', \tag{14}$$

where Q, given by Eq. (6), is the general solution of Eq. (5). By representing the C and D in Eq. (6) as

$$C=|C|e^{-i\alpha},\quad D=|D|e^{i\beta}, \tag{15}$$

where α and β are real constants, we can write R and θ more explicitly as

$$R=r/\rho=[|C|^2+|D|^2+2|CD|\cos(2\tau-\alpha-\beta)]^{1/2}$$

and

$$\theta=\tan^{-1}\left\{\frac{|C|\sin(\tau-\alpha)-|D|\sin(\tau-\beta)}{|C|\cos(\tau-\alpha)+|D|\cos(\tau-\beta)}\right\}-\frac{1}{2\epsilon}\int^t B(t')dt'. \tag{16}$$

These formulas are a simple representation of the general solution of the particle equations of motion; the constants $|C|$, $|D|$, α, and β are determined from the initial conditions.

Because of the axial symmetry of the electromagnetic field, the canonical momentum conjugate to the angle θ, defined by

$$p_\theta=Mr^2[\dot{\theta}+(1/2\epsilon)B(t)], \tag{17}$$

is also an invariant. The expression for p_θ in terms of $|C|$ and $|D|$ is

$$\begin{aligned}p_\theta&=(M/\epsilon)R^2[d(\arg Q)/d\tau]\\ &=(M/\epsilon)(|C|^2-|D|^2).\end{aligned} \tag{18}$$

The invariant I for the charged particle is a complex quantity obtained by substituting q from Eq. (12) into Eq. (2):

$$I=\tfrac{1}{2}\{\rho^{-2}(x+iy)^2+\epsilon^2[(1/M)\rho(p_x+ip_y)-\dot{\rho}(x+iy)]^2\}\times\exp\left[\frac{i}{\epsilon}\int^t B(t')dt'\right], \tag{19}$$

where p_x and p_y, the canonical momenta conjugate to x and y, respectively, are defined by

$$\begin{aligned}p_x&=M[\dot{x}-(1/2\epsilon)B(t)y],\\ p_y&=M[\dot{y}+(1/2\epsilon)B(t)x].\end{aligned} \tag{20}$$

In terms of cylindrical variables, I can be written as

$$\begin{aligned}I&=\tfrac{1}{2}\left\{\left(\frac{r}{\rho}\right)^2+\epsilon^2\left[\rho^2\frac{d}{dt}\left(\frac{r}{\rho}\right)+i\frac{p_\theta}{M}\frac{1}{(r/\rho)}\right]^2\right\}\times\exp\left\{2i\left[\theta+\frac{1}{2\epsilon}\int^t B(t')dt'\right]\right\}\\ &=\tfrac{1}{2}\left\{R^2+\left[\frac{dR}{d\tau}+i\epsilon\frac{p_\theta}{M}\frac{1}{R}\right]^2\right\}\times\exp\left\{2i\left[\theta+\frac{1}{2\epsilon}\int^t B(t')dt'\right]\right\}.\end{aligned} \tag{21}$$

It is easily verified by direct calculation of dI/dt that I is indeed an exact invariant of the particle motion. The expression for I in terms of the constants C and D is given by Eq. (7).

Of particular interest is the invariant $2[|I|^2+(\epsilon p_\theta/M)^2]^{1/2}$, which can be written as

$$2[|I|^2+(\epsilon p_\theta/M)^2]^{1/2}=(dR/d\tau)^2+R^2+(\epsilon p_\theta/M)^2(1/R^2). \quad (22)$$

To within an additive constant, this quantity is formally the same as the Hamiltonian in cylindrical coordinates for a particle moving in a *time-independent* magnetic field. Recently,[3] use has been made of the corresponding quantum-mechanical result to solve the problem of a quantum particle moving in the electromagnetic field given by Eqs. (9).

[3] H. R. Lewis, Jr. and W. B. Riesenfeld (to be published).

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General Perturbation and Variational Methods for Responses to Time-Dependent Interactions in Quantum Mechanics*

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The perturbed energy of a system in the presence of a generally time-dependent external perturbation, in the limit where all transient effects have disappeared, but still in a range where eventual real transitions can be neglected, contains a term which is proportional to the so-called "response function" giving the response to the external field. This term can be identified and separated quite generally from the total energy perturbation, and the perturbed wave function can be written in the form of a product of a time-dependent amplitude and a time-dependent phase factor involving only this part of the perturbed energy. Using this expression for the wave function, we are able to develop a time-dependent perturbation formalism in which the response to a given perturbation is obtained from just that term of the energy identified above, rather than from the expectation value of the operator corresponding to the response to be determined. A well-known formal difficulty inherent in the Dirac method of variation of constants is avoided in this way. The present method enables us to construct a variational principle by introducing a certain time-averaged energy functional, whose stationary value is precisely the average of the energy term which depends on the response function only. It can therefore be used for variational derivations of approximations for responses to dynamical perturbations. In the case of a static perturbation, the present perturbation-variation formalism is equivalent to the ordinary time-independent one. These new techniques are illustrated by a detailed discussion of the polarization of free atoms or molecules in an oscillating electric field.

I. INTRODUCTION

A WIDELY known class of problems in which time-dependent perturbation theory is used concerns the interaction of free atoms or molecules with electromagnetic radiation, in the semiclassical approximation where the radiation field is not quantized. New applications of time-dependent perturbation theory have recently arisen with the development of nonlinear optics,[1] where in particular the determination of nonlinear susceptibilities is of great interest. The present discussion will therefore be centered mainly on perturbation theory in the context of interactions of atoms with an alternating electric field. The method might, however, prove useful in more general situations, especially its variational version, which we shall develop in some detail.

The conventional procedure for the treatment of time-dependent interactions is the so-called Dirac method of variation of constants,[2] which we shall now briefly summarize for the sake of later discussions. The Hamiltonian of a system subject to a time-dependent external perturbation $V(t)$ is

$$H=H_0+V(t), \quad (1)$$

where H_0 is the time-independent Hamiltonian characterizing the unperturbed system. The properties of the system are determined by solving the Schrödinger equation[3]

$$i\frac{\partial|\psi_j(t)\rangle}{\partial t}=H|\psi_j(t)\rangle. \quad (2)$$

The perturbed state $|\psi_j(t)\rangle$, into which the system goes when the perturbation $V(t)$ acts on the unperturbed eigenstate $e^{-iE_j^{(0)}t}|j\rangle$ of energy $E_j^{(0)}$ of H_0, is expanded

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[2] P. A. M. Dirac, Proc. Roy. Soc. (London) A112, 673 (1926).
[3] Throughout this paper we use units where $\hbar=1$.