Practical Method for the Treatment of Nuclear Collective Motions*

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A practical method using projection operators is proposed for the treatment of nuclear collective motions in discrete as well as continuous energy regions. Mathematical techniques which have been developed by Feshbach and his collaborators in formulating nuclear reactions are fully made use of. As a special case this method includes an approximation procedure using a commutator of the nuclear Hamiltonian and the relevant transition operator, previously developed and applied to the study of hindered β transitions. The relationship between the present method and the one due to Tomonaga is discussed.

1. INTRODUCTION

AFTER the so-called Bohr-Mottelson model appeared, several attempts[1-3] were made to place it on a theoretical basis starting from the nuclear forces. Following the elementary theory due to Tomonaga,[1] we expand the potential part V of the nuclear Hamiltonian H into a Taylor series with respect to the collective coordinate ξ:

$$H=T+V=T+V^{(0)}+V^{(1)}\xi+\tfrac{1}{2}V^{(2)}\xi^2+\cdots. \quad (1.1)$$

When the coefficient $V^{(2)}$ of the ξ^2 term is replaced by an expectation value $\langle V^{(2)}\rangle$ over internal coordinates, the term $\frac{1}{2}\langle V^{(2)}\rangle\xi^2$ gives the potential energy for the collective motions.

For actual collective motions of nuclei, the collective coordinate ξ is supposed to be a quite complicated velocity-dependent one. It is then more practical to treat the collective state $\xi|i\rangle$ rather than ξ itself, where $|i\rangle$ represents a nuclear state. By applying the particle-hole picture[4] to the state $\xi|i\rangle$, the shell aspect can be incorporated into the collective model.

It is simple to separate the collective part from H, by defining the projection operators, P and Q, as

$$P=1-Q=\frac{\xi|i\rangle\langle i|\xi^\dagger}{\langle i|\xi^\dagger\xi|i\rangle}. \quad (1.2)$$

H can be decomposed into

$$H=H_{QQ}+(H_{PQ}+H_{QP})+H_{PP}, \quad (1.3)$$

where $H_{PP}=PHP$, etc. It is apparent that H_{PP} represents the collective Hamiltonian

$$H_{PP}=P(E_i+\Delta), \quad (1.4a)$$

where

$$\Delta=\langle i|\xi^\dagger[H,\xi]|i\rangle/\langle i|\xi^\dagger\xi|i\rangle \quad (1.4b)$$

and

$$H|i\rangle=E_i|i\rangle. \quad (1.4c)$$

The eigenstate $\xi|i\rangle$ of H_{PP} has the energy eigenvalue $E_i+\Delta$. The second term of (1.3), $H_{PQ}+H_{QP}$, represents couplings between the collective state $\xi|i\rangle$ and the other states:

$$\begin{aligned}H_{QP}&=(H_{PQ})^\dagger\\&=\{[H,\xi]-\xi\Delta\}|i\rangle\langle i|\xi/\langle i|\xi^\dagger\xi|i\rangle \quad (1.5a)\\&=Q[H,\xi]|i\rangle\langle i|\xi/\langle i|\xi^\dagger\xi|i\rangle. \quad (1.5b)\end{aligned}$$

Therefore, if the random phase approximation (RPA)[3] (or the Ahrens-Feenberg approximation[5,6])

$$\{[H,\xi]-\xi\Delta\}|i\rangle=0 \quad (1.6)$$

is valid with good accuracy, the coupling terms $H_{PQ}+H_{QP}$ vanish in (1.3). Of course, higher collective modes can also be treated, if necessary, by defining suitable projection operators to replace (1.2).

The purpose of this work is to develop a theory of collective motions starting from the Hamiltonian (1.3) with some modifications. For that purpose several manipulations using projection operators are made by following Feshbach and his collaborators,[7,8] who developed such mathematical techniques in the course of formulating nuclear reactions.

In Secs. 2 and 3 it is shown that, under suitable conditions, an approximation method using a commutator of the nuclear Hamiltonian and the relevant transition operator can be derived, as previously proposed by Ikeda and the present author[6,9,10] and extensively

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applied to the study of hindered β transitions.[6,11–15] The possibility of extending the method to diagonal nuclear matrix elements is also surveyed in Sec. 4. Relationship between the present and other methods is discussed in Sec. 4.

2. PRELIMINARY REMARKS

Before treating collective motions, we shall make remarks on some aspects of the conventional nuclear theories and show the existence of alternative approaches.

A. Conventional Method

Suppose that two model wave functions, $|i\rangle^0$ and $|f\rangle^0$, are given in place of the true ones, $|i\rangle$ and $|f\rangle$, respectively, for which

$$|i\rangle=|i\rangle^0+|\delta i\rangle, \qquad (2.1a)$$

$$|f\rangle=|f\rangle^0+|\delta f\rangle, \qquad (2.1b)$$

$$\langle i|i\rangle=\langle f|f\rangle=1, \qquad (2.1c)$$

and

$$\langle\delta i|i\rangle^0=\langle\delta f|f\rangle^0=0. \qquad (2.1d)$$

We discuss the possible error due to $|\delta i\rangle$ or $|\delta f\rangle$ in calculating $\langle f|m|i\rangle$:

$$\delta\langle f|m|i\rangle=\langle\delta f|m|i\rangle+\langle f|m|\delta i\rangle. \qquad (2.2)$$

We assume that

$$\langle\delta i|\delta i\rangle\leqq\epsilon^2 \qquad (2.3a)$$

and

$$\langle\delta f|\delta f\rangle\leqq\epsilon^2, \qquad (2.3b)$$

where ϵ is a sufficiently small number, say, of the order $\frac{1}{10}$. (See also Appendix.)

