Interior Solution for a Finite Rotating Body of Perfect Fluid*

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(Received 19 Feburary 1968)

An axially symmetric, stationary, type-D solution of Einstein's field equations has been found which represents the superposition of a Kerr-NUT metric and a rigidly rotating perfect fluid in the same space-time region, analogous to the Newtonian superposition of the gravitational fields of a mass point, or ring, and a surrounding body of distributed matter. A fairly large family of old and new metrics are thus included as special cases: Schwarzschild, NUT, and Kerr exterior solutions, and new interior solutions for a stationary body of fluid, either spherical and nonrotating, or rigidly rotating with arbitrary angular velocity.

I. INTRODUCTION

An exact interior solution for a finite rotating body of perfect fluid has been discovered as a special case of an axially symmetric, stationary, type-D solution of Einstein's field equations. The general metric can be described as the superposition of a Kerr-NUT metric[1] and a rigidly rotating perfect fluid in the same space-time region, analogous to the Newtonian superposition of the gravitational fields of a mass point, or ring, and a surrounding body of distributed matter. A fairly large family of old and new metrics is thus included as special cases: Schwarzschild, NUT,[2] and Kerr[3] exterior solutions, and new interior solutions for a stationary body of fluid, either spherical and nonrotating, or rigidly rotating with arbitrary angular velocity. The metric was obtained by solving the dyadic equations[4,5] for rigidly rotating perfect fluids with several assumptions, among them that the gravitational field is type-D. It is almost certainly the simplest solution of this class, but probably not unique.

The rotating interior solution is interesting inasmuch as it is the first exact solution for a rotating fluid body bounded by a finite surface of zero pressure. We should mention immediately, however, that it is not a possible source for the exterior Kerr metric, as we have shown by an application of the dyadic junction conditions.[6] We have not discovered an exterior solution that can be matched to it. In fact, although free of singularities, this interior solution turns out to have a defect that seems to forbid interpreting it as a model of an *isolated* rotating body; the level surfaces of constant pressure and density are prolate rather than oblate. The only obvious physical explanation for this is to ascribe the prolateness to the tidal effects of mass singularities and/or distributed matter external to the body. Since the interior solution is *stationary*, the external matter must also maintain a stationary prolate distribution and so would require a more complicated anisotropic stress tensor, capable of supporting shear stresses. The implication of the physical argument is, of course, that no *vacuum* exterior solution at all can be found to match this interior, but it remains to show this rigorously.

Another feature of interest in these solutions is the simultaneous presence of terms corresponding to the fields of point or ring singularities and of distributed matter. Because of this, they can serve as shell solutions in the interior of composite, layered bodies, and appear to be the first general-relativistic fields of this type for perfect fluids.

II. GENERAL METRIC AND PHYSICAL QUANTITIES

We write the metric in terms of comoving, pseudo-confocal, spatial coordinates (ζ,ξ,θ) which are closely related to the oblate-spheroidal coordinates in Euclidean geometry. Using rationalized gravitational units $4\pi G=c=1$, the metric form is

$$ds^2=-\frac{1}{\phi^2}(dt-A\,d\theta)^2+r_0^2(\zeta^2+\xi^2)\times\left[\frac{d\zeta^2}{(1-k^2\zeta^2)h_1}+\frac{d\xi^2}{(1+k^2\xi^2)h_2}+\frac{\delta^2h_1h_2}{(h_1-h_2)}d\theta^2\right], \quad (1)$$

where

$$\frac{1}{\phi^2}=\frac{(h_1-h_2)}{(\zeta^2+\xi^2)},\quad A=\delta r_0\left[\frac{(\xi^2h_1+\zeta^2h_2)}{(h_1-h_2)}-\xi_A{}^2\right],$$

$$h_1(\zeta)=1+\zeta^2-\frac{2m}{r_0}\zeta(1-k^2\zeta^2)^{1/2}+\frac{\zeta}{\kappa^2}\left[\zeta-\frac{1}{k}(1-k^2\zeta^2)^{1/2}\sin^{-1}(k\zeta)\right], \quad (2)$$

$$h_2(\xi)=1-\xi^2-\frac{2b}{r_0}\xi(1+k^2\xi^2)^{1/2}-\frac{\xi}{\kappa^2}\left[\xi-\frac{1}{k}(1+k^2\xi^2)^{1/2}\sinh^{-1}(k\xi)\right].$$

* This paper presents results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NAS 7-100, sponsored by the National Aeronautics and Space Administration.

