Boundary Conditions in Maxwell's Theory

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A very simple procedure to find the field components **E** and **H** in the boundary-value problems in Maxwell's theory is given. This new method is applied both to old problems in optics and to new problems in the theory of transition radiation.

1. INTRODUCTION

IN applying the boundary conditions of the continuity of the tangential components $E_{\parallel}$ and $H_{\parallel}$ and the normal components $D_{\perp}$ and $B_{\perp}$ across interfaces between different media to some simple problems (for example, the calculation of the transmission coefficient of light through a plate), one finds a strangely tedious algebra in contrast to the simplicity of the problem. The separate treatments of different polarizations parallel (p) and perpendicular (s) to the plane of incidence require two separate lengthy calculations.[1] It will be shown in this paper that there is a very simple procedure to find the field components **E** and **H** in the boundary-value problems. First we shall illustrate this new method by giving a simple unified derivation of Fresnel's reflectance formulas from a plane interface between two different media. The separate treatments of p and s polarizations will not be needed. Then we shall further apply the new method to the calculation of the transmission coefficient of light through a plate. The advantages of the method will become increasingly clear as the complexity of the problem increases. We shall show this by applying the present method to the problem of optical reflection and transmission coefficients of stratified media.[2]

This new method can be generalized to include nonfree fields, and its application to the theory of transition radiation[3–6] emitted by an electron passing through the interface of two different media is discussed. The application of transition radiation to high-energy particle detectors is currently being developed.[7] The simple results obtained by the present new approach reduce the tremendous amount of work required to carry out inverse Fourier transforms in the theory of transition radiation. Actually, this work was prompted by the need to find a way out of the labyrinth of complicated algebra encountered in the theory of transition radiation.

2. FRESNEL'S FORMULAS FOR REFLECTANCE

Let the xy plane lie in the plane interface between two different media with dielectric constants ϵ and ϵ', respectively. We choose the z axis along the normal to the interface and assume the relation $\mathbf{B}=\mathbf{H}$. If $(\mathbf{E}^i,\mathbf{H}^i)$, $(\mathbf{E}^r,\mathbf{H}^r)$, and $(\mathbf{E}^t,\mathbf{H}^t)$ denote the fields of the incident, the reflected, and the transmitted waves, respectively, then we have the following four independent boundary conditions which involve either E_z's only or H_z's only:

$$\epsilon(E_z{}^i+E_z{}^r)=\epsilon' E_z{}^t\,, \tag{1a}$$

$$k_z(E_z{}^i-E_z{}^r)=k_z' E_z{}^t\,, \tag{1b}$$

$$H_z{}^i+H_z{}^r=H_z{}^t\,, \tag{2a}$$

$$k_z(H_z{}^i-H_z{}^r)=k_z' H_z{}^t\,, \tag{2b}$$

where (1a) and (2a) are the continuity conditions of the normal components $D_{\perp}$ and $H_{\perp}$. Equation (1b) follows from the combination of the continuity condition of the tangential components $E_{\parallel}$ with the equation $\mathbf{k}\cdot\mathbf{E}=0$. Note that the k_z component of the propagation vector of $\mathbf{E}^r$ has different sign from those of $\mathbf{E}^i$ and $\mathbf{E}^t$. In the same way, one obtains (2b) by combining the continuity condition of $H_{\parallel}$ with $\mathbf{k}\cdot\mathbf{H}=0$. Now we observe that the second set of equations follows from the first by putting $\epsilon=\epsilon'=1$ (in the more general case by the replacement $\epsilon=\mu$, $\epsilon'=\mu'$, where μ and μ' are the magnetic permeabilities of the two media). Therefore, we need not write down Eqs. (2), and we can get the values of H_z's from those of E_z's simply by putting $\epsilon=\epsilon'=1$. From Eqs. (1), one obtains

$$E_z{}^r/E_z{}^i=(k_z\epsilon'-k_z'\epsilon)/(k_z\epsilon'+k_z'\epsilon)\,. \tag{3}$$

