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Causes of Sound Faster than Light in Classical Models of Ultradense Matter

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A variety of classical theories of ultradense matter give phase and group velocities (c_s) for long-wavelength sound which exceed the speed of light in vacuum. This sound speed, which depends only upon the volume dependence of the total relativistic energy density, can reflect a lack of causality (signals propagate faster than c), or a possiblity of amplification for some higher-frequency sound waves (the medium is not truly in its lowest energy state) which then cancel low-frequency effects outside the light cone. The significance of $c_s > c$ then depends upon the dynamical behavior of the system for higher frequencies. From the exact solution for a sound wave of arbitrary frequency propagating in a one-dimensional lattice of point sources of a neutral vector-meson field, it is shown that $c_s > c$ results from a breakdown of the analog of the Kramers-Kronig relation whenever the computed self-energy of a source exceeds its renormalized mass. The negative bare mass of source particles contributes the possibility of exponentially growing particle accelerations which can lower the energy of the system indefinitely. The response of the lattice can remain causal despite $c_s > c$ when such runaway modes are included in Green's functions. When they are suppressed, noncausality accompanies $c_s > c$. Classical nonlinear field theories which give $c_s > c$ are shown to be noncausal.

I. INTRODUCTION

If matter could support a pressure p greater than its total relativistic energy density ϵ, which includes both rest mass and interaction energies, then in such matter compressional waves would propagate with speed $c_s^2 = c^2 dp/d\epsilon$ exceeding that of light in vacuum. Just this situation has been shown to obtain in various classical models for superdense matter.[1] One such model consists of a system of classical particles which, when stationary, repel each other by a short-range repulsive Yukawa interaction. Although the particles interact through ordinary retarded neutral vector fields, whenever the renormalized particle mass is less than its calculated self-energy (always the case for point particles in three dimensions) then there will be a high-density domain in which $c_s^2 > c^2$. A second group of models which also lead to "superluminal" sound consists of certain classical Lorentz-invariant nonlinear field theories with positive-definite energy which, in the low-density limit, approach the canonical theory of the linear (noninteracting) Klein-Gordon field.

The calculated sound speed c_s refers to the phase velocity of a compressional wave in the long-wavelength limit: $\omega \to c_s k$ as $k \to 0$. Because the frequency ω is proportional to the wave number k, the group velocity is also c_s in this limit. This necessarily means that signal velocities also exceed c only in the usual case where the medium through which the sound wave propagates is in its lowest energy state. Otherwise superluminal high-frequency sound can be amplified sufficiently to destructively interfere with and cancel the low-frequency components outside the light cone. For the particle models of matter which can support superluminal sound, the bare-particle mass, the difference between the observable mass and the self-energy, must be negative. Therefore the model Hamiltonian is not clearly positive definite, so that if $c_s > c$ implies signal velocities greater than c can be discovered only by exploring the propagation of sound at all frequencies.

A Green's function for the propagation of a wave amplitude has poles at wave numbers $k = k(\omega)$ with a residue $f(\omega)/k(\omega)$ which depends both upon the detailed dynamics and also upon which amplitude is being described (a meson-field amplitude, the displacement of sources of this field, etc.). Then the Green's function in a homogeneous isotropic medium is

$$G(r,t) = \int_{-\infty}^{\infty} e^{ik(\omega)r - i\omega t} \frac{f(\omega)}{r} \frac{d\omega}{2\pi}. \qquad (1)$$

We can define a function $\hat{n}(\omega)$ by

$$k(\omega) \equiv \hat{n}(\omega)\omega/c \qquad (2)$$

so that $\hat{n}$ is exactly analogous to the index of refraction for light waves and, in conventional matter, obeys a similar Kramers-Kronig dispersion relation. In all of the models to be considered below, $\hat{n}(\omega) \to 1$ for $|\omega| \to \infty$ along any ray in the upper half plane in complex ω space. The causality condition, $G=0$ for $r > ct$ follows when $\hat{n}(\omega)$ [and $f(\omega)$] have no singularities in the upper half plane. The required analyticity of $\hat{n}(\omega)$, together with its asymptotic limit of unity, leads to the usual dispersion relation

$$\hat{n}(\omega) = 1 + \frac{2}{\pi} \int_0^{\infty} \frac{\omega' \, \mathrm{Im} n(\omega') d\omega'}{\omega'^2 - \omega^2}. \qquad (3)$$

