In the production of nucleons and mesons, the transverse momentum is on the order of the particle mass. As a plausible conclusion, it is, therefore, necessary to accept that the transverse momenta of the triplets produced by incident nucleons may well be on the order of the triplet mass. If this is the case, the sensitivity of the time-delay experiments will be low.

Consider as an example that a typical triplet is produced at 30 km above sea level. Shower experiments generally try to detect a massive particle delayed by about 50 nsec or more from the shower front, which corresponds to triplet momenta of no more than 30 M_T, where M_T is the triplet mass. If the transverse momentum is as large as M_T, the triplet will reach sea level 1 km from the shower core.

The arguments presented here do not prove that triplets cannot be found in experiments that require an accompanying shower for particle detection, but are meant to indicate the complementary nature of the two types of experiments and the need for carrying out the triplet search in both ways. The upper limits on the production cross section of heavy triplets of 0.1 μb given by Jones *et al.*,[10] and 10 μb given by Bjørnboe *et al.*,[10] would have to be raised considerably if the triplet were found to be produced with large transverse momenta.

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Supplementary Conditions in the Quantized Gravitational Theory*

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The role of the supplementary conditions in the quantum theory of gravitation is discussed in a simplified and unambiguous manner by means of a Lagrangian formalism involving the use of the ordered products. The consistency of the supplementary conditions for the general nonlinear gravitational field interacting with the matter field is established.

1. INTRODUCTION

THE general covariance of the gravitational-field equations is usually regarded as an especially attractive feature of Einstein's theory of gravitation. Einstein[1] himself, and others,[2] discussed the desirability of introducing coordinate conditions, but such discussions did not lead to general agreement.[3] However, a further argument in favor of the coordinate conditions or supplementary conditions was given by the present author[4] by showing that if we impose suitable supplementary conditions, we obtain a remarkable analogy between the field equations of gravitation and electromagnetism, which then enables us to quantize the gravitational field by preserving only the Lorentz covariance and gauge invariance of the gravitational theory. This approach has subsequently been adopted by many other authors.[5]

We shall now examine fully the role of the supplementary conditions in the quantum theory of gravitation. After describing a simplified treatment of the supplementary conditions for the linear gravitational field, we shall consider the general nonlinear gravitational field interacting with the matter field and establish the consistency of the supplementary conditions in general. We shall also discuss the supplementary conditions in the interaction picture.

First, we shall describe the gravitational field in terms of the metrical tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ of the Riemannian space with the line element

$$-ds^2=g_{\mu\nu}dx^\mu dx^\nu\,, \tag{1}$$

where Greek indices take the values 1,2,3,0. We shall

* Supported in part by the National Science Foundation.
[1] A. Einstein, Berliner Berichte (1918), p. 154.
[2] T. De Donder, *La Gravifique Einsteinienne* (Gauthiers-Villars, Paris, 1921); V. A. Fock, J. Phys. USSR 1, 81 (1939). For arguments in favor of the coordinate conditions in the classical theory of gravitation, especially see V. A. Fock, *Theory of Space, Time and Gravitation* (Pergamon Press, Inc., New York, 1959).
[3] L. Infeld, Helv. Phys. Acta Suppl. 4, 240 (1956).
[4] S. N. Gupta, Proc. Phys. Soc. (London) **A65**, 161 (1952); **A65**, 608 (1952).
[5] See, for instance, R. P. Feynman, Acta Phys. Polon. **24**, 697 (1963); K. Just, Nuovo Cimento **34**, 567 (1964); V. I. Ogievetskii and I. V. Polubarinov, Zh. Eksperim. i Teor. Fiz. 48, 1625 (1965) [English transl.: Soviet Phys.—JETP **21**, 1093 (1965)]; S. Weinberg, Phys. Rev. **140**, B516 (1965); B. S. DeWitt, *ibid.* **162**, 1195 (1967); **162**, 1239 (1967).

then introduce the flat-space metric tensors

$$\eta_{\mu\nu}=\eta^{\mu\nu}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\tag{2}$$

and carry out Lorentz-covariant expansions by means of the relations

$$g=|\mathfrak{g}^{\mu\nu}|,\quad g^{\mu\nu}=(-g)^{-1/2}\mathfrak{g}^{\mu\nu},\quad g_{\mu\lambda}g^{\nu\lambda}=\delta_\mu{}^\nu,\tag{3}$$

which show that if[4]

$$\mathfrak{g}^{\mu\nu}=\eta^{\mu\nu}+\kappa\gamma^{\mu\nu},\tag{4}$$

then

$$-g=1+\kappa\eta_{\mu\nu}\gamma^{\mu\nu}+\tfrac{1}{2}\kappa^2(\eta_{\mu\nu}\eta_{\lambda\rho}-\eta_{\mu\lambda}\eta_{\nu\rho})\gamma^{\mu\nu}\gamma^{\lambda\rho}+O(\kappa^3),\tag{5}$$

$$g^{\mu\nu}=\eta^{\mu\nu}+\kappa(\gamma^{\mu\nu}-\tfrac{1}{2}\eta^{\mu\nu}\eta_{\alpha\beta}\gamma^{\alpha\beta})+\kappa^2(\tfrac{1}{4}\eta^{\mu\nu}\eta_{\alpha\lambda}\eta_{\beta\rho}\gamma^{\alpha\beta}\gamma^{\lambda\rho}+\tfrac{1}{8}\eta^{\mu\nu}\eta_{\alpha\beta}\eta_{\lambda\rho}\gamma^{\alpha\beta}\gamma^{\lambda\rho}-\tfrac{1}{2}\eta_{\alpha\beta}\gamma^{\alpha\beta}\gamma^{\mu\nu})+O(\kappa^3),\tag{6}$$