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The positive definite 3-space metric beginning at r_0^2 in Eq. (1) is the comoving quotient metric on the rigidly rotating fluid body.

The familiar exterior parameters in the metric can be identified as the Schwarzschild constant m, the NUT constant b, and r_0 which is related to the Kerr constant $\bar{a}$ by $r_0^2=\bar{a}^2-b^2$. The interior parameters κ and k relate to the fluid through equations for the pressure p and energy density ρ, which are given by

$$p=\tfrac{1}{2}\rho_s\left(1-\frac{\kappa^2}{\phi^2}\right),\qquad \rho=\tfrac{1}{2}\rho_s\left(3\frac{\kappa^2}{\phi^2}-1\right). \tag{3}$$

We note that the surfaces of constant p, ρ, and ϕ coincide, and that the constant ρ_s is the energy density on the outside surface, $\phi^2=\kappa^2$, or $p=0$. The parameter k is defined by $k\equiv\kappa\rho_s^{1/2}r_0$ and in the limit $(\rho_s,k\to 0)$ the fluid disappears.

Eliminating ϕ from the equations for p and ρ, we find the relation

$$p+\tfrac{1}{3}\rho=\tfrac{1}{3}\rho_s \tag{4}$$

holding throughout the interior. Thus, the energy density must decrease inwards and this is, of course, highly unrealistic for compressible fluids. It may not be an unreasonable idealization for incompressible fluids in many cases. Even in an extreme relativistically condensed situation, where the central pressure p_c reaches one-third the central energy density ρ_c, we see from Eq. (4) that the density drops only by 50% into the center; i.e., $\rho_c=\frac{1}{2}\rho_s$. For normal situations having $p_c\ll\rho_c$, the models are virtually constant density analogous to the interior Schwarzschild solution.

The remaining constants appearing in the metric, ξ_A and δ, are determined geometrically by the other parameters. The surfaces $\xi=$ (const) are the analog of the confocal hyperboloids in flat space, and ξ_A is defined so that $\xi=\xi_A$ is the degenerate surface giving the axis of symmetry and rotation. It is implicitly determined as the solution of the equation $h_2(\xi_A)=0$, which guarantees that the metric coefficients of $dt\,d\theta$ and $d\theta^2$ vanish at the axis. The constant δ is determined so that the coordinate surfaces $\zeta=$ (const), which are the analog of the confocal spheroids in flat space, are locally flat (have no cusps) at the axis of symmetry. This condition reads

$$\delta=\pm 2\left[(1+k^2\xi_A{}^2)^{1/2}\left.\frac{dh_2}{d\xi}\right|_{\xi=\xi_A}\right]^{-1}, \tag{5}$$

and simultaneously ensures elementary flatness at the axis of the level surfaces $p=$ (const) which are tangent to $\zeta=$ (const) at the axis.

The isometry (lines of variation of t) gives the world lines of the fluid (when fluid is present). The invariants which can be formed from the acceleration a^μ and vorticity Ω^μ of this congruence can be expressed as

$$\begin{aligned} a^2&\equiv a^\mu a_\mu=\frac{\phi^2}{4\kappa^4(h_1-h_2)}(h_1g_r{}^2+h_2g_i{}^2),\\ a\cdot\Omega&\equiv a^\mu\Omega_\mu=\frac{\phi^2}{4\kappa^4}g_rg_i,\\ \Omega^2&\equiv\Omega^\mu\Omega_\mu=\frac{\phi^2}{4\kappa^4(h_1-h_2)}(h_1g_i{}^2+h_2g_r{}^2),\end{aligned} \tag{6}$$