Now let us discuss $\langle\delta f|m|i\rangle$ in (2.2). It can be shown that

$$|\langle\delta f|m|i\rangle|^2\leqq\epsilon^2\langle i|m^\dagger m|i\rangle. \qquad (2.4)$$

A proof of (2.4) is given as follows. Defining projection operators P and Q [using the same notations as in (1.2)] to be

$$P=1-Q=\frac{m|i\rangle\langle i|m^\dagger}{\langle i|m^\dagger m|i\rangle}, \qquad (2.5)$$

we obtain

$$\begin{aligned}|\langle\delta f|m|i\rangle|^2&=\langle\delta f|P|\delta f\rangle\langle i|m^\dagger m|i\rangle\\ &\leqq\langle\delta f|(P+Q)|\delta f\rangle\langle i|m^\dagger m|i\rangle\\ &=\epsilon^2\langle i|m^\dagger m|i\rangle,\end{aligned} \qquad (2.6)$$

which agrees with (2.4). If we introduce the quantity

$$\begin{aligned}\delta^2&=\langle f|P|f\rangle\\ &=|\langle f|m|i\rangle|^2/\langle i|m^\dagger m|i\rangle,\end{aligned} \qquad (2.7)$$

we can rewrite (2.6) as

$$|\langle\delta f|m|i\rangle/\langle f|m|i\rangle|^2\leqq\epsilon^2/\delta^2. \qquad (2.8)$$

The meaning of δ in (2.7) is clear from the relation

$$\delta^2=|\langle f|m|i\rangle|^2/\sum_{f'}|\langle f'|m|i\rangle|^2, \qquad (2.9)$$

which is the ratio of the relevant transition strength to the sum rule value. Thus, if δ is a quality of $O(1)$, (2.8) certainly guarantees that the possible error $|\langle\delta f|m|i\rangle/\langle f|m|i\rangle|^2$ has the order of ϵ. On the other hand, if $\delta\ll1$, the upper limit of (2.8) becomes large. It is interesting to note that the equality sign in (2.8) is valid if and only if

$$|\delta f\rangle=P|\delta f\rangle, \qquad (2.10)$$

as seen from (2.6). Since $m|i\rangle$ can be regarded as a collective state in various problems in nuclear physics, it can be stated that a large error in calculating $\langle f|m|i\rangle$ arises from the admixture of the collective state $m|i\rangle$.

B. Alternative Methods

As discussed in Ref. 9, we can construct many identities of the type

$$\langle f|m|i\rangle=\langle f|m_{\text{eff}}(E_f)|i\rangle, \qquad (2.11)$$

in which m_{eff} is some function of m, H, E_i, E_f, etc. For later convenience, only the dependence of m_{eff} on E_f is explicitly written in (2.11). As an example

$$\langle f|m_{\text{eff}}^{(1)}(E_f)|i\rangle\equiv\langle f|\frac{[H,m]-m\Delta}{E_f-E_i-\Delta}|i\rangle \qquad (2.12a)$$

$$=\frac{\langle f|Q[H,m]|i\rangle}{E_f-E_i-\Delta}, \qquad (2.12b)$$

where

$$\Delta=\frac{\langle i|m^\dagger[H,m]|i\rangle}{\langle i|m^\dagger m|i\rangle}. \qquad (2.12c)$$

In place of (2.4) we have

$$|\langle\delta f|m_{\text{eff}}(E_f)|i\rangle|^2\leqq\epsilon^2\langle i|m_{\text{eff}}^\dagger(E_f)m_{\text{eff}}(E_f)|i\rangle \qquad (2.13)$$

for a general m_{eff} in (2.11). Let us show that $\langle i|m_{\text{eff}}^\dagger m_{\text{eff}}|i\rangle$ is generally not equal to $\langle i|m^\dagger m|i\rangle$. If we write

$$\langle f|m_{\text{eff}}(E_f)|i\rangle=F^{-1}(E_f)\langle f|\hat{m}_{\text{eff}}|i\rangle, \qquad (2.14)$$

we are led to

$$\begin{aligned}\langle i|m_{\text{eff}}^\dagger(E_f)m_{\text{eff}}(E_f)|i\rangle&=\sum_{f'}|\langle f'|m_{\text{eff}}(E_f)|i\rangle|^2\\ &=|F(E_f)|^{-2}\sum_{f'}|\langle f'|\hat{m}_{\text{eff}}|i\rangle|^2 \qquad (2.15a)\\ &=|F(E_f)|^{-2}\sum_{f'}|F(E_{f'})|^2|\langle f'|m|i\rangle|^2 \qquad (2.15b)\\ &=\frac{\langle|F(E)|^2\rangle_{\text{av}}}{|F(E_f)|^2}\langle i|m^\dagger m|i\rangle, \qquad (2.15c)\end{aligned}$$