$$g_{\mu\nu}=\eta_{\mu\nu}+\kappa(\tfrac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\gamma^{\alpha\beta}-\eta_{\mu\alpha}\eta_{\nu\beta}\gamma^{\alpha\beta})+\kappa^2(\eta_{\mu\alpha}\eta_{\nu\lambda}\eta_{\beta\rho}\gamma^{\alpha\beta}\gamma^{\lambda\rho}-\tfrac{1}{4}\eta_{\mu\nu}\eta_{\alpha\lambda}\eta_{\beta\rho}\gamma^{\alpha\beta}\gamma^{\lambda\rho}+\tfrac{1}{8}\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\lambda\rho}\gamma^{\alpha\beta}\gamma^{\lambda\rho}-\tfrac{1}{2}\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\lambda\rho}\gamma^{\alpha\beta}\gamma^{\lambda\rho})+O(\kappa^3).\tag{7}$$

Subsequently, we shall employ the usual flat-space notation by writing all tensor indices as lower indices and giving Greek indices the values 1,2,3,4, with $x_\mu=(x_1,x_2,x_3,ix_0)$. We shall also take $c=\hbar=1$.

2. LAGRANGIAN FORMALISM FOR THE QUANTIZED GRAVITATIONAL FIELD

The usual classical Lagrangian density for the gravitational field interacting with the matter field is given by[6]

$$L_{\text{total}}=L+L_M,\tag{8}$$

where

$$L=\kappa^{-2}\mathfrak{g}^{\mu\nu}(\Gamma_{\mu\beta}{}^\alpha\Gamma_{\nu\alpha}{}^\beta-\Gamma_{\mu\nu}{}^\alpha\Gamma_{\alpha\beta}{}^\beta),\tag{9}$$

with

$$\Gamma_{\mu\nu}{}^\alpha=\tfrac{1}{2}g^{\alpha\lambda}(\partial_\mu g_{\nu\lambda}+\partial_\nu g_{\mu\lambda}-\partial_\lambda g_{\mu\nu}),\tag{10}$$

while L_M is the Lagrangian density of the matter field in the generally covariant form. By adopting the flat-space point of view and treating the Lagrangian density as a function of $\mathfrak{g}^{\mu\nu}$ and $\mathfrak{g}_\lambda{}^{\mu\nu}=\partial_\lambda\mathfrak{g}^{\mu\nu}$, the field equations obtained from (8) can be expressed in the form[4]

$$\partial_\alpha\partial_\beta(\eta^{\alpha\beta}\mathfrak{g}^{\mu\nu}-\eta^{\mu\alpha}\mathfrak{g}^{\nu\beta}-\eta^{\nu\alpha}\mathfrak{g}^{\mu\beta}+\eta^{\mu\nu}\mathfrak{g}^{\alpha\beta})=\kappa^2\eta^{\mu\lambda}(\mathfrak{T}_\lambda{}^\nu+\mathfrak{t}_\lambda{}^\nu),\tag{11}$$

where $\eta^{\mu\lambda}\mathfrak{T}_\lambda{}^\nu$ and $\eta^{\mu\lambda}\mathfrak{t}_\lambda{}^\nu$ are the energy-momentum tensors of the matter and gravitational fields, respectively. In the derivation of the above field equation, $\mathfrak{T}_\lambda{}^\nu$ is obtained from L_M by means of the relation

$$\tfrac{1}{2}(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}g^{\lambda\rho}T_{\lambda\rho})=\partial_\alpha\left(\frac{\partial L_M}{\partial\mathfrak{g}_\alpha{}^{\mu\nu}}\right)-\frac{\partial L_M}{\partial\mathfrak{g}^{\mu\nu}},\tag{12}$$

with

$$\mathfrak{T}_\mu{}^\nu=\mathfrak{g}^{\nu\lambda}T_{\mu\lambda},\tag{13}$$

which gives

$$\mathfrak{T}_\mu{}^\nu=2\mathfrak{g}^{\nu\lambda}\left[\partial_\alpha\left(\frac{\partial L_M}{\partial\mathfrak{g}_\alpha{}^{\mu\lambda}}\right)-\frac{\partial L_M}{\partial\mathfrak{g}^{\mu\lambda}}\right]-\delta_\mu{}^\nu\mathfrak{g}^{\lambda\rho}\left[\partial_\alpha\left(\frac{\partial L_M}{\partial\mathfrak{g}_\alpha{}^{\lambda\rho}}\right)-\frac{\partial L_M}{\partial\mathfrak{g}^{\lambda\rho}}\right].\tag{14}$$

Moreover, $\mathfrak{t}_\lambda{}^\nu$ is obtained from L by

$$\eta^{\mu\lambda}\mathfrak{t}_\lambda{}^\nu=\eta^{\mu\lambda}\mathfrak{S}_\lambda{}^\nu+\tfrac{1}{2}\partial_\rho(\mathfrak{f}^{\rho,\mu\nu}+\mathfrak{f}^{\mu,\rho\nu}+\mathfrak{f}^{\nu,\rho\mu}),\tag{15}$$

with

$$\mathfrak{S}_\mu{}^\nu=-\frac{\partial L}{\partial\mathfrak{g}_\nu{}^{\lambda\rho}}\mathfrak{g}_\mu{}^{\lambda\rho}+\delta_\mu{}^\nu L,\qquad \mathfrak{f}^{\rho,\mu\nu}=2\frac{\partial L}{\partial\mathfrak{g}_\rho{}^{\alpha\beta}}(\eta^{\alpha\mu}\mathfrak{g}^{\beta\nu}-\eta^{\alpha\nu}\mathfrak{g}^{\beta\mu}).\tag{16}$$