or

$$\begin{aligned} (a^2-\Omega^2)+2i(a\cdot\Omega)&=\frac{\phi^2}{4\kappa^4}g^2,\\ a^2+\Omega^2&=\frac{\phi^2}{4\kappa^4}g\bar{g}\left(\frac{h_1+h_2}{h_1-h_2}\right),\end{aligned} \tag{7}$$

where the bar means complex conjugate and the function g

$$g(\lambda)=g_r+ig_i=\frac{k}{r_0}\left[\cot\left(\frac{k\lambda}{r_0}\right)-\left\{\frac{2\kappa^2k}{r_0}(m-ib)+\frac{k\lambda}{r_0}\right\}\csc^2\left(\frac{k\lambda}{r_0}\right)\right] \tag{8}$$

is an analytic function of the complex coordinate λ defined by

$$\frac{k\lambda}{r_0}\equiv\sin^{-1}(k\zeta)+i\sinh^{-1}(k\xi),\quad\left[0\leq\mathrm{Re}\left(\frac{k\lambda}{r_0}\right)<\tfrac{1}{2}\pi\right]. \tag{9}$$

The two independent invariants of the type-D Weyl tensor can be expressed by the single complex variable α, which is also an analytic function of λ;

$$\begin{aligned}\alpha&=-\tfrac{1}{3}\rho_s-\frac{k}{2\kappa^2r_0}g(\lambda)\cot\left(\frac{k\lambda}{r_0}\right),\\ C^{\mu\nu\sigma\tau}C_{\mu\nu\sigma\tau}&+i\ {}^*C^{\mu\nu\sigma\tau}C_{\mu\nu\sigma\tau}\equiv 48\alpha^2.\end{aligned} \tag{10}$$

III. SPECIAL CASES

A. Reduction to Flat Space-Time

The bracketed terms in h_1 and h_2 depend only on the fluid, and vanish with k. If m and b are also set to zero, we find

$$\begin{aligned} &h_1=1+\zeta^2,\quad h_2=1-\xi^2,\\ &\xi_A=\pm1,\quad \delta=\pm1,\quad \phi=1,\quad A=0,\end{aligned} \tag{11}$$

and the metric reduces to flat space-time in oblate-spheroidal spatial coordinates with foci on the circle of radius r_0 at $\zeta=\xi=0$. The disk of the symmetry plane inside the circle is given by $\zeta=0$; the exterior region of the symmetry plane by $\xi=0$.

B. The Kerr-NUT Exterior Solution

Keeping $k=0$ but restoring m and b, we find a nice form for the Kerr-NUT metric with $r_0^2=\bar{a}^2-b^2$; the metric with $b>\bar{a}$ results from the transformations $r_0=iz_0$, $\zeta=-i\eta$, $\xi=-i\mu$. Writing the first case only, we have

$$\xi_A=-\left(\frac{b\pm\bar{a}}{r_0}\right),\qquad \delta=\pm\frac{r_0}{\bar{a}},$$
$$h_1=1+\zeta^2-\frac{2m}{r_0}\zeta,\quad h_2=1-\xi^2-\frac{2b}{r_0}\xi,$$
$$\frac{1}{\phi^2}=1-\frac{2}{r_0}\frac{(m\zeta-b\xi)}{(\zeta^2+\xi^2)},\tag{12}$$
$$A=\delta r_0\left\{\phi^2\left[1-\frac{2\zeta\xi(m\xi+b\zeta)}{r_0(\zeta^2+\xi^2)}\right]-\xi_A{}^2\right\}.$$

In this $k=0$ limit, the complex variable λ becomes

$$\lambda=r_0(\zeta+i\xi),\tag{13}$$

and has previously been written[1] in polar-coordinate form (r,χ) obtained by setting

$$\zeta=\frac{r}{r_0},\quad \xi=-\frac{1}{r_0}(b+\bar{a}\cos\chi).\tag{14}$$

The function $g(\lambda)$ becomes

$$g(\lambda)=-2\kappa^2(m-ib)/\lambda^2,\tag{15}$$

giving

$$\alpha=(m-ib)/\lambda^3.\tag{16}$$

The expression for α reveals that the only true (curvature) singularity occurs on the "ring" $\zeta=\xi=0$. An analysis of Eq. (10) shows that this is also the only singularity of the general space-time with fluid present, if we restrict the domain to the finite physical region bounded by $p=0$. Further, the singular terms are then still proportional to m and b, so that for $m=b=0$, the manifold is free of singularities.

