This agrees closely with the ratio of neutron reduced widths for the $^{15}$C(g.s.) to $^{15}$C(0.745 MeV), viz., 0.093/0.045. The ratio of neutron to proton reduced widths is about 0.6. The Clebsch-Gordan coefficients for coupling isospin would lead one to expect a value of 3 for this ratio, so that our experimental value disagrees by a factor of about 5. However, Macfarlane and French[16] have pointed out that the reduced width deduced from stripping data is "usually smaller by a factor of 4 or 5 than would be expected on the basis of some reasonable potential-well model of the nucleon transfer process, even in cases in which the overlap factor $S$ should be close to unity." Thus the apparent "disagreement" by a factor of 5 may be regarded as additional empirical justification for the statement by Macfarlane and French.

We wish to make it clear that this level (produced at $E_p=2.49$ MeV) is distinct from the nearby $\frac{5}{2}^+$ level, which is excited at $E_p=2.46$ MeV, since there has been some confusion in the literature on this point. Sanders,[7] using earlier data which he cites, gave a $\frac{3}{2}^-$ assignment to the 2.46-MeV level, but later analysis of $^{11}$B$(\alpha,p)^{14}$C at $E_\alpha=2.06$ MeV yields[18] a specific $\frac{5}{2}^+$ assignment and requires $T=\frac{1}{2}$.

We therefore conclude that there are two neighboring states in $^{15}$N, at $E_x=12.51$ and 12.54 MeV, and that each of these is a $\frac{5}{2}^+$ state; however, the isospins are, respectively, $\frac{1}{2}$ and $\frac{3}{2}$.

[18] L. L. Lee, Jr., and J. P. Schiffer, Phys. Rev. **115**, 160 (1959)

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# Ground State of Three Alpha Particles*

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The Faddeev equation is applied to solve for the energy of three $\alpha$ particles using a local static two-body potential obtained by fitting scattering phase shifts. A single bound state is found with a binding energy of 2.79 MeV. Although this value is not very close to the ground-state energy of C$^{12}$, the existence of a 3-$\alpha$ bound state with a binding energy of several MeV indicates that the main structure of C$^{12}$ is like a composite of three $\alpha$ particles. The effect of inelastic processes is estimated in a rough approximation by using a two-channel two-body potential. It is shown that they can easily increase the binding energy of the 3-$\alpha$ bound state by several MeV.

A THREE-$\alpha$-PARTICLE model of C$^{12}$ has been suggested by Harrington,[1] who solved the Faddeev equation with a separable (nonlocal) two-body potential obtained by fitting the $s$-wave $\alpha$-$\alpha$ scattering length and effective range with the Coulomb effects removed.

Recently, a systematic method for solving the Faddeev equation with local potentials was presented by Ball and one of us (D. Y. W.).[2] Here we apply this method to investigate the possibility of 3-$\alpha$ bound states using the phenomenological $\alpha$-$\alpha$ potential obtained by Darriulat *et al.*[3] by fitting the scattering phase shifts up to 120-MeV laboratory kinetic energy. This potential has the form

$$V(r)=U_1\{1+\exp[(r-r_1)/a_1]\}^{-1} - U_2\{1+\exp[(r-r_2)/a_2]\}^{-1} + 4e^2/r + iW(r)\Theta(E_L-40\text{ MeV}). \quad (1)$$

We remark here that the Faddeev equation requires the knowledge of the two-body $T$ matrix at energies below the threshold and therefore the imaginary part is absent. However, the $\Theta$ function is not analytic and the error in the continuation of the potential as a function of the energy can be a major source of uncertainty in the value of the three-body binding energy.

In this paper, we are addressing ourselves to the question of how closely the ground state of C$^{12}$ can be described as a composite of three rigid $\alpha$ particles. Hence, we must use a static two-body potential such as the real part of that given by (1). Since the absorption term originates from other channels, such as (Li$^7$+$p$),

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[1] D. R. Harrington, Phys. Rev. **147**, 685 (1966).
[2] J. S. Ball and D. Y. Wong (to be published).
[3] P. Darriulat *et al.*, Phys. Rev. **137**, B315 (1965).