Let us now consider a modified Lagrangian density of the form

$$L_{\text{total}}=L+L'+L_M,\tag{17}$$

where L' is some function of $\mathfrak{g}^{\mu\nu}$ and $\mathfrak{g}_\lambda{}^{\mu\nu}$. It is convenient to treat L' in the same manner as L_M, which yields, in place of (11),

$$\partial_\alpha\partial_\beta(\eta^{\alpha\beta}\mathfrak{g}^{\mu\nu}-\eta^{\mu\alpha}\mathfrak{g}^{\nu\beta}-\eta^{\nu\alpha}\mathfrak{g}^{\mu\beta}+\eta^{\mu\nu}\mathfrak{g}^{\alpha\beta})=\kappa^2\eta^{\mu\lambda}(\mathfrak{T}_\lambda{}^\nu+\mathfrak{T}'_\lambda{}^\nu+\mathfrak{t}_\lambda{}^\nu),\tag{18}$$

where

$$\mathfrak{T}'_\mu{}^\nu=2\mathfrak{g}^{\nu\lambda}\left[\partial_\alpha\left(\frac{\partial L'}{\partial\mathfrak{g}_\alpha{}^{\mu\lambda}}\right)-\frac{\partial L'}{\partial\mathfrak{g}^{\mu\lambda}}\right]-\delta_\mu{}^\nu\mathfrak{g}^{\lambda\rho}\left[\partial_\alpha\left(\frac{\partial L'}{\partial\mathfrak{g}_\alpha{}^{\lambda\rho}}\right)-\frac{\partial L'}{\partial\mathfrak{g}^{\lambda\rho}}\right].\tag{19}$$

We further put

$$\mathfrak{t}_\lambda{}^\nu=\mathfrak{t}_{G,\lambda}{}^\nu-\mathfrak{t}'_\lambda{}^\nu,\tag{20}$$

where $\eta^{\mu\lambda}\mathfrak{t}_{G,\lambda}{}^\nu$ is the gravitational energy-momentum tensor corresponding to the Lagrangian density $L+L'$, while $\mathfrak{t}'_\lambda{}^\nu$ is given by

$$\eta^{\mu\lambda}\mathfrak{t}'_\lambda{}^\nu=\eta^{\mu\lambda}\mathfrak{S}'_\lambda{}^\nu+\tfrac{1}{2}\partial_\rho(\mathfrak{f}'^{\rho,\mu\nu}+\mathfrak{f}'^{\mu,\rho\nu}+\mathfrak{f}'^{\nu,\rho\mu}),\tag{21}$$

with

$$\mathfrak{S}'_\mu{}^\nu=-\frac{\partial L'}{\partial\mathfrak{g}_\nu{}^{\lambda\rho}}\mathfrak{g}_\mu{}^{\lambda\rho}+\delta_\mu{}^\nu L',\qquad \mathfrak{f}'^{\rho,\mu\nu}=2\frac{\partial L'}{\partial\mathfrak{g}_\rho{}^{\alpha\beta}}(\eta^{\alpha\mu}\mathfrak{g}^{\beta\nu}-\eta^{\alpha\nu}\mathfrak{g}^{\beta\mu}).\tag{22}$$

This enables us to express (18) as

$$\partial_\alpha\partial_\beta(\eta^{\alpha\beta}\mathfrak{g}^{\mu\nu}-\eta^{\mu\alpha}\mathfrak{g}^{\nu\beta}-\eta^{\nu\alpha}\mathfrak{g}^{\mu\beta}+\eta^{\mu\nu}\mathfrak{g}^{\alpha\beta})=\kappa^2\Theta^{\mu\nu}+\kappa^2\eta^{\mu\lambda}(\mathfrak{T}'_\lambda{}^\nu-\mathfrak{t}'_\lambda{}^\nu),\tag{23}$$

where $\Theta^{\mu\nu}$ represents the total energy-momentum

[6] L. Landau and E. Lifshitz, *Classical Theory of Fields* (Addison-Wesley Publishing Co., Inc., London, 1951).

tensor

$$\Theta^{\mu\nu}=\eta^{\mu\lambda}(\mathfrak{T}_\lambda{}^\nu+\mathfrak{t}_{G,\lambda}{}^\nu). \quad (24)$$

When L' is chosen as

$$L'=-\tfrac{1}{2}\kappa^{-2}\eta_{\mu\nu}\mathfrak{g}_\alpha{}^{\mu\alpha}\mathfrak{g}_\beta{}^{\nu\beta}, \quad (25)$$

the field equation (23) becomes, after it is expressed in terms of $\gamma^{\mu\nu}$ by the relation (4),

$$\begin{aligned}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\gamma^{\mu\nu}=\kappa\Theta^{\mu\nu}+\kappa[(\partial_\alpha\gamma^{\mu\nu})(\partial_\beta\gamma^{\alpha\beta})-(\partial_\alpha\gamma^{\mu\alpha})(\partial_\beta\gamma^{\nu\beta})\\ -\tfrac{1}{2}\eta^{\mu\nu}\eta_{\lambda\rho}(\partial_\alpha\gamma^{\lambda\alpha})(\partial_\beta\gamma^{\rho\beta})+\gamma^{\mu\nu}(\partial_\alpha\partial_\beta\gamma^{\alpha\beta})\\ -\gamma^{\mu\alpha}(\partial_\alpha\partial_\beta\gamma^{\nu\beta})-\gamma^{\nu\alpha}(\partial_\alpha\partial_\beta\gamma^{\mu\beta})], \quad (26)\end{aligned}$$