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where

$$\langle |F(E)|^2\rangle_{\rm av} = \sum_{f'} |F(E_{f'})|^2 |\langle f'|m|i\rangle|^2 / \sum_{f'} |\langle f'|m|i\rangle|^2. \quad (2.16)$$

At the step from (2.15a) to (2.15b) we made use of the relation,

$$\langle f'|m|i\rangle = \langle f'|m_{\rm eff}(E_{f'})|i\rangle = F^{-1}(E_{f'})\langle f'|\hat{m}_{\rm eff}|i\rangle, \quad (2.17a)$$

but it should be noticed that in general

$$\langle f'|m|i\rangle \neq \langle f'|m_{\rm eff}(E_f)|i\rangle. \quad (2.17b)$$

In the special case (2.12a), (2.15c) becomes

$$\langle i|m_{\rm eff}{}^{(1)\dagger}(E_f)m_{\rm eff}{}^{(1)}(E_f)|i\rangle = \frac{M_2}{(E_f-E_i-\Delta)^2}\langle i|m^\dagger m|i\rangle, \quad (2.18a)$$

where M_2 represents the second moment of transition strength distribution:

$$M_2 = \frac{\sum_{f'} (E_{f'}-E_i-\Delta)^2 |\langle f'|m|i\rangle|^2}{\sum_{f'} |\langle f'|m|i\rangle|^2}. \quad (2.18b)$$

From (2.13) and (2.18a) we can conclude that, for the value of E_f satisfying $M_2 \ll (E_f-E_i-\Delta)^2$, the possible error $|\langle\delta f|m_{\rm eff}|i\rangle|^2$ is expected to be much smaller than $|\langle\delta f|m|i\rangle|^2$.

Another example is given in (2.11b) of Ref. 9;

$$\langle f|m_{\rm eff}{}^{(2)}(E_f)|i\rangle \equiv \frac{\langle f|Q[H,[H,m]]|i\rangle}{(E_f-E_i)^2-(\Delta^2+M_2)}, \quad (2.19)$$

for which

$$\langle i|m_{\rm eff}{}^{(2)\dagger}(E_f)m_{\rm eff}{}^{(2)}(E_f)|i\rangle = \frac{\langle |(E-E_i)^2-(\Delta^2+M_2)|^2\rangle_{\rm av}}{[(E_f-E_i)^2-(\Delta^2+M_2)]^2}\langle i|m^\dagger m|i\rangle = \frac{M_4+4M_3\Delta+M_2(4\Delta^2-M_2)}{[(E_f-E_i)^2-(\Delta^2+M_2)]^2}\langle i|m^\dagger m|i\rangle. \quad (2.20)$$

The present argument is based on the assumption that $|i\rangle$, E_i, E_f, and Δ are somehow known or calculated from H with sufficient accuracy, and it was shown that, in this situation, it is better to use $m_{\rm eff}$ in place of m to prevent a large possible error $\langle\delta f|m|i\rangle$ arising from $|\delta f\rangle$. In other words, some higher-order effects are taken into account in terms of Δ and a numerator of (2.12) which is different from the original transition operator m.

In principle, if we find $m_{\rm eff}$ such that $\langle i|m_{\rm eff}{}^\dagger m_{\rm eff}|i\rangle \ll \langle i|m^\dagger m|i\rangle$, it gives a good estimate for $\langle f|m|i\rangle$. However, in practice, a complicated effective transition operator like $m_{\rm eff}{}^{(2)}$ in (2.19a) can hardly be convenient to treat. Therefore, instead of examining the general $m_{\rm eff}$ further we shall try to make the meaning of the identity (2.12) clearer, and find the condition for which (2.12) become useful as an approximation method in the next section.

It should be remarked here that most of the arguments in this section are also valid if $|i\rangle$ is replaced by a model wave function $|i\rangle^0$ in every place where $|i\rangle$ appears, as seen in the next section.

In the case of Gamow-Teller matrix elements, for which the identity (2.12) was used as the basis of an approximation method, Δ is essentially equal to the single-particle Coulomb displacement Δ_c, having the order of magnitude 15 MeV for heavy nuclei, and the largest contribution to $[H,m]$ comes from spin-orbit forces which are estimated to be considerably smaller than $|E_f-E_i-\Delta|$. Therefore the relation $M_2 \ll (E_f-E_i-\Delta)^2$ is fairly well satisfied.

3. FORMULATION OF COLLECTIVE MOTIONS

If the RPA (1.6) for a chosen ξ is valid with good accuracy, the original decomposition of H, (1.3), should give a good starting point for the treatment of collective motions. However, as discussed in Ref. 5 the RPA is known to be valid very well only for the Coulomb part of H. Therefore we start from another decomposition of H corresponding to the conventional shell-model approach or its variations:

$$H = H_{pp} + (H_{pq}+H_{qp}) + H_{qq}, \quad (3.1)$$

where p and q are a set of projection operators satisfying $p+q=1$ and $pq=qp=0$, and $H_{pq}=pHq$, etc. We assume that explicit calculations are carried out in the subspace projected out by the projection operator p. However, as pointed out in Sec. 1 such calculations might have large errors if collective effects exist in the q subspace. Our basic idea is to introduce the collective decomposition similar to (1.3) only for a part of H, H_{qq}:

$$H_{qq} = H_{Qq,qQ} + (H_{Pq,qQ}+H_{Qq,qP}) + H_{Pq,qP}, \quad (3.2)$$

where

$$H_{Pq,qQ} = PqHqQ, \quad \text{etc.}$$

For simplicity we assume that all the relevant states have a discrete energy spectrum, and the continuous case is discussed in the next section.