TABLE I. Numerical values of the parameters for the $\alpha$-$\alpha$ potential as defined in (1).

| $l$ | $U_1$ (MeV) | $a_1$ (F) | $r_1$ (F) | $U_2$ (MeV) | $a_2$ (F) | $r_2$ (F) |
|---|---|---|---|---|---|---|
| 0 | 150 | 0.1 | 1.65 | 9.2 | 0.4 | 3.72 |
| 2 | 150 | 0.05 | 1.63 | 16.0 | 0.3 | 3.55 |

($He^3+He^5$), etc., one must solve a many-channel problem for $C^{12}$ as well as for the two-$\alpha$ states if the velocity dependence of the two-body potential were to be taken into account properly. Instead we solve the static problem and let the difference between our result and the experimental value be a measure of the importance of the absorptive channels. At the end we shall briefly discuss an approximation to the multichannel problem and show that inelasticity tends, as predicted, to increase the binding energy.

The parameters of the potential (1) are taken from Ref. 3 and are tabulated in Table I. We mention here that the Coulomb force presents no difficulty because the three-body energy is below threshold.

Using this potential we first compute the two-body $T$ matrix as follows:

$$V_l(p,q)\equiv -m^{3/2}\int_0^\infty dr\, r^2 V(r) j_l(m^{1/2}pr) j_l(m^{1/2}qr), \quad (2)$$

where $j_l(x)$ are spherical Bessel functions, $p$ and $q$ are the center-of-mass momenta divided by $m^{1/2}$, and $m$ is the mass of the $\alpha$ particle.

$$t_l(p,q;E)=V_l(p,q)+\frac{1}{\pi}\int_0^\infty dk^2 \frac{kV_l(p,k)t_l(k,q;E)}{k^2-E}. \quad (3)$$

Let $\varphi_{nl}(p;E)$ be eigenfunctions satisfying

$$\lambda_n(E)\varphi_{nl}(p;E)=\frac{1}{\pi}\int_0^\infty dk^2 \frac{kV_l(p,k)\varphi_{nl}(k;E)}{k^2-E}. \quad (4)$$

Then, for negative values of $E$, the $\varphi_{nl}(p;E)$ form a complete orthonormal[4] set and can be obtained by standard numerical methods. The two-body $T$ matrix is then given by

$$t_l(p,q;E)=\sum \frac{\lambda_{nl}(E)}{1-\lambda_{nl}(E)}\varphi_{nl}(p;E)\varphi_{nl}(q;E). \quad (5)$$

The on-shell $T$ matrix is normalized so that

$$t_l(p,p;p^2)=(e^{i\delta_l}\sin\delta_l)/p.$$

In the three-body Faddeev equation the two-body $T$ matrix plays the part of a potential in the two-body Lippmann-Schwinger equation. Following the notation of Ref. 2, we have for the zero total angular momentum state a set of coupled integral equations:

$$\Psi_l(p,q;s)=\Phi_l(p,q;s)+2\sum_{l'}\int_0^\infty dq'^2\int_{\frac{1}{3}(q'-2q)^2}^{\frac{1}{3}(q'+2q)^2} dp'^2 \times\frac{[(2l+1)(2l'+1)]^{1/2}P_l(z)P_{l'}(z')t_l(p,\bar{p};s-q^2)}{\sqrt{3}\pi q(p'^2+q'^2-s)}\times\Psi_{l'}(p',q';s), \quad (6)$$

where $\Psi_l(p,q;s)$ is the three-body $T$-matrix element with a final state consisting of two particles in a relative $l$ orbital state and the third particle in an $l$ state relative to the center of mass of the first two. The quantity $m^{1/2}p$ is the magnitude of the relative momentum of the first two particles, $m^{1/2}q$ is the magnitude of the momentum of the third particle in the three-body center-of-mass frame, and $s$ is the total energy of the three particles. The initial state is arbitrary.