which can be written in the usual flat-space notation as

$$\begin{aligned}\Box^2\gamma_{\mu\nu}=\kappa\Theta_{\mu\nu}+\kappa[(\partial_\alpha\gamma_{\mu\nu})(\partial_\beta\gamma_{\alpha\beta})-(\partial_\alpha\gamma_{\mu\alpha})(\partial_\beta\gamma_{\nu\beta})\\ -\tfrac{1}{2}\delta_{\mu\nu}(\partial_\alpha\gamma_{\lambda\alpha})(\partial_\beta\gamma_{\lambda\beta})+\gamma_{\mu\nu}(\partial_\alpha\partial_\beta\gamma_{\alpha\beta})\\ -\gamma_{\mu\alpha}(\partial_\alpha\partial_\beta\gamma_{\nu\beta})-\gamma_{\nu\alpha}(\partial_\alpha\partial_\beta\gamma_{\mu\beta})]. \quad (27)\end{aligned}$$

The Lagrangian formalism for the quantized gravitational field differs from the usual classical treatment in two respects. First, we must modify the Lagrangian density as described above, which yields the wave equation $\Box^2\gamma_{\mu\nu}=0$ for $\kappa=0$. Second, we must treat the Lagrangian density as an ordered product of the field operators in accordance with our general approach to the quantum theory of fields.[7] Thus the appropriate Lagrangian density for the quantized gravitational field interacting with the matter field is

$$L_{\text{total}}=L_G+L_M, \quad (28)$$

where

$$\begin{aligned}L_G=\kappa^{-2}:[\mathfrak{g}^{\mu\nu}(\Gamma_{\mu\beta}{}^\alpha\Gamma_{\nu\alpha}{}^\beta-\Gamma_{\mu\nu}{}^\alpha\Gamma_{\alpha\beta}{}^\beta)\\ -\tfrac{1}{2}\eta_{\mu\nu}\mathfrak{g}_\alpha{}^{\mu\alpha}\mathfrak{g}_\beta{}^{\nu\beta}]:, \quad (29)\end{aligned}$$

and L_M is the ordered product of the usual generally covariant Lagrangian density of the matter field. The resulting gravitational-field equations in the flat-space notation are

$$\begin{aligned}\Box^2\gamma_{\mu\nu}=\kappa\Theta_{\mu\nu}+\kappa:[(\partial_\alpha\gamma_{\mu\nu})(\partial_\beta\gamma_{\alpha\beta})-(\partial_\alpha\gamma_{\mu\alpha})(\partial_\beta\gamma_{\nu\beta})\\ -\tfrac{1}{2}\delta_{\mu\nu}(\partial_\alpha\gamma_{\lambda\alpha})(\partial_\beta\gamma_{\lambda\beta})+\gamma_{\mu\nu}(\partial_\alpha\partial_\beta\gamma_{\alpha\beta})\\ -\gamma_{\mu\alpha}(\partial_\alpha\partial_\beta\gamma_{\nu\beta})-\gamma_{\nu\alpha}(\partial_\alpha\partial_\beta\gamma_{\mu\beta})]:, \quad (30)\end{aligned}$$

where $\Theta_{\mu\nu}$ is an ordered product representing the total energy-momentum tensor.

In the Lagrangian formalism of the gravitational field, it is customary to treat all the components of $\mathfrak{g}^{\mu\nu}$ or $\gamma_{\mu\nu}$ as independent. This is more convenient and evidently does not affect the field equations or the energy-momentum tensor. On the other hand, it is well known that the usual form of commutation relations can be used only if the symmetry property of $\gamma_{\mu\nu}$ is taken into account in obtaining its canonical conjugate.[4] However, it is again more convenient to derive the canonical conjugate of $\gamma_{\mu\nu}$ from the relation

$$\pi_{\mu\nu}=\partial L/\partial(\partial_0\gamma_{\mu\nu}) \quad (31)$$

by treating all the components of $\gamma_{\mu\nu}$ as independent, and then choosing the commutation relations in the symmetrized form

$$[\gamma_{\mu\nu}(\mathbf{r},x_0),\pi_{\lambda\rho}(\mathbf{r}',x_0)]=\tfrac{1}{2}i(\delta_{\mu\lambda}\delta_{\nu\rho}+\delta_{\mu\rho}\delta_{\nu\lambda})\delta(\mathbf{r}-\mathbf{r}'). \quad (32)$$

3. SUPPLEMENTARY CONDITIONS FOR THE LINEAR GRAVITATIONAL FIELD

We shall first describe the role of the supplementary conditions in the quantization of the gravitational field in the linear approximation and the absence of the matter field.

The Lagrangian density (29) gives in the linear approximation

$$\begin{aligned}L_0=-\tfrac{1}{4}:[(\partial_\alpha\gamma_{\mu\nu})(\partial_\alpha\gamma_{\mu\nu})-\tfrac{1}{2}(\partial_\alpha\gamma_{\mu\mu})(\partial_\alpha\gamma_{\nu\nu})]:\\ +\tfrac{1}{2}:[(\partial_\alpha\gamma_{\mu\beta})(\partial_\beta\gamma_{\mu\alpha})-(\partial_\alpha\gamma_{\mu\alpha})(\partial_\beta\gamma_{\mu\beta})]:, \quad (33)\end{aligned}$$

which yields the field equation

$$\Box^2\gamma_{\mu\nu}=0, \quad (34)$$

the energy

$$\begin{aligned}\mathcal{H}=\tfrac{1}{4}\int:[2(\partial_0\gamma_{\mu\nu})(\partial_0\gamma_{\mu\nu})-(\partial_0\gamma_{\mu\mu})(\partial_0\gamma_{\nu\nu})\\ +(\partial_\lambda\gamma_{\mu\nu})(\partial_\lambda\gamma_{\mu\nu})-\tfrac{1}{2}(\partial_\lambda\gamma_{\mu\mu})(\partial_\lambda\gamma_{\nu\nu})]:d\tau, \quad (35)\end{aligned}$$

and the commutation relations

$$[\gamma_{\mu\nu}(x),\gamma_{\lambda\rho}(x')]=i(\delta_{\mu\lambda}\delta_{\nu\rho}+\delta_{\mu\rho}\delta_{\nu\lambda}-\delta_{\mu\nu}\delta_{\lambda\rho})D(x-x'). \quad (36)$$