C. Nonrotating Spherical Limit

A nonrotating, spherically symmetric fluid body with point mass singularity $4\pi m$ at the origin can be obtained from the general metric by putting $b=0$, $\zeta=r/r_0$, $\xi=\cos\chi$ and letting $r_0\to 0$. In the limit, we find

$$ds^2=-\frac{1}{\phi^2}dt^2+\frac{\phi^2}{(1-\kappa^2\rho_s r^2)}dr^2+r^2(d\chi^2+\sin^2\chi d\theta^2),\tag{17}$$

with

$$\frac{1}{\phi^2}=1-\frac{2m}{r}(1-\kappa^2\rho_s r^2)^{1/2}+\frac{1}{\kappa^2}\left[1-\frac{(1-\kappa^2\rho_s r^2)^{1/2}}{\kappa\rho_s{}^{1/2}r}\sin^{-1}(\kappa\rho_s{}^{1/2}r)\right],\tag{18}$$

and

$$g=g_r=-\frac{\kappa^2 r}{(1-\kappa^2\rho_s r^2)^{1/2}}\left[\rho_s+\frac{1}{r^2}\left(1-\frac{1}{\phi^2}\right)\right],$$
$$a=\frac{r\phi}{(1-\kappa^2\rho_s r^2)^{1/2}}\left[\tfrac{1}{2}\rho_s+\frac{1}{2r^2}\left(1-\frac{1}{\phi^2}\right)\right],\tag{19}$$
$$\alpha=\tfrac{1}{6}\rho_s+\frac{1}{2r^2}\left(1-\frac{1}{\phi^2}\right).$$

For $\rho_s\to 0$, we recover, of course, the exterior Schwarzschild solution. For $m=0$, we obtain a complete fluid sphere whose structure, in the case of small central pressure, is similar to the interior Schwarzschild solution. Near the center

$$\frac{1}{\phi^2}\approx 1+\tfrac{1}{3}\rho_s r^2\tag{20}$$

so that, at $r=0$,

$$\phi_c{}^2=1,\quad \alpha_c=0,$$
$$p_c=\tfrac{1}{2}\rho_s(1-\kappa^2),\quad \rho_c=\tfrac{1}{2}\rho_s(3\kappa^2-1).\tag{21}$$

This type-D interior space-time is conformally flat only at the center, of course, unlike the Schwarzschild interior, which is conformally flat throughout. Whereas the interior Schwarzschild matches to an exterior Schwarzschild with mass parameter $M=\frac{1}{3}\rho_s R_s{}^3$ (the missing 4π results from using rationalized units), the present sphere requires a parameter

$$M=\tfrac{1}{2}\left(\rho_s R_s{}^3-\frac{(1-\kappa^2)}{\kappa^2}R_s\right),\tag{22}$$

where R_s is determined from the equation for the outside surface $\phi_s{}^2=\kappa^2$, viz.,

$$\kappa^3\rho_s{}^{1/2}R_s=(1-\kappa^2\rho_s R_s{}^2)^{1/2}\sin^{-1}(\kappa\rho_s{}^{1/2}R_s).\tag{23}$$

For $\kappa\rho_s{}^{1/2}R_s\ll 1$, we find, from the preceding equation,

$$\tfrac{1}{3}\rho_s R_s{}^2\approx\frac{1-\kappa^2}{\kappa^2},\tag{24}$$

and the mass parameters then agree.