As in (2.1) the wave functions can be rewritten as

$$|i\rangle = |pi\rangle + |qi\rangle \quad (3.3a)$$

and

$$|f\rangle = |pf\rangle + |qf\rangle, \quad (3.3b)$$

where $|pi\rangle \equiv p|i\rangle$ and $|pf\rangle \equiv p|f\rangle$. The present formalism can be applied to any choice of p, but an example of p common to conventional theories is to project out the states corresponding to few lowest levels of a harmonic-oscillator potential, the closed core being assumed to be inert.

Two kinds of basic assumptions can be adopted.

(i) H is given, p being specified. Then we can solve the equation

$$(H_{pp}-\tilde{E}_f)|p\tilde{f}\rangle=0, \tag{3.4}$$

and similarly for $|p\tilde{i}\rangle$.

(ii) In addition to H, H' as stated below is also given, p being specified: According to Feshbach's procedure,[7] we can obtain $|pf\rangle$ and E_f by solving

$$(H_{pp}'-E_f)|pf\rangle=0, \tag{3.5a}$$

where

$$H_{pp}'=H_{pp}+H_{pq}(E_f-H_{qq})^{-1}H_{qp}, \tag{3.5b}$$

and similarly for $|pi\rangle$ and E_i.

In this section we adopt the latter assumption (ii), and the model wave function $|f\rangle^0$ in (2.1b) is identified with $|pf\rangle$ in (3.5a). Another case will be discussed in the next section.

Let us define the quantities ϵ_i and ϵ_f by

$$\langle qf|qf\rangle=\epsilon_f^2 \tag{3.6a}$$

and

$$\langle qi|qi\rangle=\epsilon_i^2, \tag{3.6b}$$

analogously to (2.3). These are assumed to be considerably smaller than 1. (See also Appendix.) Since we have the relation[7]

$$|qf\rangle=(E_f-H_{qq})^{-1}H_{qp}|pf\rangle, \tag{3.7}$$

the relations (3.6) can be rewritten as

$$\langle pf|H_{pq}(E_f-H_{qq})^{-2}H_{qp}|pf\rangle=\epsilon_f^2 \tag{3.8a}$$

and

$$\langle pi|H_{pq}(E_i-H_{qq})^{-2}H_{qp}|pi\rangle=\epsilon_i^2. \tag{3.8b}$$

Using (3.7) we obtain the conventional formulas written in the projection operator formalism,

$$\langle f|H|f\rangle=\langle pf|H(1+(E_f-H_{qq})^{-1}H_{qp})|pf\rangle/\langle pf|pf\rangle, \tag{3.9}$$

$$\langle f|m|f\rangle=\langle pf|(1+H_{pq}(E_f-H_{qq})^{-1})m\times(1+(E_f-H_{qq})^{-1}H_{qp})|pf\rangle, \tag{3.10}$$

and

$$\langle f|m|i\rangle=\langle pf|(1+H_{pq}(E_f-H_{qq})^{-1})m|i\rangle \tag{3.11a}$$

$$=\langle pf|(1+H_{pq}(E_f-H_{qq})^{-1})m\times(1+(E_i-H_{qq})^{-1}H_{qp})|pi\rangle \tag{3.11b}$$

for $i\neq f$. Although (3.9) is a special case of (3.10), it is separately written in a different way because H is special. It should be remembered that $|f\rangle$ is normalized, $\langle f|f\rangle=1$, but $|pf\rangle$ is not, in general, and similarly for $|i\rangle$.

In the following we examine the above three cases individually, estimating possible errors due to $|qf\rangle$ or $|qi\rangle$. If the errors are large because of "collective effects," we introduce the decomposition of H_{qq} in (3.2), in which the projection operator P, defined in (1.2), corresponds to relevant collective motions in nuclei.

A. Expectation Values of Energy

In this subsection the formula (3.9) is examined. The possible error is given by

$$\left|\langle f|H|f\rangle-\frac{\langle pf|H|pf\rangle}{\langle pf|pf\rangle}\right|^2=\left|\frac{\langle pf|H_{pq}|qf\rangle}{\langle pf|pf\rangle}\right|^2\leqq\frac{\epsilon_f^2}{1-\epsilon_f^2}\frac{\langle pf|H_{pq}H_{qp}|pf\rangle}{\langle pf|pf\rangle} \tag{3.12}$$

as in (2.4). The equality sign in (3.12) is valid if and only if

$$|qf\rangle=P_H|qf\rangle, \tag{3.13a}$$

where[16]

$$P_H=\frac{H_{qp}|pf\rangle\langle pf|H_{pq}}{\langle pf|H_{pq}H_{qp}|pf\rangle}. \tag{3.13b}$$