As noted above, by setting $\epsilon'=\epsilon=1$, we obtain

$$H_z{}^r/H_z{}^i=(k_z-k_z')/(k_z+k_z')\,. \tag{4}$$

Because both $E_z{}^r$ and $E_z{}^i$ are in the same medium, $|E_z{}^r/E_z{}^i|^2$ gives directly the reflectance for the p polarization. Similarly $|H_z{}^r/H_z{}^i|^2$ yields the reflectance for the s polarization. In fact the substitutions

$$k_z=k\cos\theta\,,\quad k_z'=k'\cos\theta'\,,$$

and

$$(\epsilon'/\epsilon)^{1/2}=k'/k=\sin\theta/\sin\theta'\,,$$

[1] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Inc., New York, 1959), p. 38.
[2] Reference 1, p. 51.
[3] V. L. Ginzburg and I. M. Frank, Zh. Eksperim. i Teor. Fiz. **16**, 15 (1945).
[4] G. M. Garibian, Zh. Eksperim. i Teor. Fiz. **33**, 1403 (1958) [English transl.: Soviet Phys.—JETP **6**, 1079 (1958)].
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[6] F. G. Bass and V. M. Yakovenko, Usp. Fiz. Nauk **86**, 189 (1965) [English transl.: Soviet Phys.—Usp. **8**, 420 (1965)]. This review article contains a bibliography on transition radiation up to 1965.
[7] J. Oostens, S. Prünster, C. L. Wang, and Luke C. L. Yuan, Phys. Rev. Letters **19**, 541 (1967).

where θ and θ' are the angles of incidence and refraction, respectively, will give the familiar forms

$$E_z{}^r/E_z{}^i = \tan(\theta-\theta')/\tan(\theta+\theta'),$$
$$H_z{}^r/H_z{}^i = -\sin(\theta-\theta')/\sin(\theta+\theta').$$

3. TRANSMISSION COEFFICIENT OF LIGHT THROUGH A PLATE

As discussed above, we need to write down only the boundary conditions that contain E_z's:

$$\epsilon(E_z{}^i+E_z{}^r)=\epsilon'(E_z{}^m+E_z{}^{m'}), \quad (5a)$$
$$k_z(E_z{}^i-E_z{}^r)=k_z'(E_z{}^m-E_z{}^{m'}), \quad (5b)$$
$$\epsilon'(E_z{}^m e^{ik_z'd}+E_z{}^{m'}e^{-ik_z'd})=\epsilon E_z{}^t e^{ik_zd}, \quad (5a')$$
$$k_z'(E_z{}^m e^{ik_z'd}-E_z{}^{m'}e^{-ik_z'd})=k_zE_z{}^t e^{ik_zd}, \quad (5b')$$

where $\mathbf{E}^m$ and $\mathbf{E}^{m'}$ are two waves inside the plate, and d is the thickness of the plate. One can easily obtain from these equations

$$E_z{}^t/E_z{}^i = 4\epsilon\epsilon' k_zk_z'/[(k_z\epsilon'+k_z'\epsilon)^2e^{-ik_z'd} - (k_z\epsilon'-k_z'\epsilon)^2e^{ik_z'd}]. \quad (6)$$

By setting $\epsilon=\epsilon'=1$, we get

$$H_z{}^t/H_z{}^i = 4k_zk_z'/[(k_z+k_z')^2e^{-ik_z'd}-(k_z-k_z')^2e^{ik_z'd}]. \quad (7)$$

The absolute squares of (6) and (7) yield the transmission coefficients of light through a plate for p and s polarizations, respectively. This simple and unified derivation preserves the simplicity of the problem under study and that of Maxwell's theory.

4. OPTICAL REFLECTION AND TRANSMISSION COEFFICIENTS OF STRATIFIED MEDIA

Now we proceed to the more complicated problem of calculating the optical reflection and transmission coefficients of multiple layers. Consider $n+1$ planes parallel to the xy plane, located at $z=D_i$ $(i=1, \cdots, n+1)$. The thickness d_p of the pth layer between $z=D_p$ and $z=D_{p+1}$ is $d_p=D_{p+1}-D_p$ and its dielectric and magnetic constants (ϵ_p,μ_p). Let $\mathbf{E}^p$ and $\mathbf{E}^{p'}$ denote the two waves inside the pth layer with the z components of the wave vector $k_z{}^p$ and $-k_z{}^p$, respectively. The incident wave $\mathbf{E}^i$ and the reflected wave $\mathbf{E}^r$ are in the space $z\leq D_1=0$ and the transmitted $\mathbf{E}^t$ in the space $z\geq D_{n+1}$. The dielectric and magnetic constants of these two semi-infinite spaces are assumed to be the same (ϵ,μ). The boundary conditions which correspond to (1a) and (1b) at the first, the $(p+1)$th, and $(n+1)$th interfaces are given by Eqs. (8), (9), and (10), respectively:

$$\epsilon(E_z{}^i+E_z{}^r)=\epsilon_1(E_z{}^1+E_z{}^{1'}), \quad (8a)$$
$$k_z(E_z{}^i-E_z{}^r)=k_z{}^1(E_z{}^1-E_z{}^{1'}), \quad (8b)$$
$$\epsilon_p(E_z{}^pe^{ik_z{}^pD_{p+1}}+E_z{}^{p'}e^{-ik_z{}^pD_{p+1}}) = \epsilon_{p+1}(E_z{}^{p+1}e^{ik_z{}^{p+1}D_{p+1}}+E_z{}^{p+1'}e^{-ik_z{}^{p+1}D_{p+1}}), \quad (9a)$$
$$k_z{}^p(E_z{}^pe^{ik_z{}^pD_{p+1}}-E_z{}^{p'}e^{-ik_z{}^pD_{p+1}}) = k_z{}^{p+1}(E_z{}^{p+1}e^{ik_z{}^{p+1}D_{p+1}}-E_z{}^{p+1'}e^{-ik_z{}^{p+1}D_{p+1}}), \quad (9b)$$
$$\epsilon_n(E_z{}^ne^{ik_z{}^nD_{n+1}}+E_z{}^{n'}e^{-ik_z{}^nD_{n+1}})=\epsilon E_z{}^te^{ik_zD_{n+1}}, \quad (10a)$$
$$k_z{}^n(E_z{}^ne^{ik_z{}^nD_{n+1}}-E_z{}^{n'}e^{-ik_z{}^nD_{n+1}})=k_zE_z{}^te^{ik_zD_{n+1}}. \quad (10b)$$

From (9a) and (9b), the propagation matrix follows immediately:

$$\begin{pmatrix} E_z{}^pe^{ik_z{}^pD_{p+1}} \\ E_z{}^{p'}e^{-ik_z{}^pD_{p+1}} \end{pmatrix} = \frac{1}{2\epsilon_pk_z{}^p}\begin{pmatrix} a_pe^{-ik_z{}^{p+1}d_{p+1}} & b_pe^{ik_z{}^{p+1}d_{p+1}} \\ b_pe^{-ik_z{}^{p+1}d_{p+1}} & a_pe^{ik_z{}^{p+1}d_{p+1}} \end{pmatrix} \times \begin{pmatrix} E_z{}^{p+1}e^{ik_z{}^{p+1}D_{p+2}} \\ E_z{}^{p+1'}e^{-ik_z{}^{p+1}D_{p+2}} \end{pmatrix}, \quad (11)$$

where

$$a_p=\epsilon_{p+1}k_z{}^p+\epsilon_pk_z{}^{p+1}, \quad b_p=\epsilon_{p+1}k_z{}^p-\epsilon_pk_z{}^{p+1}.$$

If we denote the matrix in (11) by M_p, then we obtain

$$\begin{pmatrix} E_z{}^i \\ E_z{}^r \end{pmatrix} = \prod_{p=0}^{n}\frac{M_p}{2\epsilon_pk_z{}^p}\begin{pmatrix} \epsilon k_z{}^n+\epsilon_nk_z \\ \epsilon k_z{}^n-\epsilon_nk_z \end{pmatrix} E_z{}^te^{ik_zD_{n+1}}, \quad (12)$$

where $\epsilon_0=\epsilon$, $k_z{}^0=k_z$. The reflection coefficient $|E_z{}^r/E_z{}^i|^2$ and the transmission coefficient $|E_z{}^t/E_z{}^i|^2$ for the p polarization follow immediately from (12). The replacements $\epsilon_p\rightarrow\mu_p$ give the corresponding coefficients for the s polarization.

5. E_x, E_y COMPONENTS

In some other applications, for example, in the theory of transition radiation, one needs to know the values E_x and E_y in addition to E_z. However, the following equations give a very simple solution:

$$k_xE_x+k_yE_y=-k_zE_z, \quad (13a)$$
$$-k_yE_x+k_xE_y=(\omega/c)H_z, \quad (13b)$$

where (13b) is the third equation of Faraday's law, and (13a) follows from $\mathbf{k}\cdot\mathbf{E}=0$. Therefore, we have a simple procedure to find all field quantities: (1) Write down the conditions which involve the E_z's only and correspond to (1a) and (1b). (2) The values of the H_z's follow from those of the E_z's by putting $\epsilon=\mu$, $\epsilon'=\mu'$. (3) E_x and E_y can be written down immediately in terms of E_z and H_z by Eqs. (13).