For a medium in its lowest energy state, energy conservation demands $\mathrm{Im}\hat{n}(\omega') \geqslant 0$, so that

$$\hat{n}(0) = 1 + \frac{2}{\pi} \int_0^{\infty} \frac{\mathrm{Im}\hat{n}(\omega') d\omega'}{\omega'} \geqslant 1 \qquad (4)$$

and

$$c_s = c/\hat{n}(0) \leqslant c. \qquad (5)$$

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[1] S. Bludman and M. Ruderman, Phys. Rev. **170**, 1176 (1968).

Violation of the inequality (5) implies any of four possibilities:

(i) the breakdown of causality, so that Eq. (3) is invalid,

(ii) the validity of Eq. (3) (*signals* do not propagate faster than c) but $\mathrm{Im}\hat{n}(\omega)$ is negative for some positive frequencies,

(iii) the simultaneous violation of both Eq. (3) and the positive definiteness of $\mathrm{Im}\hat{n}(\omega)$,

(iv) the existence of a causal Green's function in which neither Eq. (3) nor Eq. (1) is valid. This occurs, for example, when among the possible permitted motions of the system there are modes in which particles accelerate exponentially for infinite times. Such "runaway" solutions are discussed in Sec. IV.

Green's functions are generally not uniquely defined even by the outgoing wave condition that no amplitude exists in the remote past. Then (iv) and (i)–(iii) are not exclusive.

II. NONCAUSALITY FOR EXACT SOLUTIONS OF CLASSICAL PARTICLE MODELS

The ground state for the system of classical particles interacting through neutral vector-meson fields is an ordered lattice.[2] In three dimensions, the dispersion relation for such a lattice is simply exhibited in closed form only for certain limiting domains such as infinitesimal interparticle spacing or the opposite one of solely nearest-neighbor interactions. The Green's function which describes the interaction of a particle with its neighbor a distance a away oscillating with frequency ω is proportional to

$$G(a,t)=\int_{-\infty}^{\infty}\frac{\exp[-(\mu^2-\omega^2)^{1/2}a]}{a}e^{-i\omega t}\frac{d\omega}{2\pi}, \quad (6)$$

with μ the inverse-meson Compton wavelength. When $\mu a\gg 1$, the nearest-neighbor approximation is accurate for the static lattice and thus for c_s, but it is not sensible for high ω when $\omega^2\gtrsim\mu^2$. Similarly, in the opposite approximation of very high density, the infinitesimal particle spacing will ultimately be a bad approximation as $\omega\to\infty$ along the upper imaginary axis.

However, all of the superluminal properties of dense three-dimensional classical matter can also be exhibited for the one-dimensional lattice whenever the renormalized particle mass is less than the (in this case finite) computed self-energy for an isolated particle; and the exact dispersion relation for the one-dimensional lattice can be exhibited in closed form. We consider a one-dimensional lattice composed of particles whose mass (i.e., mass/unit area) in the absence of interactions is m and which have equilibrium separations a. Each particle is coupled to a neutral vector-meson field of mass μ with coupling strength g so that the static nearest-neighbor interaction potential is $g^2e^{-\mu a}$. An infinitesimal longitudinal disturbance of the form

$$y_n=ye^{ikan}e^{-i\omega t} \quad (7)$$

is imposed on the lattice, where $n=0,\pm1,\pm2,\cdots$ designates the nth lattice site. Then the net force on the $n=0$ particle consists of four terms, each multiplied by $ye^{-i\omega t}$. The sum

$$2g^2\sum_{n=1}^{\infty}\left[\mu^2e^{-\mu na}-\frac{(\mu^2-\omega^2)\mu}{(\mu^2-\omega^2)^{1/2}}\times\{\exp[-(\mu^2-\omega^2)^{1/2}na]\cos kna\}\right] \quad (8a)$$