If we introduce the quantities

$$\delta^2=|\langle pf|H|pf\rangle|^2/(\langle pf|H^2|pf\rangle\langle pf|pf\rangle) \tag{3.14a}$$

and[17]

$$\kappa^2=\langle pf|H_{pq}H_{qp}|pf\rangle/\langle pf|H^2|pf\rangle, \tag{3.14b}$$

then (3.12) becomes

$$\left|\frac{\langle f|H|f\rangle\langle pf|pf\rangle}{\langle pf|H|pf\rangle}-1\right|^2\leqq\frac{\epsilon_f^2}{(1-\epsilon_f^2)^2}\frac{\kappa^2}{\delta^2}. \tag{3.15}$$

Therefore, we can draw a conclusion from (3.15) that if $\delta=O(1)$, the error is of the order $\epsilon_f\chi$; if $|\delta/\kappa\epsilon_f|\ll1$, the error could be as large as $|\kappa\epsilon_f/\delta|$.

In the latter case we introduce (3.2) in estimating the possible error. The following formulas are useful:

$$\frac{1}{E_f-H_{qq}}P=\frac{1}{E_f-H_{Pq,qP}}P+\frac{1}{E_f-H_{qq}}H_{Qq,qP}\frac{1}{E_f-H_{Pq,qP}}P, \tag{3.16a}$$

$$P\frac{1}{E_f-H_{qq}}Q=\frac{1}{E_f-H_{Pq,qP}}H_{Pq,qQ}Q\frac{1}{E_f-H_{qq}}Q, \tag{3.16b}$$

and, (3.16b) being inserted into (3.16a),

$$P\frac{1}{E_f-H_{qq}}P=\frac{1}{E_f-H_{Pq,qP}}P\left(1+H_{Pq,qQ}\times Q\frac{1}{E_f-H_{qq}}QH_{Qq,qP}\frac{1}{E_f-H_{Pq,qP}}\right). \tag{3.16c}$$

[16] In place of (3.13b) we can define P_H by $P_H=H|pf\rangle\langle pf|H/\langle pf|H^2|pf\rangle$. In such a case (3.13a) is replaced by $|qf\rangle=qP_H|qf\rangle$. Similar arguments can be applied to P_m in (3.28).

[17] If we use the closure approximation for (3.8a), we obtain $\kappa^2=\epsilon_f^2(E_f-\langle H_{qq}\rangle)^2/\langle pf|H^2|pf\rangle$. Therefore, the right-hand side of (3.15) usually has the order ϵ_f^4.

Assuming that $P=P_H$ defined in (3.13b), we obtain from (3.16a)

$$\langle pf|H|qf\rangle=\langle pf|H_{pq}\frac{1}{E_f-H_{qq}}H_{qp}|pf\rangle$$
$$=\langle pf|H_{pq}(E_f-H_{qq})^{-1}PH_{qp}|pf\rangle$$
$$=\langle pf|H_{pq}(E_f-H_{Pq,qP})^{-1}H_{qp}|pf\rangle$$
$$+\langle pf|H_{pq}(E_f-H_{qq})^{-1}H_{Qq,qP}\times(E_f-H_{Pq,qP})^{-1}H_{qp}|pf\rangle. \quad (3.17)$$

If we introduce the quantity $\langle H_{qq}\rangle$, defined by

$$H_{Pq,qP}=\langle H_{qq}\rangle P, \quad (3.18a)$$

where

$$\langle H_{qq}\rangle=\frac{\langle pf|H_{pq}H_{qq}H_{qp}|pf\rangle}{\langle pf|H_{pq}H_{qp}|pf\rangle}, \quad (3.18b)$$

we obtain

$$\langle f|H|f\rangle=\frac{1}{\langle pf|pf\rangle}\left\{\langle pf|H|pf\rangle+\frac{\langle pf|H_{pq}H_{qp}|pf\rangle}{E_f-\langle H_{qq}\rangle}+\frac{\langle qf|QH_{qq}PH_{qp}|pf\rangle}{E_f-\langle H_{qq}\rangle}\right\}. \quad (3.19)$$

Now, let us examine the last term in the parentheses of (3.19):

$$\left|\frac{\langle qf|QH_{qq}PH_{qp}|pf\rangle}{E_f-\langle H_{qq}\rangle}\right|^2 \leqq \frac{\epsilon_f^2}{(E_f-\langle H_{qq}\rangle)^2}\times\langle pf|H_{pq}PH_{qq}QH_{qq}PH_{qp}|pf\rangle, \quad (3.20a)$$
$$=\frac{\epsilon_f^2}{(E_f-\langle H_{qq}\rangle)^2}\times\langle pf|H_{pq}[H_{qq}-\langle H_{qq}\rangle]^2H_{qp}|pf\rangle, \quad (3.20b)$$

where we used the relation,

$$H_{Pq,qQ}=PH_{qq}Q=P(H_{qq}-\langle H_{qq}\rangle). \quad (3.20c)$$

In the limit of RPA being valid, we have $H_{Pq,qQ}=0$, the right-hand side of (3.20) vanishes. If we define the quantity μ_H by the relation

$$\mu_H^2=\frac{\langle pf|H_{pq}(H_{qq}-\langle H_{qq}\rangle)^2H_{qp}|pf\rangle}{|\langle pf|H_{pq}H_{qp}|pf\rangle|^2}, \quad (3.21a)$$

