Exact Internal Solutions for Dense Massive Stars

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A model is proposed in which the central pressure and density are infinitely large, but the mass contained in an arbitrary radius ϵ ($m_\epsilon=\int_0^\epsilon 4\pi\rho r^2dr$) is finite. A new m/a ratio has been found: $m/a=n/(2n+1)$, where n can take values $0<n\leqslant 2$. Further, it is proved that the Schwarzschild radius is unattainable.

I. INTRODUCTION

STATIC massive spheres with isotropic pressure were first considered in general relativity by Schwarzschild, considering a sphere of homogeneous density throughout.[1] In no star, however, can the density remain constant throughout. The density and pressure near the surface will be very small compared to their central values. The assumptions made here are made with a view to solving Einstein's equations in a special case, i.e., for extremely dense stars.

The general assumptions made are

(a) The system is spherically symmetrical and static.

(b) Space-time is everywhere regular. The origin is taken as the center of the spherical symmetry.

(c) The space is empty outside a finite region of radius a.

(d) At $r=a$ the internal and exterior Schwarzschild solutions give the same value. This is necessary for continuity.

(e) The pressure and density are everywhere finite except at the center, where $p_c=\infty$ and $\rho_c=\infty$.

The justification for such an assumption lies in the fact that in any volume surrounding the origin

(i) the mass is finite;

(ii) $\int p dV$, i.e., the expression corresponding to potential energy, is finite;

(iii) the total internal energy $U=\int (3p+\rho)e^{\nu/2}dV$ is finite.

The choice of infinite pressure at the center is deliberate, since we are considering a special case of extremely dense bodies.

(f) The pressure vanishes at $r=a$.

In general relativity the restriction for infinite pressure is guided by our classical concept of physics, but one cannot say anything about the actual behavior of matter at $p=\infty$. It may be that massive bodies may continue to exist at infinite central pressure, and we have tried to solve one particular such case. Taking the velocity of light $c=1$ and the gravitational constant $G=1$, the relations between the density ρ, the pressure p, and the energy-momentum tensor of a perfect fluid are given by

$$\rho=T_0{}^0,\quad p=-T_1{}^1=-T_2{}^2=-T_3{}^3. \tag{1}$$

II. FIELD EQUATIONS AND THEIR SOLUTIONS IN SPHERICAL COORDINATES

The line element is given by

$$ds^2=g_{00}dt^2+g_{kl}dx^kdx^l,\quad (k,l=1,2,3) \tag{2}$$

where

$$g_{00}=e^{\nu(r)},\quad g_{11}=-e^{\lambda(r)},\quad g_{22}=-r^2,$$
$$g_{33}=-r^2\sin^2\theta,\quad g_{kl}=0\quad \text{for}\quad k\neq l.$$

ν and λ are functions of r alone. Application of the field equations yields

$$-8\pi T_1{}^1=8\pi p=e^{-\lambda}\left(\frac{1}{r}\frac{d\nu}{dr}+\frac{1}{r^2}\right)-\frac{1}{r^2}, \tag{3}$$

$$-8\pi T_2{}^2=-8\pi T_3{}^3=8\pi p=e^{-\lambda}\left[\frac{1}{2}\frac{d^2\nu}{dr^2}+\frac{1}{4}\left(\frac{d\nu}{dr}\right)^2-\frac{1}{4}\frac{d\lambda d\nu}{drdr}+\frac{1}{2r}\left(\frac{d\nu}{dr}-\frac{d\lambda}{dr}\right)\right], \tag{4}$$

$$-8\pi T_0{}^0=-8\pi\rho=\frac{1}{r^2}-e^{-\lambda}\left(\frac{1}{r}\frac{d\lambda}{dr}-\frac{1}{r^2}\right). \tag{5}$$

Avoiding the singularity at the center, we have the solution

$$e^{-\lambda}=1-\frac{8\pi}{r}\int\rho r^2dr. \tag{6}$$

Now let us assume the density

$$\rho=\frac{1}{8\pi r^2}\left[\frac{2n-n^2}{2n+1-n^2}+\frac{n^2(3+5n-2n^2)}{(n+1)(2n+1)(2n+1-n^2)}\left(\frac{r}{a}\right)^{2(2n+1-n^2)/(n+1)}\right], \tag{7}$$

[1] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, London, 1934), pp. 247-250.

where a is the radius of the sphere. This value of density is positive everywhere and outside the sphere it is zero. As we approach the center, the density $\rho_c \to \infty$ in such a way that $m_\epsilon = \int_0^\epsilon 4\pi\rho r^2 dr$ is positive and finite for an arbitrary positive number ϵ. This value of density gives, from (6),

$$e^{-\lambda} = \frac{1}{2n+1-n^2} - \frac{n^2}{(2n+1)(2n+1-n^2)}\left(\frac{r}{a}\right)^{2(2n+1-n^2)/(n+1)}. \quad (8)$$

Now at $r=a$, $e^{-\lambda}$ must be equal to its value given by the Schwarzschild exterior solution, i.e.,

$$1-2m/a = 1/(2n+1-n^2) - n^2/(2n+1)(2n+1-n^2)$$

or

$$m/a = n/(2n+1) \quad (9)$$

and

$$e^{\nu/2} = Kr^n. \quad (10)$$

For continuity we must have at $r=a$

$$(e^\nu)_{r=a} = (K^2 r^{2n})_{r=a} = K^2 a^{2n} = 1-2m/a$$

or

$$K^2 = (1-2m/a)/a^{2n}.$$

Therefore

$$e^\nu = (1-2m/a)(r/a)^{2n}. \quad (11)$$

The pressure will be given by (3), (4), (8), and (10):

$$p = \frac{n^2}{8\pi r^2(2n+1-n^2)}\left[1-(r/a)^{2(2n+1-n^2)/(n+1)}\right], \quad (12)$$

which is zero at $r=a$. The complete metric is given by

$$ds^2 = \left(1-\frac{2m}{a}\right)\left(\frac{r}{a}\right)^{2n} dt^2 - \left[\frac{1}{2n+1-n^2} - \frac{n^2}{(2n+1-n^2)(2n+1)}\left(\frac{r}{a}\right)^{2(2n+1-n^2)/(n+1)}\right]^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \quad (13)$$

III. DISCUSSIONS

(*i*) *Value of n.* From Eq. (12) we see that the pressure is non-negative throughout, as long as the value of $2n+1-n^2>0$. Also, from Eq. (13) it can be observed that in order to have a positive, finite, and nonzero value of e^λ everywhere, we must always have the condition

$$2n+1-n^2>0.$$

Further, no mass will exist if $n=0$; hence the complete condition is

$$0<n<\sqrt{2}+1.$$

(*ii*) *Density restriction.* From the expression for the density, Eq. (10), we see that for $2<n<\sqrt{2}+1$, (a) the density in the central region is negative; (b) the density near the surface is positive. This means that the density in the same volume changes from negative to positive—a very peculiar phenomenon, allowing negative and positive mass to exist simultaneously inside the same volume. This is physically not possible. Hence the value of n cannot be greater than 2. So, finally, we have

$$0<n\leqq 2. \quad (14)$$

(*iii*) The stars with the ratio $m/a=\frac{1}{2}$ are opaque. The radius $a=2m$ is called the Schwarzschild radius. According to (9) and (14), the maximum m/a ratio is 2/5; hence the Schwarzschild radius is unattainable.

We have thus found the exact solution of Einstein's equations for the case of all spherically symmetric static massive bodies of extremely high density. These solutions will be of great utility in discussing the behavior of quasistellar sources and neutron stars.