is the net force due to the change in position of the zeroth particle in the potential of all its neighbors together with the effect of the oscillation in position of its neighbors. The sum

$$-2g^2\sum_{n=1}^{\infty}\left[-\frac{\omega^2\mu}{(\mu^2-\omega^2)^{1/2}}\times\{\exp[-(\mu^2-\omega^2)^{1/2}na]\cos kna\}\right] \quad (8b)$$

is the force on the zeroth particle from the oscillating currents of its neighbors. (It is the analog of the $\dot{\mathbf{A}}$ force of electrodynamics.) The term

$$-g^2[\mu^2-(\mu^2-\omega^2)\mu/(\mu^2-\omega^2)^{1/2}] \quad (8c)$$

is the force a charged mass exerts upon itself due to its own oscillating potential. It is the special case of the sum (8a) for the single term $n=0$ and without the factor 2 from pairing of neighbors.

Finally, the term

$$-g^2[-\omega^2\mu/(\mu^2-\omega^2)^{1/2}] \quad (8d)$$

is the self-force from the changing vector potential of the mass sheet interacting with itself.[3] From Eqs. (8a)–

[2] The classical-particle models discussed in this paper and in Ref. 1 describe systems with only a single sign of the g charge. A relativistic quantum field theory for the sources of a neutral vector-meson field must, of course, contain particles of both signs of the charge even in the limit $h\to0$. The condition $\frac{1}{2}g^2>m_rc^2$, which leads to $c_s>c$, means that the maximum binding energy of a classical plus-minus charged pair (g^2) exceeds the sum of their rest masses. Therefore the inclusion, in a classical theory, of pairs with such a large g^2 coupling to the meson field would mean that neither the lattice nor even the vacuum represents a lowest energy state. Indeed, there no longer is one. The enormous qualitative change between the classical theory and the quantum theory of ultradense matter lies in the existence of oppositely charged particles in the latter and their absence in the former. [See Ref. 1; see also M. Broido and J. G. Taylor (to be published) for a discussion of a deduction of causality from relativistic invariance in quantum field theory.]

[3] Equations (8a)–(8d) can be derived from the three-dimensional equations [(2.54 and (2.55)] of Ref. 1 by smearing the neighbors into appropriately oriented sheets. The same result is, of course, obtained by solving the one-dimensional equations directly with the use of the Green's function

$$\delta_{\mu\nu}\int_{-\infty}^{\infty}d\omega\ \{\exp[-(\mu^2-\omega^2)^{1/2}|x|]\}/(\mu^2-\omega^2)^{1/2}$$

applied to the conserved particle currents. Equations (8c) and (8d) include the one-dimensional analog of the Bhabha term for a point source coupled to a meson field in three dimensions.

(8c), the dispersion relation for small-amplitude sound waves (in which particle motion is always nonrelativistic) in the one-dimensional lattice is

$$\frac{M(\omega)\omega^2}{2g^2\mu^2}=\frac{1}{e^{\mu a}-1}-\frac{\mu}{2(\mu^2-\omega^2)}\times\left\{\frac{1}{\exp[(\mu^2-\omega^2)^{1/2}a-ika]}+\text{c.c.}\right\},\quad(9)$$

with

$$M(\omega)=m+\frac{g^2}{\omega^2}\left[\frac{\mu}{(\mu^2-\omega^2)^{1/2}}-1\right].\quad(10)$$

The use of retarded Green's functions dictates that the square-root branch in Eqs. (6) and in the dispersion relations (9) and (10) be chosen as $-i(\omega^2-\mu^2)^{1/2}$ for ω real and greater than μ. Significantly, however, the dispersion relation at low frequencies is obviously independent of which branch of the square root is chosen.

For $\omega^2\to 0$, we have

$$M(\omega)\to m+\tfrac{1}{2}g^2\equiv m_r.\quad(11)$$

Thus, the effective inertial mass for small accelerations, i.e., the renormalized particle mass is $m+\frac{1}{2}g^2$. When $m_r<\frac{1}{2}g^2$, the bare mass is negative.