(3.20) can be rewritten as

$$\left|\frac{\langle qf|QH_{qq}PH_{qp}|pf\rangle}{E_f-\langle H_{qq}\rangle}\right|^2 \leqq \epsilon_f^2\mu_H^2\left|\frac{\langle pf|H_{pq}H_{qp}|pf\rangle}{E_f-\langle H_{qq}\rangle}\right|^2. \quad (3.21b)$$

From (3.19) and (3.21b) we are led to the formula[18]

$$\langle f|H|f\rangle=\frac{1}{\langle pf|pf\rangle}\left\{\langle pf|H|pf\rangle+\frac{\langle pf|H_{pq}H_{qp}|pf\rangle}{E_f-\langle H_{qq}\rangle}[1+O(\epsilon_f\mu_H)]\right\}. \quad (3.22)$$

It can be noticed that the second term in the parentheses of (3.22) is obtained also by adopting the closure approximation

$$\langle pf|H_{pq}\frac{1}{E_f-H_{qq}}H_{qp}|pf\rangle=\frac{\langle pf|H_{pq}H_{qp}|pf\rangle}{E_f-\langle H_{qq}\rangle}. \quad (3.23)$$

The magnitude of error involved in this approximation is prescribed by the parameter μ_H defined by (3.21a); if $\mu_H\epsilon_f\ll 1$, the contribution of the third term in the parentheses of (3.19) may be neglected.

B. Diagonal Matrix Elements

We start from the formula (3.10). The possible error can be estimated by the relation,

$$|\langle f|m|f\rangle-\langle pf|m|pf\rangle|\leqq|\langle pf|m|qf\rangle|+|\langle qf|m|pf\rangle|+|\langle qf|m|qf\rangle|, \quad (3.24)$$

where

$$|\langle qf|m|pf\rangle|^2\leqq\epsilon_f^2\langle pf|m^\dagger qm|pf\rangle \quad (3.25a)$$

and

$$|\langle qf|m|qf\rangle|^2\leqq\epsilon_f^4\frac{\langle qf|m^\dagger qm|qf\rangle}{\langle qf|qf\rangle}. \quad (3.25b)$$

We can rewrite (3.25a) as

$$\left|\frac{\langle qf|m|pf\rangle}{\langle pf|m|pf\rangle}\right|^2\leqq\epsilon_f^2\kappa_f^2/\delta_f^2, \quad (3.26a)$$

where

$$\delta_f^2=\frac{|\langle pf|m|pf\rangle|^2}{\langle pf|m^\dagger m|pf\rangle} \quad (3.26b)$$

and

$$\kappa_f^2=\frac{\langle pf|m^\dagger qm|pf\rangle}{\langle pf|m^\dagger m|pf\rangle}, \quad (3.26c)$$

provided that $\langle pf|m|pf\rangle\neq 0$. By making use of the formula (3.16a), we can proceed as follows:

$$\langle qf|m|pf\rangle=\langle pf|H_{pq}(E_f-H_{qq})^{-1}m|pf\rangle$$
$$=\langle pf|H_{pq}P_H(E_f-H_{qq})^{-1}P_mqm|pf\rangle, \quad (3.27)$$

where P_H is given by (3.13b) and

$$P_m=\frac{qm|pf\rangle\langle pf|m^\dagger q}{\langle pf|m^\dagger qm|pf\rangle}; \quad (3.28)$$

[18] From (3.20a) we can see that $\epsilon_f\mu_H$ gives the upper limit of the possible correction term. If $\epsilon_f\mu_H\ll 1$, the correction term must be small. However, if $\epsilon_f\mu_H$ is not small, the third term in the parentheses (3.19) must be examined more carefully.

assuming $P=P_m$ and using (3.16a), we get

$$\langle qf|m|pf\rangle = \frac{\langle pf|H_{pq}m|pf\rangle}{E_f-\langle H_{qq}\rangle}+\frac{\langle qf|QH_{qq}Pqm|pf\rangle}{E_f-\langle H_{qq}\rangle}, \quad (3.29a)$$

where

$$\langle H_{qq}\rangle = \frac{\langle pf|m^\dagger H_{qq}m|pf\rangle}{\langle pf|m^\dagger m|pf\rangle}. \quad (3.29b)$$

Then, it can be shown that

$$|\langle qf|QH_{qq}Pqm|pf\rangle|^2 \leqq \epsilon_f^2\langle pf|m^\dagger(H_{qq}-\langle H_{qq}\rangle)^2qm|pf\rangle. \quad (3.30)$$

If we define the quantity μ_m by

$$\mu_m^2=\frac{\langle pf|m^\dagger(H_{qq}-\langle H_{qq}\rangle)^2qm|pf\rangle}{|\langle pf|H_{pq}m|pf\rangle|^2}, \quad (3.31a)$$

we are led to the formula

$$\langle qf|m|pf\rangle=\frac{\langle pf|H_{pq}m|pf\rangle}{E_f-\langle H_{qq}\rangle}[1+O(\epsilon_f\mu_m)]. \quad (3.31b)$$

From the symmetry between P_m and P_H in (3.27) it is clear that the same type of formula is valid for P_m if $\langle H_{qq}\rangle$ is given by (3.18b) and μ_m is replaced by μ_H.