The derivation of the above results can be simplified by using the Lagrangian density

$$L_0=-\tfrac{1}{4}:[(\partial_\alpha\gamma_{\mu\nu})(\partial_\alpha\gamma_{\mu\nu})-\tfrac{1}{2}(\partial_\alpha\gamma_{\mu\mu})(\partial_\alpha\gamma_{\nu\nu})]:, \quad (37)$$

which is obtained from (33) by dropping divergence terms, but the form (33) is preferable in general.

By carrying out the Fourier expansion

$$\begin{aligned}\gamma_{\mu\nu}=V^{-1/2}\textstyle\sum_{\mathbf{k}}(2k_0)^{-1/2}\\ \times[a_{\mu\nu}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-k_0x_0)}+a_{\mu\nu}{}^*(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}-k_0x_0)}], \quad (38)\end{aligned}$$

with

$$k_0=|\mathbf{k}|, \quad (39)$$

we obtain from (35) and (36)

$$\mathcal{H}=\textstyle\sum_{\mathbf{k}}k_0[\tfrac{1}{2}a_{\mu\nu}{}^*(\mathbf{k})a_{\mu\nu}(\mathbf{k})-\tfrac{1}{4}a_{\mu\mu}{}^*(\mathbf{k})a_{\nu\nu}(\mathbf{k})], \quad (40)$$

$$[a_{\mu\nu}(\mathbf{k}),a_{\lambda\rho}{}^*(\mathbf{k})]=\delta_{\mu\lambda}\delta_{\nu\rho}+\delta_{\mu\rho}\delta_{\nu\lambda}-\delta_{\mu\nu}\delta_{\lambda\rho}. \quad (41)$$

Further, by putting

$$\begin{aligned}a_+(\mathbf{k})&=(1/\sqrt{8})[a_{11}(\mathbf{k})-a_{22}(\mathbf{k})]-(i/\sqrt{2})a_{12}(\mathbf{k}),\\ a_-(\mathbf{k})&=(1/\sqrt{8})[a_{11}(\mathbf{k})-a_{22}(\mathbf{k})]+(i/\sqrt{2})a_{12}(\mathbf{k}),\\ a'_{30}(\mathbf{k})&=\tfrac{1}{2}[a_{33}(\mathbf{k})+a_{00}(\mathbf{k})], \quad (42)\\ a(\mathbf{k})&=(1/\sqrt{8})[a_{11}(\mathbf{k})+a_{22}(\mathbf{k})+a_{33}(\mathbf{k})-a_{00}(\mathbf{k})],\\ a'(\mathbf{k})&=(1/\sqrt{8})[a_{11}(\mathbf{k})+a_{22}(\mathbf{k})-a_{33}(\mathbf{k})+a_{00}(\mathbf{k})],\end{aligned}$$

[7] S. N. Gupta, Phys. Rev. 107, 1722 (1957).

it is possible to express (40) and (41) as

$$\mathcal{H}=\sum_{\mathbf{k}} k_0[a_+^*(\mathbf{k})a_+(\mathbf{k})+a_-^*(\mathbf{k})a_-(\mathbf{k}) +a_{13}^*(\mathbf{k})a_{13}(\mathbf{k})-a_{10}^*(\mathbf{k})a_{10}(\mathbf{k}) +a_{23}^*(\mathbf{k})a_{23}(\mathbf{k})-a_{20}^*(\mathbf{k})a_{20}(\mathbf{k}) +a'_{30}{}^*(\mathbf{k})a'_{30}(\mathbf{k})-a_{30}^*(\mathbf{k})a_{30}(\mathbf{k}) +a'^*(\mathbf{k})a'(\mathbf{k})-a^*(\mathbf{k})a(\mathbf{k})], \quad (43)$$

with

$$\begin{aligned} &[a_+(\mathbf{k}),a_+^*(\mathbf{k})]=1, \quad [a_-(\mathbf{k}),a_-^*(\mathbf{k})]=1, \\ &[a_{13}(\mathbf{k}),a_{13}^*(\mathbf{k})]=1, \quad [a_{10}(\mathbf{k}),a_{10}^*(\mathbf{k})]=-1, \\ &[a_{23}(\mathbf{k}),a_{23}^*(\mathbf{k})]=1, \quad [a_{20}(\mathbf{k}),a_{20}^*(\mathbf{k})]=-1, \quad (44) \\ &[a'_{30}(\mathbf{k}),a'_{30}{}^*(\mathbf{k})]=1, \quad [a_{30}(\mathbf{k}),a_{30}^*(\mathbf{k})]=-1, \\ &[a'(\mathbf{k}),a'^*(\mathbf{k})]=1, \quad [a(\mathbf{k}),a^*(\mathbf{k})]=-1. \end{aligned}$$