From Eqs. (9) and (10) the wave number k is given as a function of ω by

$$\cos ka=\tfrac{1}{2}\{\exp[(\mu^2-\omega^2)^{1/2}a]+\exp[-(\mu^2-\omega^2)^{1/2}a]\}-\left[\frac{\exp[(\mu^2-\omega^2)^{1/2}a]-\exp[-(\mu^2-\omega^2)^{1/2}a]}{2(\mu^2-\omega^2)^{1/2}}\right]\times\left[\frac{2\mu}{e^{\mu a}-1}-\frac{m_r\omega^2}{g^2\mu}+\mu+\frac{\omega^2}{2\mu}\right]^{-1}.\quad(12)$$

For $g^2=0$, Eq. (12) gives both $\omega=(k^2+\mu^2)^{1/2}$, the dispersion relation for the free-meson field, and $\omega\equiv 0$ for the uncoupled lattice; for finite g^2, it describes the propagation of the coupled lattice and field. For $\omega\to 0$, Eq. (12) reduces to

$$c_s^2=\eta(\mu a)^2\left(\frac{e^{\mu a}+1}{e^{\mu a}-1}\right)\frac{e^{\mu a}}{(e^{\mu a}-1)^2+\eta[e^{\mu a}(\mu a+1)-1]},\quad(13)$$

with

$$\eta\equiv g^2/m_r.\quad(14)$$

As it must, this agrees with the sound speed calculated in Ref. 1 directly from the static energy per particle and its derivative with respect to the interparticle spacing a.

In order that the one-dimensional Green's function, which is analogous to Eq. (2), correspond to retarded (outgoing) waves, that root of Eq. (12) is chosen for which real k and ω have the same sign at low frequencies. For $\omega\to\infty$ in the upper half plane, it then follows from Eq. (12) that $k(\omega)\to\omega/c$. For the Green's function to vanish outside the light cone (everywhere for the meson field and at all lattice points for sound-wave amplitude describing the source displacement), $k(\omega)$ should be analytic in the upper half plane. From Eq. (12), the singularities of $k(\omega)$ are as follows:

(i) When $m_r>\frac{1}{2}g^2$, $k(\omega)$ is analytic in the upper half plane. $\text{Im}\hat{n}(\omega)=\text{Im}(k/\omega)$ is never negative for real positive ω. The quantity $\text{Im}\hat{n}$ vanishes for real ω except for ω in the neighborhood of $\omega=\pm W$, where

$$W^2=\mu^2\left(\frac{e^{\mu a}+1}{e^{\mu a}-1}\right)\left(\frac{2g^2}{2m_r-g^2}\right).\quad(15)$$

Therefore, a conventional Kramers-Kronig-type dispersion relation for $\hat{n}(\omega)$ guarantees causality.

(ii) When $m_r<\frac{1}{2}g^2$, $\text{Im}\hat{n}(\omega)=0$ for all real ω. There are now branch points in $k(\omega)$ and $e^{ik|x|}$ on that part of the upper imaginary ω axis where $\cos ka=\pm 1$, below the point $\omega=i|W|$ where $\cos ka$ has a pole. Therefore, in this case, $c_s^2>c^2$ directly reflects a breakdown of causality rather than any high-frequency negative absorbtivity. [In the high-frequency limit, $k(\omega)\to(\omega^2-\mu^2)^{1/2}$ for either sign of $m_r-\frac{1}{2}g^2$.]