The third term of the right-hand side of (3.24) can be treated similarly, and leads to (3.25b):

$$|\langle qf|m|qf\rangle|^2=|\langle qf|m(E_f-H_{qq})^{-1}H_{qp}|pf\rangle|^2 \quad (3.32a)$$

$$\leqq \epsilon_f^2\langle pf|H_{qp}(E_f-H_{qq})^{-1}m^\dagger qm \times(E_f-H_{qq})^{-1}H_{qp}|pf\rangle$$

$$=\epsilon_f^4\langle qf|m^\dagger qm|qf\rangle/\langle qf|qf\rangle, \quad (3.32b)$$

in which the equality sign is valid if and only if $|qf\rangle$ is an eigenstate of m. If RPA is assumed to be valid for $P=P_H$, we get an estimate from (3.32a),

$$\langle qf|m|qf\rangle=\frac{\langle pf|H_{pq}mH_{qp}|pf\rangle}{(E_f-\langle H_{qq}\rangle)^2}. \quad (3.32c)$$

In the usual situation, $\langle qf|m|qf\rangle$ is, of course, expected to have the order ϵ_f^2.

Thus, we are led to the formula

$$\langle f|m|f\rangle=\langle pf|m|pf\rangle+\frac{\langle pf|(H_{pq}m+m^\dagger H_{qp})|pf\rangle}{E_f-\langle H_{qq}\rangle} \times[1+O(\epsilon_f\mu_m)]+O(\epsilon_f^2). \quad (3.33)$$

If RPA is valid for $P=P_m$, the $O(\epsilon_f\mu_m)$ term should be small.

C. Nondiagonal Matrix Elements

We examine the formula (3.11b). Conventional theories are based on the assumption that each term in the following expression,

$$\langle f|m|i\rangle=\langle pf|m|pi\rangle+\langle qf|m|pi\rangle +\langle pf|m|qi\rangle+\langle qf|m|qi\rangle, \quad (3.34)$$

has the relative order of magnitude $O(1)$, $O(\epsilon_f)$, $O(\epsilon_i)$, and $O(\epsilon_i\epsilon_f)$, respectively. This assumption can be examined by the relation

$$\left|\frac{\langle qf|m|pi\rangle}{\langle pf|m|pi\rangle}\right|^2 \leqq \epsilon_f^2\kappa_i^2/\delta_i^2, \quad (3.35a)$$

where

$$\delta_i^2=\frac{|\langle pf|m|pi\rangle|^2}{\langle pi|m^\dagger m|pi\rangle} \quad (3.35b)$$

and

$$\kappa_i^2=\frac{\langle pi|m^\dagger qm|pi\rangle}{\langle pi|m^\dagger m|pi\rangle}, \quad (3.35c)$$

and similarly for $|\langle pf|m|qi\rangle|^2$. Therefore, the expansion in (3.34) can be expressed symbolically as

$$\langle f|m|i\rangle=\langle pf|m|pi\rangle[1+O(\epsilon_f\kappa_i/\delta_i) +O(\epsilon_i\kappa_f/\delta_f)+O(\epsilon_i\epsilon_f)], \quad (3.36)$$

provided that $\langle pf|m|pi\rangle\neq 0$. Namely, if $\kappa_i/\delta_i=O(1)$, the term, $\langle qf|m|pi\rangle$, can be considered to be the quantity of $O(\epsilon_f)$.

Now let us examine the term of $O(\epsilon_f)$ or $O(\epsilon_i)$ in detail. Using the formula (3.16a) again, we obtain

$$\langle qf|m|pi\rangle =\langle pf|H_{pq}(E_f-H_{qq})^{-1}m|pi\rangle$$

$$=\langle pf|H_{pq}(E_f-H_{Pq,qP})^{-1}Pqm|pi\rangle +\langle qf|QH_{qq}P(E_f-H_{Pq,qP})^{-1}qm|pi\rangle, \quad (3.37a)$$

in which

$$P=\frac{qm^\dagger|pi\rangle\langle pi|mq}{\langle pi|m^\dagger qm|pi\rangle}. \quad (3.37b)$$

If the quantity Δ_q is defined by

$$E_i+\Delta_q=\langle H_{qq}\rangle=\frac{\langle pi|m^\dagger H_{qq}m|pi\rangle}{\langle pi|m^\dagger qm|pi\rangle}, \quad (3.38)$$

(3.37a) can be expressed as

$$\langle qf|m|pi\rangle=\frac{\langle pf|H_{pq}m|pi\rangle}{E_f-E_i-\Delta_q}[1+O(\epsilon_f\mu_i)], \quad (3.39a)$$

where

$$\mu_i^2=\frac{\langle pi|m^\dagger q(H_{qq}-\langle H_{qq}\rangle)^2m|pi\rangle}{|\langle pf|H_{pq}m|pi\rangle|^2}. \quad (3.39b)$$