The above results indicate that the quantization of the gravitational field requires the use of an indefinite metric, which can be introduced most conveniently by a generalization of the definition of the Hermitian conjugate quantities.[8] The states of negative norm in such a formalism can be eliminated by imposing the supplementary conditions

$$\partial_\nu \gamma_{\mu\nu}{}^+\Psi=0, \quad (45)$$

where the superscript + denotes the positive-frequency part. For, on substituting (38) into (45) and choosing the x_3 axis along $\mathbf{k}$, we obtain

$$[a_{13}(\mathbf{k})-a_{10}(\mathbf{k})]\Psi=0, \quad [a_{23}(\mathbf{k})-a_{20}(\mathbf{k})]\Psi=0, \quad (46)$$

$$[a_{33}(\mathbf{k})-a_{30}(\mathbf{k})]\Psi=0, \quad [a_{03}(\mathbf{k})-a_{00}(\mathbf{k})]\Psi=0, \quad (47)$$

and, by subtracting and adding the two relations in (47),

$$[a'_{30}(\mathbf{k})-a_{30}(\mathbf{k})]\Psi=0, \quad [a'(\mathbf{k})-a(\mathbf{k})]\Psi=0. \quad (48)$$

As in quantum electrodynamics,[9] the relations (46) and (48), when applied to the states of the system described by (43) and (44), ensure that only $a_+(\mathbf{k})$ and $a_-(\mathbf{k})$ correspond to the observable gravitons.

For practical purposes it is convenient to define

$$h_{\mu\nu}=\gamma_{\mu\nu}-\tfrac{1}{2}\delta_{\mu\nu}\gamma_{\lambda\lambda}, \quad (49)$$

where, according to (36),

$$[h_{\mu\nu}(x),h_{\lambda\rho}(x')]=i(\delta_{\mu\lambda}\delta_{\nu\rho}+\delta_{\mu\rho}\delta_{\nu\lambda}-\delta_{\mu\nu}\delta_{\lambda\rho})D(x-x'). \quad (50)$$

Note that the above commutation relations for $h_{\mu\nu}$ are identical with those in the earlier papers,[4] although the commutation relations for $\gamma_{\mu\nu}$ are now different because $\gamma_{\mu\nu}$ and $\gamma_{\lambda\lambda}$ are not treated independently in the present simplified procedure.

4. SUPPLEMENTARY CONDITIONS FOR THE GENERAL GRAVITATIONAL FIELD

In order to formulate the supplementary conditions for the general gravitational field interacting with the matter field, we shall express the conditions (45) of Sec. 3 in a different form.

Let us consider the supplementary conditions

$$\langle :f\Omega_\mu:\rangle=0, \quad \langle :f\Omega_\mu':\rangle=0, \quad (51)$$

with

$$\Omega_\mu\equiv\partial_\nu\gamma_{\mu\nu}, \quad (52)$$

where f represents an arbitrary function of the gravitational field operators, and a prime denotes differentiation with respect to $x_0=t$. By carrying out the Fourier decomposition of Ω_μ and Ω_μ' with the help of (38), it is easy to see that (51) is equivalent to

$$\langle :f\Omega_\mu{}^+:\rangle=0, \quad \langle :f\Omega_\mu{}^-:\rangle=0,$$

or

$$\Psi^*:f:\Omega_\mu{}^+\Psi=0, \quad \Psi^*\Omega_\mu{}^-:f:\Psi=0.$$

Since $:f:$ is arbitrary, it follows that

$$\Omega_\mu{}^+\Psi=0, \quad \Psi^*\Omega_\mu{}^-=0, \quad (53)$$

which establishes the equivalence of the supplementary conditions (45) and (51) for the linear interaction-free gravitational field.

We shall now show that the supplementary conditions (51) can also be applied in a consistent manner to the general gravitational field interacting with the matter field.

Since the Lagrangian formalism ensures the vanishing of the divergence of the total energy-momentum tensor $\Theta_{\mu\nu}$, we obtain by differentiating (30) with respect to x_ν

$$\Box^2\partial_\nu\gamma_{\mu\nu}=-\kappa:[(\partial_\alpha\gamma_{\nu\alpha})(\partial_\mu\partial_\beta\gamma_{\nu\beta})+(\partial_\alpha\gamma_{\nu\alpha})(\partial_\nu\partial_\beta\gamma_{\mu\beta}) +\gamma_{\nu\alpha}(\partial_\nu\partial_\alpha\partial_\beta\gamma_{\mu\beta})]: \quad (54)$$

or

$$\Box^2\Omega_\mu=-\kappa:[\Omega_\nu(\partial_\mu\Omega_\nu)+\Omega_\nu(\partial_\nu\Omega_\mu)+\gamma_{\nu\alpha}(\partial_\nu\partial_\alpha\Omega_\mu)]:, \quad (55)$$

which can be expressed as

$$\Omega_\mu''=\partial_i\partial_i\Omega_\mu+\kappa:[\Omega_\nu(\partial_\mu\Omega_\nu)+\Omega_\nu(\partial_\nu\Omega_\mu) +2\gamma_{i\nu}(\partial_i\partial_\nu\Omega_\mu)-\gamma_{ij}(\partial_i\partial_j\Omega_\mu)-\gamma_{44}\Omega_\mu'']:. \quad (56)$$

The above relation shows that Ω_μ'' can be expanded by repeated substitutions as an infinite series of the form

$$\Omega_\mu''=\partial_i\partial_i\Omega_\mu+\sum_{n=1}^{\infty}(-1)^{n-1}\kappa^n \times:(\gamma_{44})^{n-1}[\Omega_\nu(\partial_\mu\Omega_\nu)+\Omega_\nu(\partial_\nu\Omega_\mu) +2\gamma_{i\nu}(\partial_i\partial_\nu\Omega_\mu)-\gamma_{ij}(\partial_i\partial_j\Omega_\mu)-\gamma_{44}\partial_i\partial_i\Omega_\mu]:, \quad (57)$$

where the right side contains at most the first time derivative of Ω_μ.