When both ρ_s and m are nonzero, this metric can be considered as a shell solution appropriate for a range of r such that $\frac{1}{3}\leqq\kappa^2/\phi^2\leqq 1$, keeping p and ρ finite and positive. We can, for instance, build a composite sphere with a core of interior Schwarzschild solution and an envelope of this new solution with a value of m determined from matching to the core. Let $\tilde{\rho}$ be the constant energy density in the core and $\tilde{R}$ the Schwarzschild coordinate of its surface. The required value of m

in the envelope is found to be

$$m=\tfrac{1}{2}\tilde{R}(1-\kappa^2\rho_s\tilde{R}^2)^{-1/2}\left\{1+\frac{1}{\kappa^2}\left[1-\frac{(1-\kappa^2\rho_s\tilde{R}^2)^{1/2}}{\kappa\rho_s^{1/2}\tilde{R}}\times\sin^{-1}(\kappa\rho_s^{1/2}\tilde{R})\right]-\frac{(1-\frac{2}{3}\tilde{\rho}\tilde{R}^2)}{(1-\kappa^2\rho_s\tilde{R}^2)}\right\} \quad (25)$$

or, for $\kappa\rho_s^{1/2}\tilde{R}\ll1$,

$$m\approx\tfrac{1}{3}\tilde{\rho}\tilde{R}^3+\tfrac{1}{6}\rho_s(1-3\kappa^2)\tilde{R}^3. \quad (26)$$

If $\rho_s=0$, we find the expected relation for a match between interior and exterior Schwarzschild solutions. Referring to Eq. (21) for the density at the center of a complete sphere of envelope solution, we can rewrite Eq. (26) in a more suggestive form as

$$m\approx\tfrac{1}{3}(\tilde{\rho}-\rho_c)\tilde{R}^3, \quad (27)$$

so that it has the significance of scooping out a hole in the middle of a complete sphere of envelope solution which is then filled with interior Schwarzschild solution. We note also that for a model with $\tilde{\rho}<\rho_c$, we have a valid example of a *negative* Schwarzschild mass parameter. The exterior Schwarzschild metric surrounding this composite body will have a mass parameter which is given again by Eq. (22), but the equation determining R_s differs from Eq. (23). In the present case, the equation for the surface reads

$$\kappa^3\rho_s^{1/2}R_s=(1-\kappa^2\rho_sR_s^2)^{1/2}\times[\sin^{-1}(\kappa\rho_s^{1/2}R_s)+2m\kappa^3\rho_s^{1/2}], \quad (28)$$

which, for $\kappa\rho_s^{1/2}R_s\ll1$, leads to the expected result

$$M\approx\tfrac{1}{3}\rho_sR_s^3+m. \quad (29)$$

D. Rotating Fluid Body

Finally, there is the singularity-free interior solution for a rigidly rotating fluid body obtained by setting $m=b=0$. Reasonably simple expressions for the physical quantities in terms of the parameters are found at two locations: the center ($\zeta=0$, $\xi=\xi_A$) and the coordinate ring ($\zeta=0$, $\xi=0$). Denoting these with subscripts c and 0, respectively, we have

$$\begin{aligned}
&\phi_c^2=\xi_A^2, && \phi_0^2=1,\\
&p_c=\tfrac{1}{2}\rho_s\left(1-\frac{\kappa^2}{\xi_A^2}\right), && p_0=\tfrac{1}{2}\rho_s(1-\kappa^2),\\
&a_c=0, && a_0=\tfrac{1}{3}\rho_sr_0,\\
&\Omega_c=\tfrac{1}{2}\rho_sr_0\frac{|\kappa^2(\xi_A^2-1)-k^2\xi_A^4|}{k^2\xi_A^2(1+k^2\xi_A^2)^{1/2}}, && \Omega_0=\tfrac{1}{3}\rho_sr_0,\\
&\alpha_c=\tfrac{1}{6}\rho_s\left[1-3\frac{\kappa^2}{k^2}\frac{(\xi_A^2-1)}{\xi_A^4}\right], && \alpha_0=0.
\end{aligned} \quad (30)$$