The inhomogeneous term $\Phi_l(p,q;s)$ is the symmetrized $T$ matrix with one noninteracting particle. $t_l(p,\bar{p};s-q^2)$ is the two-body $T$ matrix as defined above:

$$\bar{p}^2=p'^2+q'^2-q^2,$$
$$z=[(q'^2-q^2)+3(q^2-p'^2)]/(2\sqrt{3}\bar{p}q),$$
$$z'=(-4q^2+3p'^2+q'^2)/(2\sqrt{3}p'q').$$

By using the separable expansion of $t_l$ given by (5), the Faddeev equation (6) is reduced into a system of coupled integral equations with only one integration variable. Since the $p$ dependence of $(\Psi_l-\Phi_l)$ is given entirely by $\varphi_{nl}(p,s-q^2)$, we can write $\Psi_l$ in the form

$$\Psi_l(p,q;s)=\Phi_l(p,q;s)+\sum_n\left[\frac{\lambda_{nl}(s-q^2)}{1-\lambda_{nl}(s-q^2)}\right]\varphi_{nl}(p;s-q^2)\chi_{nl}(q;s), \quad (7)$$

with the function $\chi_{nl}(q;s)$ satisfying the integral equations

$$\chi_{nl}(q;s)=\eta_{nl}(q;s)+\sum_{n',l'}\int_0^\infty dq'^2 K_{nl,n'l'}(q,q';s)\chi_{n'l'}(q';s), \quad (8)$$

where

$$\eta_{nl}(q;s)=2\sum_{l'}\int_0^\infty dq'^2\int_{\frac{1}{3}(q'-2q)^2}^{\frac{1}{3}(q'+2q)^2} dp'^2\frac{[(2l+1)(2l'+1)]^{1/2}P_l(z)P_{l'}(z')\varphi_{nl}(\bar{p};s-q^2)\Phi_{l'}(p',q';s)}{\sqrt{3}\pi q(p'^2+q'^2-s)}, \quad (9)$$

$$K_{nl,n'l'}(q,q';s)=2\int_{\frac{1}{3}(q'-2q)^2}^{\frac{1}{3}(q'+2q)^2} dp'^2\frac{[(2l+1)(2l'+1)]^{1/2}P_l(z)P_{l'}(z')\varphi_{nl}(\bar{p};s-q^2)\lambda_{n'l'}(s-q'^2)\varphi_{n'l'}(p';s-q'^2)}{\sqrt{3}\pi q(p'^2+q'^2-s)(1-\lambda_{n'l'}(s-q'^2))}. \quad (10)$$

[4] See Eq. 4 of Ref. 2 for the definition of the orthonormality property of the $\varphi_{nl}$.

TABLE II. Values of the $C^{12}$ ground-state energy as a function of the number of terms kept in the expansion of the two-body $T$ matrix and of the number of the two-body partial waves.

| Two-body $l$ values | Maximum $n$ | $E_B$ (MeV) |
|---|---|---|
| 0 | 1 | 2.15 |
| | 2 | 2.20 |
| | 3 | 2.20 |
| | 4 | 2.20 |
| (0,2) | (2,1) | 2.79 |

Keeping a finite number of terms in the sum over $n'$ and $l'$, Eq. (8) becomes an ordinary Fredholm equation, which can be solved by numerical methods. In particular, one can find the energy $s$ for which the eigenvalue of the kernel is unity and, therefore, the homogeneous equation possesses a solution. This energy would then correspond to a bound state of the three-body system.