6. TRANSITION RADIATION

In the theory of transition radiation, the appearance of nonfree fields requires a slight modification of the present method. First, we have to find the bound fields $\mathbf{E}^b$, $\mathbf{H}^b$ carried by a uniformly moving electron from the following Fourier transforms of Maxwell's equations:

$$\mathbf{k}\times\mathbf{E}^b(\mathbf{k},\omega)=(\omega/c)\mathbf{H}^b(\mathbf{k},\omega), \quad (14a)$$
$$\mathbf{k}\times\mathbf{H}^b(\mathbf{k},\omega)=-(\omega/c)\epsilon\mathbf{E}(\mathbf{k},\omega)+(4\pi/ic)\mathbf{j}(\mathbf{k},\omega), \quad (14b)$$
$$\mathbf{k}\cdot\mathbf{E}^b(\mathbf{k},\omega)=4\pi\rho(\mathbf{k},\omega)/i\epsilon, \quad (14c)$$
$$\mathbf{k}\cdot\mathbf{H}^b(\mathbf{k},\omega)=0, \quad (14d)$$

where

$$\rho(\mathbf{k},\omega)=(e/2\pi)\delta(\omega-\mathbf{k}\cdot\mathbf{v})$$

and

$$\mathbf{j}(\mathbf{k},\omega)=(e\mathbf{v}/2\pi)\delta(\omega-\mathbf{k}\cdot\mathbf{v})$$

are the Fourier transforms of the charge density $\rho(\mathbf{r},t)=e\delta(\mathbf{r}-\mathbf{v}t)$ and current density $\mathbf{j}(\mathbf{r},t)=e\mathbf{v}\delta(\mathbf{r}-\mathbf{v}t)$. From (14), we obtain the bound fields $\mathbf{E}^b$ and $\mathbf{H}^b$:

$$\mathbf{E}^b=\frac{2e}{i}\frac{(\omega/v^2)\mathbf{v}-(1/\epsilon)\mathbf{k}}{(\omega^2/c^2)\epsilon-k^2}\delta(\omega-\mathbf{k}\cdot\mathbf{v}), \tag{15a}$$

$$\mathbf{H}^b=\frac{2e}{i}\frac{(1/c)\mathbf{k}\times\mathbf{v}}{(\omega^2/c^2)\epsilon-k^2}\delta(\omega-\mathbf{k}\cdot\mathbf{v}). \tag{15b}$$

$\mathbf{E}^{b\prime}$ and $\mathbf{H}^{b\prime}$ denote the corresponding fields in a medium where the dielectric constant ϵ is replaced by ϵ'.

As in Sec. 2, let the xy plane lie in the plane interface between two different media with dielectric constants ϵ and ϵ', respectively. The bound fields $(\mathbf{E}^b,\mathbf{H}^b)$ and $(\mathbf{E}^{b\prime},\mathbf{H}^{b\prime})$ on opposite sides of the interface do not satisfy the boundary conditions, and consequently additional free fields $(\mathbf{E},\mathbf{H})$ and $(\mathbf{E}',\mathbf{H}')$ are created at the interface to fulfill the boundary conditions. These free fields are called transition radiation. Because of the factor $\delta(\omega-\mathbf{k}\cdot\mathbf{v})$ there is only an ∞^3 manifold of bound fields. Free fields are also restricted to an ∞^3 manifold by the relation $\epsilon\omega^2/c^2=k^2$. The boundary conditions must be satisfied everywhere on the interface at any instant and this requires that the frequency ω and the tangential propagation vector $\mathfrak{k}=(k_x,k_y)$ must be the same for both free and bound fields. In the following, two-dimensional vectors on the xy plane like $\mathfrak{B}=(V_x,V_y)$, $\mathfrak{E}^b=(E_x{}^b,E_y{}^b)$ will be designated by German boldface. The normal components of the propagation vector $\mathbf{k}$ of $\mathbf{E}^b$ and $\mathbf{E}^{b\prime}$ is determined by $k_z=(\omega-\mathfrak{k}\cdot\mathfrak{B})/v_z$, while those of $\mathbf{E}$ and $\mathbf{E}'$ are given by $-\lambda=-(\epsilon\omega^2/c^2-\mathfrak{k}^2)^{1/2}$ and $\lambda'=(\epsilon'\omega^2/c^2-\mathfrak{k}^2)^{1/2}$, respectively. The signs show that the wave $(\mathbf{E},\mathbf{H})$ propagates in the negative z direction, while $(\mathbf{E}',\mathbf{H}')$ moves in the positive z direction.