Thus there is great simplicity at the ring, where the space-time is conformally flat, the magnitudes of the acceleration and angular velocity vectors are equal, and interestingly, the expression for a_0 is precisely the acceleration of gravity at the surface of a homogeneous Newtonian sphere of density ρ_s and radius r_0. Inside the ring on the equatorial plane we have $\Omega/a>1$, while outside, $\Omega/a<1$. The equation for p_0 shows, however, that the ring itself may be either inside ($\kappa^2<1$) or outside ($\kappa^2>1$) the body. Having the ring inside the body, so that $\Omega/a<1$ at the equator, implies *very* slow rotation for normal objects. For instance, picking values appropriate to a body like the Earth in size and density, one finds that the period of rotation required in order to have the ring at the equator is about 6 yr. To achieve a 1-day period of rotation, the radius of the ring must be approximately 2000 times the radius of the Earth. This peculiar situation results from the fact that, as pointed out previously by Synge,[7] even though rotation is not a very large effect dynamically at the Earth's equator, the dimensionless ratio $\Omega/a\approx2000$.

A rough criterion for the dynamical importance of rotation in these models is found in the rotational velocity parameter v_0:

$$v_0\equiv\Omega_0r_0=\frac{\Omega_0^2r_0}{a_0}=\frac{\Omega_0^2R}{a_0(R/r_0)}\approx\frac{\Omega_R^2R}{a_R}\equiv\frac{\Omega_R}{a_R}v_R, \quad (31)$$

where R is an equatorial radius, and the validity of the approximation step depends on the facts that Ω is nearly constant for rigid rotation, and a is proportional to radius for nearly constant density. Thus, if rotation is important, v_0 must be of order unity, and since usually $v_R\ll1$, the ring must be far outside the physical surface.

We remark in passing that this suggests there may be two essentially different kinds of situations for which the singularity parameters m and b might be left in without harm to provide rotating "shell" solutions.[8] First, the case of very slow rotation with the singular ring inside the inner boundary of a shell or envelope. And second, the rapid-rotation case with the ring far outside the outer boundary of a shell or core. We have not investigated these possibilities in detail, and it may be that the peculiar multiple-valuedness of Kerr-type singularities[9] prohibits either or both of them. A specific difficulty appears in the fact that these parts of h_1 and h_2 contribute terms of first degree in ζ and ξ; such terms tend to produce cusps in the level surfaces on the equatorial plane.

Returning again to the $m=b=0$ case, we can obtain

[7] J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing Co., Amsterdam, 1960), p. 139.

[8] Again, since the matter is perfect fluid, these are not *free-standing* shells; they cannot be self-supporting and must be part of a composite body.

[9] R. H. Boyer and R. W. Lindquist, J. Math. Phys. 8, 265 (1967).

an approximate equation for the $p=0$ surface which is valid for very slow rotation. The details of the calculation are given in the Appendix. It is shown there that for $v_0 \ll 1$, the basic parameters of the metric are given to first order in v_0 by

$$\xi_A{}^2 \approx 1+v_0\,, \quad \kappa^2 = \frac{k^2}{3v_0} \approx \frac{1+v_0}{1+x_P}\,, \quad \delta^2 \approx 1+\frac{3v_0 x_P}{1+x_P}\,,$$

where the constant x_P is defined by

$$\frac{2p_c}{\rho_s} \equiv \frac{x_P}{1+x_P}.$$

We treat the case $v_0 \ll x_P$, which corresponds to a very slowly rotating body with the ring inside ($\kappa^2<1$). If $x_P \ll 1$ also, we have the limit of a slowly rotating Newtonian body.

The intrinsic shape of the $p=0$ surface can best be appreciated by embedding this 2-surface of revolution in Euclidean 3-space. When this is done, the equation for the surface, as shown by Eq. (52) of the Appendix, becomes in cylindrical coordinates (r,z)

$$r^2+(1-\tfrac{4}{5}v_0)z^2 = R^2(1+\tfrac{2}{5}v_0)\,,$$

where R is the radius of the associated spherical body for $v_0=0$. This is the equation of a prolate spheroid, and, since it is prolate even in the Newtonian limit, it is not possible to explain away this result by appealing to some peculiar relativistic effect. There seems to be no escape from the physical conclusion that external matter must be present.

ACKNOWLEDGMENT

It is a pleasure to express my appreciation to Frank Estabrook for constant encouragement and many valuable suggestions during the course of this work.