With the two-body potential given above, it is found that the three-body ground-state energy can be calculated to $\sim 95\%$ accuracy by keeping only $n=1$, 2 for $l=0$, $n=1$ for $l=2$, and neglecting $l\geq 4$ contributions. Qualitatively, the $l=0$ two-body eigenvalues $\lambda_{n0}$ changes sign for $n$ around 3 because of the cancellation between the attractive and the repulsive nuclear forces. Contributions from $n\geq 3$ become very small. For the $d$ and higher waves, the centrifugal-barrier effect appears squared because when two of the particles are in a relative $l$ state the third particle must also be in an $l$ state relative to the center of mass of the first two in order to have a zero total angular momentum. This reduces drastically the higher waves contribution. In fact, the $d$-wave contribution is quite small even though the two-body potential gives rise to a rather low-energy (2.9 MeV) $d$-wave resonance. The results for the three-body ground-state energy are summarized in Table II. We find one and only one bound state and the binding energy is $\sim 2.79$ MeV, to be compared with the experimental value of 7.28 MeV.[5]

Although the binding energy we calculated is not very close to that of $C^{12}$ with respect to disintegration into three $\alpha$'s, the existence of a three-$\alpha$ bound state is of some interest. (a) With the same two-body potential, one finds that the two-$\alpha$ system is not bound but the three-$\alpha$ system is. (b) A small percentage increase ($\sim 10\%$) of the attractive part of the phenomenological potential will shift the binding energy close to that of $C^{12}$. One may argue that the effect of two-body forces in a three-body system is more attractive than that for the two body because of the closed-channel contribution. (c) The three-$\alpha$ system may be used as a first-order approximation for describing the $C^{12}$ nucleus. It seems likely that by including a higher-mass channel such as $(\alpha,L_i^7,p)$ with the two-body interactions adjusted to fit both the real and the imaginary parts of the $\alpha$-$\alpha$ phase shift, the structure of $C^{12}$ will be rather well represented by the coupled three-body systems.

[5] F. Ajzenberg-Selove and T. Lauritsen, Nucl. Phys. **11**, 1 (1959).

TABLE III. Numerical values of the parameters of the two-channel $\alpha$-$\alpha$ potential as defined in (11).

| $\lambda_1^2$ | $\lambda_2^2$ | $U_1$ (MeV) | $a_1$ (F) | $r_1$ (F) | $U_2$ (MeV) | $a_2$ (F) | $r_2$ (F) |
|---|---|---|---|---|---|---|---|
| 0.65 | 0.35 | 240 | 0.06 | 1.5 | 12 | 0.4 | 3.72 |

In order to obtain a rough estimate of the influence of inelastic processes in the binding energy of the three-$\alpha$ system, we have fitted the $\alpha$-$\alpha$ elastic and inelastic $s$-wave phase shifts by using a phenomenological two-channel potential of the form

$$V_{ij}(r)=\lambda_i\lambda_j V(r)\,, \tag{11}$$

where $i$ and $j=1,2$ and $V(r)$ has the same form as the real part of the potential defined by (1).

Then the Lippman-Schwinger equation transforms into

$$\tilde{t}_l(p,q;E)=V_l(p,q)+\sum_{n=1}^{2}\frac{\lambda_n^2}{\pi}\int_0^\infty dk^2\frac{kV_l(p,k)\tilde{t}_l(k,q;E)}{k^2-E-\epsilon_n}\,, \tag{12}$$

where $V_l(p,q)$ is the Fourier-Bessel transform of $V(r)$ as defined in (2); $\epsilon_2=20$ MeV is the center-of-mass threshold energy of the second channel and $\epsilon_1=0$. The two-body $T$ matrix is

$$(t_l)_{ij}(p,q;E)=\lambda_i\lambda_j\tilde{t}_l(p,q;E)\,.$$

The values of $\lambda_1$, $\lambda_2$, and the parameters of the potential $V(r)$ are given in Table III. With this two-body $T$ matrix, the Faddeev equation is solved neglecting the interaction between the $\alpha$ particle and the constituents of the second channel, but keeping the complete two-body $T$ matrix. The energy of the ground state of $C^{12}$ thus obtained is $\sim 5.0$ MeV, which indicates that the inelastic processes tend, as expected, to increase the binding energy of the three-$\alpha$ particles closer to the experimental value of the ground-state energy of $C^{12}$.