Now the boundary conditions which correspond to Eqs. (1a), (1b), (2a), and (2b) are

$$\epsilon(E_z{}^b+E_z)=\epsilon'(E_z{}^{b\prime}+E_z'), \tag{16a}$$

$$\mathfrak{k}\cdot\mathfrak{E}^b-\lambda E_z=\mathfrak{k}\cdot\mathfrak{E}^{b\prime}+\lambda' E_z', \tag{16b}$$

$$H_z{}^b+H_z=H_z{}^{b\prime}+H_z', \tag{17a}$$

$$k_zH_z{}^b-\lambda H_z=k_zH_z{}^{b\prime}+\lambda' H_z'. \tag{17b}$$

For bound fields the relation $\mathbf{k}\cdot\mathbf{H}^b=0$ still holds, while $\mathbf{k}\cdot\mathbf{E}^b\neq 0$. The latter introduces a slight modification in (16b), but $\mathfrak{k}\cdot\mathfrak{E}^b$ and $\mathfrak{k}\cdot\mathfrak{E}^{b\prime}$ are known quantities and are given by (15a), while all unknown quantities contain only z components, so Eqs. (16) and (17) can be solved easily:

$$E_z=(\lambda' A-\epsilon' B)/(\lambda'\epsilon+\epsilon'\lambda),$$
$$E_z'=(\lambda A+\epsilon B)/(\lambda'\epsilon+\epsilon'\lambda),$$
$$H_z=h_z(\lambda'-k_z)/(\lambda'+\lambda),$$
$$H_z'=-h_z(\lambda'+k_z)/(\lambda'+\lambda),$$

where $A=\epsilon' E_z{}^{b\prime}-\epsilon E_z{}^b$, $B=\mathfrak{k}\cdot(\mathfrak{E}^{b\prime}-\mathfrak{E}^b)$, and $h_z=H_z{}^{b\prime}-H_z{}^b$ are given by (15a) and (15b).

Further steps to find E_x and E_y are the same as described in Sec. 5. In the case of normal incidence, $h_z=H_z=H_z'=0$ and one obtains from (13a) and (13b)

$$E_x=k_x\lambda E_z/\mathfrak{k}^2,$$
$$E_y=k_y\lambda E_z/\mathfrak{k}^2.$$

Therefore, the vector $\mathbf{E}$ lies in the plane of observation. The transition radiation emitted by electrons normally incident on a plane interface is polarized in the plane of observation. This most important feature of transition radiation is valid for the general case of multiple layers of an arbitrary number of different media separated by parallel interfaces. The present method gives the simplest proof of this important result. Equations (17a) and (17b) and similar equations for each interface contain only $H_z{}^b$'s and H_z's. For normal incidence one sees, from Eq. (15b), that all $H_z{}^b$'s in each layer vanish, and obtains from Eqs. (17) that all the H_z's$=0$. This, combined with Eqs. (13a) and (13b), implies that the tangential vectors $\mathfrak{E}=(E_x,E_y)$ are all proportional to $\mathfrak{k}=(k_x,k_y)$. Therefore, the emitted radiation will be polarized in the plane of observation.

In the case of oblique incidence, the present method allows easy separation of the components of the electric vector parallel and perpendicular to the plane of observation and reduces the tremendous amount of work required to carry out the inverse Fourier transforms. A detailed exposition of the application of the present method to the theory of transition radiation will be given elsewhere.

7. CONCLUSION

As described above, there is a very simple procedure to find the field quantities in the boundary-value problems in Maxwell's theory, where additional free fields are required to fulfill the boundary conditions. This new method preserves the simplicity of the problem under study and that of Maxwell's theory. In the optical problems, the usual separate treatments of p and s polarizations are no longer required in the present approach. In other problems, like transition radiation, this method yields results by a very simple algebra.