APPENDIX: SHAPE OF THE $p=0$ SURFACE

We adopt the comoving symmetry coordinate θ as one of the coordinates in the surface. Using a subscript S as before to denote general quantities evaluated on $p=0$, we define a second surface coordinate x (orthogonal to θ) by setting

$$x \equiv h_2(\xi_s)+\frac{\xi_s{}^2}{\kappa^2}-1 = \frac{\xi_s{}^2}{\kappa^2}\left[(1+k^2\xi_s{}^2)^{1/2}\frac{\sinh^{-1}(k\xi_s)}{k\xi_s}-\kappa^2\right]. \quad (32)$$

From the equation for the surface $\phi_s{}^2=\kappa^2$, and the general expression for ϕ^2, Eq. (2), it follows that also

$$x = h_1(\zeta_s)-\frac{\zeta_s{}^2}{\kappa^2}-1 = \frac{\zeta_s{}^2}{\kappa^2}\left[\kappa^2-(1-k^2\zeta_s{}^2)^{1/2}\frac{\sin^{-1}(k\zeta_s)}{k\zeta_s}\right]. \quad (33)$$

The equator, which is described by $\xi_s=0$ or $\zeta_s=0$, depending on its size relative to the ring, is thus given in either case by $x=0$. At the pole ($\xi_s=\xi_A$, $h_{2s}=0$), we find from Eq. (32)

$$1+x_P = \frac{\xi_A{}^2}{\kappa^2}\,, \quad (34)$$

and from the equation for p_c, Eq. (30), we then have

$$\frac{2p_c}{\rho_s} = \frac{x_P}{1+x_P}\,, \quad (35)$$

relating x_P to physical parameters. For $0 \leqq p_c \leqq \frac{1}{3}\rho_c$, the limits on x_P are $0 \leqq x_P \leqq \frac{1}{2}$.

The problem now is to invert Eq. (32) and Eq. (33) to find $\xi_s(x)$ and $\zeta_s(x)$. We do this in an expansion to first order in the rotation parameter v_0. The parameters k, κ, ξ_A, and δ, however, also depend on v_0 and we first find them to this order. Introducing v_0 and using Eq. (34), we can write $k^2=3v_0\kappa^2=3v_0\xi_A{}^2/(1+x_P)$. Treating x_P and v_0 as independent, the defining equation for ξ_A can now be written: $h_2(\xi_A, x_P, v_0)=0$, and we find

$$\xi_A{}^2|_{v_0=0}=1\,, \quad \left.\frac{\partial(\xi_A{}^2)}{\partial v_0}\right|_{v_0=0}=1\,. \quad (36)$$

Thus to first order in v_0 we have

$$\xi_A{}^2=1+v_0\,, \quad \kappa^2=\frac{k^2}{3v_0}=\frac{1+v_0}{1+x_P}\,, \quad \delta^2=1+\frac{3v_0x_P}{1+x_P}. \quad (37)$$

We shall treat the case $v_0 \ll x_P$ which implies a slowly rotating body with the ring inside ($\kappa^2<1$) and near the center. At the equator, then, $\xi_s=0$, while the value of $k\zeta_s$ there is given by the vanishing of the bracket on the right-hand side of Eq. (33). For $v_0=0$, we define

$$k\zeta_s|_{x=0,\, v_0=0} \equiv \sin\mu\,, \quad (38)$$

so that in this case the bracket gives

$$(1+x_P)\mu = \tan\mu\,. \quad (39)$$

In the expansions to follow we shall write the terms of zero order in v_0 exactly, but for simplicity in the first-order terms we use the approximate solution of Eq. (39) for small x_P:

$$\begin{aligned}\mu &\approx (3x_P)^{1/2}(1-\tfrac{3}{5}x_P)\,,\\ \sin\mu &\approx (3x_P)^{1/2}[1-(11/10)x_P].\end{aligned} \quad (40)$$

Only the leading terms of order v_0/x_P and v_0 are retained; all others starting with $v_0 x_P$ are dropped. We thus require in addition x_P sufficiently small; if $v_0 \ll x_P \ll \frac{1}{2}$, we have the limit of a very slowly rotating Newtonian body.