Further, we expand $:f\Omega_\mu:$ in powers of t for arbitrary values of the space coordinates as

$$:f\Omega_\mu:=[:f\Omega_\mu:]_{t=0}+t[:(f'\Omega_\mu+f\Omega_\mu'):]_{t=0} +(t^2/2!)[:(f''\Omega_\mu+2f'\Omega_\mu'+f\Omega_\mu''):]_{t=0} +(t^3/3!)[:(f'''\Omega_\mu+3f''\Omega_\mu' +3f'\Omega_\mu''+f\Omega_\mu'''):]_{t=0}+\cdots. \quad (58)$$

[8] S. N. Gupta, Can. J. Phys. **35**, 961 (1957).
[9] S. N. Gupta, Proc. Phys. Soc. (London) **A63**, 681 (1950).

Since by means of (57) we can convert $[\Omega_\mu'']_{t=0}$, $[\Omega_\mu''']_{t=0}, \cdots$ into ordered products involving the factors $[\Omega_\mu]_{t=0}$ and $[\Omega_\mu']_{t=0}$, it follows that if for arbitrary f we have

$$\langle :f\Omega_\mu:\rangle=0\,,\quad \langle :f\Omega_\mu':\rangle=0,\quad \text{at } t=0\,, \tag{59}$$

then

$$\langle :f\Omega_\mu:\rangle=0 \tag{60}$$

for all values of t, which gives, on differentiation with respect to t,

$$\langle :f\Omega_\mu':\rangle=0 \tag{61}$$

for all values of t. This proves the consistency of the supplementary conditions (51).

5. SUPPLEMENTARY CONDITIONS IN THE INTERACTION PICTURE

It would be interesting to see what form the supplementary conditions acquire when we pass over from the Heisenberg picture to the interaction picture, and for this purpose we shall take

$$L_M=-\tfrac{1}{2}:(-g)^{1/2}[g^{\mu\nu}(\partial_\mu U)(\partial_\nu U)+m^2UU]:\,, \tag{62}$$

which corresponds to a neutral spinless matter field. The total Lagrangian density for the gravitational field interacting with the matter field can then be expressed as

$$\begin{aligned}L_{\text{total}}=&-\tfrac{1}{4}:[(\partial_\lambda\gamma_{\mu\nu})(\partial_\lambda\gamma_{\mu\nu})-\tfrac{1}{2}(\partial_\lambda\gamma_{\alpha\alpha})(\partial_\lambda\gamma_{\beta\beta})-2(\partial_\alpha\gamma_{\mu\beta})(\partial_\beta\gamma_{\mu\alpha})+2(\partial_\alpha\gamma_{\mu\alpha})(\partial_\beta\gamma_{\mu\beta})]:\\&-\tfrac{1}{2}:[(\partial_\mu U)(\partial_\mu U)+m^2UU]:-\tfrac{1}{2}\kappa:\gamma_{\mu\nu}[\tfrac{1}{2}(\partial_\mu\gamma_{\lambda\rho})(\partial_\nu\gamma_{\lambda\rho})-\tfrac{1}{4}(\partial_\mu\gamma_{\alpha\alpha})(\partial_\nu\gamma_{\beta\beta})+(\partial_\lambda\gamma_{\mu\rho})(\partial_\rho\gamma_{\nu\lambda})\\&+\tfrac{1}{2}(\partial_\lambda\gamma_{\alpha\alpha})(\partial_\lambda\gamma_{\mu\nu})-(\partial_\lambda\gamma_{\mu\rho})(\partial_\lambda\gamma_{\nu\rho})]:-\tfrac{1}{2}\kappa:\gamma_{\mu\nu}[(\partial_\mu U)(\partial_\nu U)+\tfrac{1}{2}\delta_{\mu\nu}m^2UU]:+O(\kappa^2)\,.\end{aligned} \tag{63}$$

The field equations obtained from the above Lagrangian density are

$$\begin{aligned}\Box^2(\gamma_{\mu\nu}-\tfrac{1}{2}\delta_{\mu\nu}\gamma_{\alpha\alpha})=&\tfrac{1}{2}\kappa:[(\partial_\mu\gamma_{\lambda\rho})(\partial_\nu\gamma_{\lambda\rho})-\tfrac{1}{2}(\partial_\mu\gamma_{\alpha\alpha})(\partial_\nu\gamma_{\beta\beta})+2(\partial_\lambda\gamma_{\mu\rho})(\partial_\rho\gamma_{\nu\lambda})+(\partial_\lambda\gamma_{\alpha\alpha})(\partial_\lambda\gamma_{\mu\nu})-2(\partial_\lambda\gamma_{\mu\rho})(\partial_\lambda\gamma_{\nu\rho})]:\\&-\tfrac{1}{2}\kappa:\partial_\lambda[2\gamma_{\lambda\rho}(\partial_\rho\gamma_{\mu\nu})-\delta_{\mu\nu}\gamma_{\lambda\rho}(\partial_\rho\gamma_{\alpha\alpha})+2\gamma_{\mu\rho}(\partial_\nu\gamma_{\rho\lambda})+2\gamma_{\nu\rho}(\partial_\mu\gamma_{\rho\lambda})+\gamma_{\mu\nu}(\partial_\lambda\gamma_{\alpha\alpha})\\&+\delta_{\mu\nu}\gamma_{\rho\sigma}(\partial_\lambda\gamma_{\rho\sigma})-2\gamma_{\mu\rho}(\partial_\lambda\gamma_{\rho\nu})-2\gamma_{\nu\rho}(\partial_\lambda\gamma_{\rho\mu})]:+\kappa:[(\partial_\mu U)(\partial_\nu U)+\tfrac{1}{2}\delta_{\mu\nu}m^2UU]:+O(\kappa^2)\,,\end{aligned} \tag{64}$$