III. OTHER CASES

That a lack of causality follows from Eq. (12) is not surprising, since, according to Eqs. (9) and (10), even a single isolated mass may preaccelerate before a sharp impulse reaches it. When self-interaction is included, then the acceleration of a single isolated mass in one dimension from an external force $F\delta(t)$ is

$$\ddot{y}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{Fd\omega\,e^{-i\omega t}}{m_r-\frac{1}{2}g^2+(g^2/\omega^2)[\mu(\mu^2-\omega^2)^{-1/2}-1]}.\quad(16)$$

For $m_r>\frac{1}{2}g^2$, a contour integration gives $\ddot{y}(t)=0$ for all $t<0$. But for $0<m_r<\frac{1}{2}g^2$, the case which also gives superluminal sound in the very dense lattice, the residue from the pole on the imaginary axis where the denominator of the integrand in Eq. (16) vanishes gives an exponentially growing preacceleration at negative t. This is, of course, a well-known phenomenon in three-dimensional classical electrodynamics, where preacceleration (or an indefinitely increasing "runaway" acceleration) is the response of an electron to a sharp impulse.

However, although it is present in the equation of motion of each lattice particle, this microscopic preacceleration is not a sufficient cause for $c_s>c$. In classical electrodynamics, it has indeed been shown that over macroscopic distances an array of electrons does not propagate a signal faster than light; in that case the net effect of preacceleration on propagating a light signal outside of the light cone diminishes as the inverse square root of the number of electrons through which a signal has been transmitted.[4] For the one-dimensional lattice with meson interactions, the frequency-dependent parts of the self-interaction which cause the nonlocality for single-particle response can be omitted without changing c_s^2 or necessarily restoring causality. For example, suppose that, instead of Eq. (10), we ignore this nonlocality by approximating

$$M(\omega)=m+\tfrac{1}{2}g^2=m_r, \tag{17}$$

which omits the effects of the dependence of the range and amplitude of the self-field on frequency and, at higher frequencies, all consequences of radiation-damping. The response of such a particle to an external impulse no longer has any preacceleration. In terms of m_r, c_s^2 is unchanged. The wave vector $k(\omega)$ is now given by

$$\cos ka=\tfrac{1}{2}\{\exp[(\mu^2-\omega^2)^{1/2}a]+\exp[-(\mu^2-\omega^2)^{1/2}a]\}$$
$$-\tfrac{1}{2}\{\exp[(\mu^2-\omega^2)^{1/2}a]-\exp[-(\mu^2-\omega^2)^{1/2}a]\}$$
$$\times\left[1+2(\mu^2-\omega^2)^{1/2}\left(\frac{1}{e^{\mu a}-1}-\frac{m_r\omega^2}{2g^2\mu^2}\right)\right]^{-1}. \tag{18}$$

In this case, the qualitative behavior of $e^{ik(\omega)a}$ in the upper half plane is independent of the ratio (m_r/g^2). Regardless of whether $m=m_r-\frac{1}{2}g^2$ is positive or negative, there always exist branch points in the upper half plane and an $\mathrm{Im}k(\omega)$ which, for real positive ω, has negative parts. Thus, for real $\omega\to\infty$, we have

$$k(\omega)\to\omega-i\frac{\mu^2}{a\omega^2}\left(\frac{g^2\mu^2}{m_r\omega^2}\right)^2. \tag{19}$$

A pair of branch points of $e^{ik|x|}$ in the upper half plane approach the real axis as $\mu a\to 0$. For $\mu a\ll 1$, they are at

$$\omega=\pm(g^2\mu/m_r a)^{1/2}+\tfrac{1}{2}i\mu^2 a. \tag{20}$$

Therefore, the reason for $c_s^2>c^2$ lies in a combination of noncausality and $\mathrm{Im}\hat{n}(\omega)$ becoming negative. When $m_r<\frac{1}{2}g^2$ the noncausality remains in the above approximation, even though c_s^2 is now less than c^2. For contrived choices of the frequency-dependent self-interaction other than those of Eqs. (10) and (17), the dispersion relation for $\hat{n}(\omega)$ can be retained and the entire burden for $c_s^2>c^2$ put into negative regions for $\mathrm{Im}\hat{n}(\omega)$ rather than any breakdown of causality. However, such *ad hoc* choices are not suggested by the physical models.