With these stipulations we solve Eqs. (32) and (33), obtaining

$$\begin{aligned} \xi_s^2 &\approx \frac{x}{x_P}\left[1+\frac{v_0}{x_P}\left(1+x_P-\frac{x}{x_P}\right)\right], \\ k\zeta_s &\approx \sin\mu\left[1-\frac{1}{2}\frac{v_0}{x_P}\left(1-\frac{6}{5}x_P-\frac{x}{x_P}\right)\right], \end{aligned} \tag{41}$$

and using these, find for h_{2s} and h_{1s}

$$\begin{aligned} h_{2s} &\approx \left(1-\frac{x}{x_P}\right)\left\{1-\frac{v_0}{x_P}(1+x_P)\frac{x}{x_P}\right\}, \\ v_0 h_{1s} &\approx \tfrac{1}{3}(1+x_P)^2\sin^2\mu\left\{1-\frac{v_0}{x_P}\left[\tfrac{3}{5}x_P-(1+x_P)\frac{x}{x_P}\right]\right\}. \end{aligned} \tag{42}$$

From the comoving quotient metric on the rigidly rotating body,

$$dl^2 = r_0^2(\zeta^2+\xi^2)\left[\frac{d\zeta^2}{(1-k^2\zeta^2)h_1}+\frac{d\xi^2}{(1+k^2\xi^2)h_2} +\delta^2\frac{h_1h_2}{(h_1-h_2)}d\theta^2\right], \tag{43}$$

we find the radius of curvature R_s of a symmetry circle [$x=$ (const), θ varies] on the outside surface to be

$$R_s^2 = \delta^2 r_0^2 \kappa^2 h_{1s} h_{2s} \tag{44}$$

and interval along a meridian [$\theta=$ (const), x varies] to be

$$(dl_s)^2 = r_0^2(\zeta_s^2+\xi_s^2)\left[\frac{(d\zeta_s)^2}{(1-k^2\zeta_s^2)h_{1s}}+\frac{(d\xi_s)^2}{(1+k^2\xi_s^2)h_{2s}}\right]. \tag{45}$$

Using the above expansions, we obtain the equations

$$R_s^2 \approx \frac{(1+x_P)\sin^2\mu}{\rho_s}\left(1-\frac{x}{x_P}\right)(1+\tfrac{2}{5}v_0), \tag{46}$$

$$\left(\frac{dR_s}{dx}\right)^2 \approx \frac{(1+x_P)\sin^2\mu}{4\rho_s x_P^2[1-(x/x_P)]}(1+\tfrac{2}{5}v_0), \tag{47}$$

$$\left(\frac{dl_s}{dx}\right)^2 \approx \frac{(1+x_P)\sin^2\mu}{4\rho_s x_P x[1-(x/x_P)]}\left[1+\tfrac{2}{5}v_0\left(3-2\frac{x}{x_P}\right)\right]. \tag{48}$$

The ratio of Eqs. (47) and (48) with x/x_P replaced by R_s^2 from Eq. (46) gives an intrinsic characterization of this surface of revolution. We note that

$$\left.\left(\frac{dR_s}{dl_s}\right)^2\right|_{x=x_P} = 1, \tag{49}$$

which is the necessary and sufficient condition for elementary flatness at the pole.

Rather than dealing directly with the intrinsic ratio dR_s/dl_s, we can obtain a more perspicuous representation of the surface by embedding it in a Euclidean 3-space, giving an integrated equation for a surface having the same intrinsic ratio. Accordingly, we define a Euclidean coordinate z_s by

$$dz_s^2 \equiv dl_s^2 - dR_s^2, \tag{50}$$

and integrate to find

$$z_s^2 = \frac{(1+x_P)\sin^2\mu}{\rho_s}(1+6/5v_0)\frac{x}{x_P}. \tag{51}$$

Eliminating x between Eqs. (46) and (51), we have then

$$R_s^2 + (1-\tfrac{4}{5}v_0)z_s^2 = \frac{(1+x_P)\sin^2\mu}{\rho_s}(1+\tfrac{2}{5}v_0), \tag{52}$$

which is the equation of a prolate spheroid in cylindrical coordinates.