$$(\Box^2-m^2)U=\kappa:[\tfrac{1}{2}m^2\gamma_{\alpha\alpha}U-\partial_\mu(\gamma_{\mu\nu}\partial_\nu U)]:+O(\kappa^2)\,, \tag{65}$$

while the canonical conjugates of $\gamma_{\mu\nu}$ and U are given by

$$\begin{aligned}\pi_{\mu\nu}=&\tfrac{1}{2}(\partial_0\gamma_{\mu\nu})-\tfrac{1}{4}\delta_{\mu\nu}(\partial_0\gamma_{\alpha\alpha})+\tfrac{1}{2}\delta_{\mu i}(\partial_i\gamma_{\nu 0})\\&+\tfrac{1}{2}\delta_{\nu i}(\partial_i\gamma_{\mu 0})+\tfrac{1}{2}i\delta_{\mu 4}(\partial_i\gamma_{\nu i})+\tfrac{1}{2}i\delta_{\nu 4}(\partial_i\gamma_{\mu i})\\&+\tfrac{1}{2}\kappa:[\tfrac{1}{2}\gamma_{\mu\nu}(\partial_0\gamma_{\alpha\alpha})-\gamma_{\mu\beta}(\partial_0\gamma_{\beta\nu})-\gamma_{\nu\beta}(\partial_0\gamma_{\beta\mu})\\&+\tfrac{1}{2}\delta_{\mu\nu}\gamma_{\rho\sigma}(\partial_0\gamma_{\rho\sigma})-\gamma_{0\rho}(\partial_\rho\gamma_{\mu\nu})-\gamma_{\mu\beta}(\partial_\nu\gamma_{\beta 0})\\&-\gamma_{\nu\beta}(\partial_\mu\gamma_{\beta 0})+\tfrac{1}{2}\delta_{\mu\nu}\gamma_{0\rho}(\partial_\rho\gamma_{\beta\beta})]:+O(\kappa^2)\,,\end{aligned} \tag{66}$$

$$\Pi=\partial_0U-\kappa:[\gamma_{0\nu}(\partial_\nu U)]:+O(\kappa^2)\,, \tag{67}$$

respectively.

With the help of the above relations we can express $\partial_\nu\gamma_{\mu\nu}$ and $\partial_0\partial_\nu\gamma_{\mu\nu}$ in terms of the canonical field variables and their space derivatives, and then transform to the interaction picture by replacing these canonical variables by the corresponding variables for free fields. Thus, by transforming (51), we find that the supplementary conditions in the interaction picture are

$$\begin{aligned}\langle :f(\partial_\nu\gamma_{\mu\nu}+\kappa P_{\mu 4,4}):\rangle=0\,,\\ \langle :f(\partial_0\partial_\nu\gamma_{\mu\nu}+i\kappa\partial_i P_{\mu i,4}+i\kappa Q_{\mu 4}+i\kappa\Theta_{\mu 4}):\rangle=0\,,\end{aligned} \tag{68}$$

where

$$\begin{aligned}P_{\mu\nu,4}=&:[\gamma_{\mu\alpha}(\partial_4\gamma_{\alpha\nu})+\gamma_{\nu\alpha}(\partial_4\gamma_{\alpha\mu})-\gamma_{\mu\alpha}(\partial_\nu\gamma_{\alpha 4})\\&-\gamma_{\nu\alpha}(\partial_\mu\gamma_{\alpha 4})-\gamma_{4\alpha}(\partial_\alpha\gamma_{\mu\nu})-\tfrac{1}{2}\gamma_{\mu\nu}(\partial_4\gamma_{\alpha\alpha})]:\\&+\delta_{\mu\nu}:[\gamma_{\alpha\beta}(\partial_\alpha\gamma_{\beta 4})-\tfrac{1}{2}\gamma_{\alpha\beta}(\partial_4\gamma_{\alpha\beta})\\&+\tfrac{1}{4}\gamma_{\alpha\alpha}(\partial_4\gamma_{\beta\beta})]:+O(\kappa)\,,\end{aligned} \tag{69}$$

$$\begin{aligned}Q_{\mu\nu}=&:[(\partial_\alpha\gamma_{\mu\nu})(\partial_\beta\gamma_{\alpha\beta})-(\partial_\alpha\gamma_{\mu\alpha})(\partial_\beta\gamma_{\nu\beta})\\&-\tfrac{1}{2}\delta_{\mu\nu}(\partial_\alpha\gamma_{\lambda\alpha})(\partial_\beta\gamma_{\lambda\beta})+\gamma_{\mu\nu}(\partial_\alpha\partial_\beta\gamma_{\alpha\beta})\\&-\gamma_{\mu\alpha}(\partial_\alpha\partial_\beta\gamma_{\nu\beta})-\gamma_{\nu\alpha}(\partial_\alpha\partial_\beta\gamma_{\mu\beta})]:+O(\kappa)\,,\end{aligned} \tag{70}$$

while $\Theta_{\mu\nu}$ is the total energy-momentum tensor and all operators refer to the interaction picture.

It should be observed that it is really not necessary to obtain the supplementary conditions explicitly in the interaction picture, because the proof of the consistency of the supplementary conditions in the Heisenberg picture given in Sec. 4 ensures their consistency in all pictures.[10]

[10] It has been pointed out by Feynman (Ref. 5) that the S-matrix elements for closed loops in gravitational interactions appear to violate the unitary condition, and he has suggested an interesting device to overcome this difficulty. We have made no attempt to resolve this problem here.