IV. ORIGIN OF NONCAUSALITY: RUNAWAY SOLUTIONS

In the models of Secs. II and III, the Kramers-Kronig dispersion relation for $k(\omega)$ failed because of branch cuts in the upper half ω plane beginning where $\cos ka=\pm 1$. These correspond to runaway lattice modes with real k, for which particle displacements, which vary in phase among themselves, grow exponentially in time. The particle displacement amplitude is given by

$$y_n\sim e^{iakn}e^{-i\mathrm{Re}\,\omega t}e^{|\mathrm{Im}\,\omega|t}. \tag{21}$$

Such runaway modes occur because the lattice is really not the lowest possible energy state of a system of particles whose bare mass is negative. Although a lower energy state is not achieved for any geometrical rearrangement or steady-state motions, it is achieved by a sufficiently rapid particle acceleration which partially decouples the particle from its self-field and thus reveals the negative mechanical mass. That this will occur follows from the spatial dependence of the oscillating field amplitude ϕ caused by a (point) source oscillating with frequency ω:

$$\phi=g^2\exp[-(\mu^2-\omega^2)^{1/2}r]/r. \tag{22}$$

Real frequencies ($\omega^2>0$) expand the range of the self-field and increase the total integrated field energy so that it exceeds that of the source at rest. But, if the source instead of oscillating accelerates exponentially ($\omega^2<0$), then the range of the field amplitude and consequently the integrated field energy decrease. In a one-dimensional world, both the range and amplitude decrease (cf. Ref. 3). For sufficiently large imaginary ω, the sum of the negative bare mass and the diminishing positive field self-energy becomes negative, so that the source responds like a particle of negative mass. In particular, its kinetic energy approaches negative infinity as it continues to accelerate and radiate.

In writing the Green's function in the form of Eq. (1), it is assumed that relevant motions of the lattice have Fourier transforms in space and time. This excludes runaway modes. However, the motion of the lattice particles is not completely described by simple second-order differential equations and there is a variety of independent Green's functions. A completely causal one can be constructed by including the runaway solutions. It is only by invoking the final (future) condition, that motions of the lattice will not ultimately diverge, that the causal Green's function is suppressed and the noncausal one given unique status.

An analogous and mathematically more transparent situation arises in the classical electrodynamics of point particles. The response of an electron of renormalized mass m_r and charge e to a weak electric field E is given by[5]

$$e\mathbf{E}=m_r\ddot{\mathbf{x}}-\frac{2e^2}{3c^3}\frac{d^3\mathbf{x}}{dt^3}. \tag{23}$$

Then a transverse electromagnetic wave of frequency ω and wave number k moving in an uncorrelated gas

[4] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945).

[5] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938); see also F. Rohrlich, *Classical Charged Particles* (Addison-Wesley Publishing Co., Reading, Mass., 1965), and also Ref. 4.

of n such electrons per unit volume satisfies

$$[c^2k^2-\omega^2+\omega_p^2/(1+i\omega\tau)]\mathbf{E}=0\,, \tag{24}$$

where

$$\omega_p^2=4\pi ne^2/m_r\,, \tag{25}$$

and

$$\tau=2e^2/3m_rc^3. \tag{26}$$

The wave number dispersion relation $k=k(\omega)$ gives an (infinite) branch point at $\omega=i/\tau$, so that the conventional Green's function of the form of Eq. (1) is noncausal. The space integral of the Green's function which is obtained by Fourier-transforming Eq. (24) with a $\delta(t)$ source on the right-hand side is

$$\int d\mathbf{r}\, G(\mathbf{r},t)=1-\tau\frac{d}{dt}G_0(t)\,, \tag{27}$$

with

$$G_0(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega t}d\omega}{-\omega^2(1+i\omega\tau)+\omega_p^2}. \tag{28}$$

In the regime $\omega_p\tau\ll1$,

$$\begin{aligned} G_0(t)&=\tau e^{t/\tau}\,, && \text{for} \quad t<0 \\ &=e^{-\omega_p^2\tau t/2}\left[\frac{\sin\omega_pt}{\omega_p}+\tau\cos\omega_pt\right], && \text{for} \quad t>0. \end{aligned} \tag{29}$$

A causal Green's function which satisfies the same differential equation has, instead of Eq. (29),

$$\begin{aligned} G_0^c(t)&=0\,, && \text{for} \quad t<0 \\ &=\tau\cos\omega_pt+\frac{\sin\omega_pt}{\omega_p}-\tau e^{t/\tau}, && \text{for} \quad t>0. \end{aligned} \tag{30}$$

This causal Green's function has no Fourier transform.

We have then the situation that $c_s>c$, caused by negative bare mass for particles, certainly implies a highly peculiar sort of matter. Exactly what kind depends both upon the description of higher-frequency modes and upon what lattice motions are acceptable in the theory. The negative bare mass, which is sufficient for the static result $p>\epsilon$ and $c_s>c$, also causes runaway modes with no lower bound for the particle's kinetic energy. Admitting these modes into the construction of Green's functions permits the retention of causality. They can be arbitrarily excluded only by admitting noncausality into the theory. General arguments that $c_s\leqslant c$ which assume a lowest energy[2] or steady asymptotic states amenable to thermodynamic considerations[6] would seem to have uncertain application to matter with negative bare masses which has this sort of instability.

[6] E. C. G. Stueckelberg de Breidenbach, Helv. Phys. Acta **35**, 568 (1962). I am indebted to Dr. John Bell for this reference.

V. CLASSICAL NONLINEAR FIELD THEORIES

In Ref. 1, it was shown that the Lagrangian $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_I$, where

$$\mathcal{L}_0=\psi_\mu{}^\dagger\partial^\mu\phi+\partial_\mu\phi^\dagger\psi_\mu-|\psi_\mu|^2-\mu^2|\phi|^2\,, \tag{31}$$

$$\mathcal{L}_I=-gf(j_\mu{}^2)\,, \tag{32}$$

with conserved current

$$j_\mu=\tfrac{1}{2}i(\phi^\dagger\psi_\mu-\psi_\mu{}^\dagger\phi)\,, \tag{33}$$

gives a positive-definite energy, the Klein-Gordon equation in the low-density limit, and a sound wave for which $c_s^2>c^2$ at ultrahigh densities for particular choices of the function f. The acoustic mode satisfies the frequency–wave-number relation

$$\omega^2=\tfrac{1}{2}(\Omega_1-\Omega_2)\,, \tag{34}$$

where

$$\Omega_1=4\mu^2+(D+c^2)k^2\,, \tag{35}$$

$$\Omega_2=\left\{[4\mu^2+(c^2-D)k^2]^2+16\frac{c^2k^2\mu^3}{\omega_0}\right\}^{1/2}, \tag{36}$$

$$D=c_s^2+c^2\mu/\omega_0\,, \tag{37}$$

and

$$\omega_0=\mu+gj_0f(j_0^2)>\mu. \tag{38}$$

Then

$$k^2(\omega)=\{\omega^2D+\omega^2c^2-4\mu^2c_s^2+[(\omega^2D+\omega^2c^2-4\mu^2c_s^2)^2-4c^2D(\omega^4-4\mu^2\omega^2)]^{1/2}\}/2c^2D. \tag{39}$$

In the limit $\omega\to\infty$, $k(\omega)\to\omega/c$. The wave number $k(\omega)$ is real for all real ω, regardless of whether c_s^2 is greater than or less than c^2. Therefore, since $\hat{n}(\omega)$ is not unity for all ω, the function $k(\omega)$ must have singularities in the upper half ω plane, even if the density is sufficiently low that the sound wave is not superluminal.

For $c_s^2<c^2$, $k(\omega)$ has a pair of branch points on the upper imaginary axis of ω space. These come together at

$$\omega=i2(\omega_0\mu c)^{1/2} \tag{40}$$

when the density is increased sufficiently that $c_s^2=c^2$. At higher densities, where $c_s^2>c^2$, the branch points move off to the right and left of the imaginary axis. Therefore the classical nonlinear field theory which gives $c_s^2>c^2$ at ultrahigh densities is not only noncausal in that regime but also in the low-density one, where $\omega<ck$ for all real k. There are no runaway solutions ($\text{Im}\,\omega>0$) for k real.

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