Similarly, we have

$$\langle pf|m|qi\rangle=\frac{\langle pf|mH_{qp}|pi\rangle}{E_i-E_f-\Delta_q'}[1+O(\epsilon_i\mu_f)], \quad (3.40)$$

where Δ_q' and μ_f are defined analogously to Δ_q and μ_i, respectively.[19] From (3.37a) and (3.40) we obtain

$$\langle f|m|i\rangle=\langle pf|m|pi\rangle+\frac{\langle pf|H_{pq}m|pi\rangle}{E_f-E_i-\Delta_q}[1+O(\epsilon_f\mu_i)]+\frac{\langle pf|mH_{qp}|pi\rangle}{E_i-E_f-\Delta_q'}[1+O(\epsilon_i\mu_f)]+O(\epsilon_i\epsilon_f). \quad (3.41)$$

Let us examine the special case, $qm^\dagger|pf\rangle=0$, but $qm|pi\rangle\neq0$. Such an asymmetry between initial and final states is frequently seen[6,9] for the actual treatment of heavy nuclei with $(N-Z)\neq0$. (See Sec. 4 C.) In this case $\kappa_f^2=0$ and the third term of the right-hand side of (3.41) may be omitted. The first and second terms of (3.41) can be combined to yield

$$\langle pf|m|pi\rangle+\frac{\langle pf|H_{pq}m|pi\rangle}{E_f-E_i-\Delta_q}=\frac{\langle pf|(E_f-E_i-\Delta_q)m+H_{pq}m|pi\rangle}{E_f-E_i-\Delta_q}, \quad (3.42)$$

(3.5) being inserted into (3.42),

$$=\frac{\langle pf|[H,m]-m\Delta_q|pi\rangle}{E_f-E_i-\Delta_q}+\frac{\langle qf|H_{qp}m|pi\rangle-\langle pf|mH_{pq}|qi\rangle}{E_f-E_i-\Delta_q}. \quad (3.43)$$

The second term of (3.43) is expected to be $O(\epsilon_f^2)$ or $O(\epsilon_i^2)$ as shown by the example in Sec. 4 C.

Thus, in the case of $qm^\dagger|pf\rangle=0$ we obtain, in place of (3.41), an approximate formula,

$$\langle f|m|i\rangle=\frac{\langle pf|[H,m]-m\Delta_q|pi\rangle}{E_f-E_i-\Delta_q}, \quad (3.44)$$

which is very similar to (2.12a).

Now, let us examine the possible error in the formula (3.44) by starting from the identity (2.12a), provided that E_f, E_i, and Δ are known with sufficiently good accuracy. To Δ the arguments in Sec. 3 B can be applied, and we assume that, as in the example of Sec. 4 C,

$$\Delta=\Delta_q+O(\epsilon_i). \quad (3.45)$$

The relation (3.45) implies that q can be replaced by 1 in the definition of Δ_q, (3.38). Then the identity (2.12a) can be rewritten as

$$\langle f|m|i\rangle=\frac{\langle f|[H,m]-m\Delta_q|i\rangle+(\Delta_q-\Delta)\langle f|m|i\rangle}{E_f-E_i-\Delta}$$

$$=\frac{\langle f|[H,m]-m\Delta_q|i\rangle}{E_f-E_i-\Delta_q} \quad (3.46a)$$

$$=\frac{\langle pf|R|pi\rangle+\langle qf|R|pi\rangle+\langle pf|R|qi\rangle+\langle qf|R|qi\rangle}{E_f-E_i-\Delta_q}, \quad (3.46b)$$

where

$$R=[H,m]-m\Delta_q. \quad (3.46c)$$

The second term in the parentheses of the numerator of (3.46) satisfies the following inequality:

$$|\langle qf|R|pi\rangle|^2\leqq\epsilon_f^2\langle pi|R^\dagger qR|pi\rangle. \quad (3.47a)$$

Similarly, we have

$$|\langle pf|R|qi\rangle|^2\leqq\epsilon_i^2\langle pf|RqR^\dagger|pf\rangle. \quad (3.47b)$$

The relation (3.47a) can be rewritten as

$$\left|\frac{\langle qf|R|pi\rangle}{\langle pf|R|pi\rangle}\right|^2\leqq\epsilon_f^2\bar{\kappa}_i^2/\bar{\delta}_i^2, \quad (3.48a)$$

provided that $\langle pf|R|pi\rangle\neq0$, where

$$\bar{\delta}_i^2=\frac{|\langle pf|R|pi\rangle|^2}{\langle pi|R^\dagger R|pi\rangle} \quad (3.48b)$$

and

$$\bar{\kappa}_i^2=\frac{\langle pi|R^\dagger qR|pi\rangle}{\langle pi|R^\dagger R|pi\rangle}. \quad (3.48c)$$

Since $\langle pf|R|qi\rangle$ can be treated similarly, we can write (3.46) symbolically as

$$\langle f|m|i\rangle=\frac{\langle pf|R|pi\rangle[1+O(\epsilon_f\bar{\kappa}_i/\bar{\delta}_i)+O(\epsilon_i\bar{\kappa}_f/\bar{\delta}_f)+O(\epsilon_i\epsilon_f)]}{E_f-E_i-\Delta_q}. \quad (3.49)$$

Summarizing this subsection we can make several remarks:

(a) The second and third terms in the parenthesis of (3.36) can be rewritten as in the right-hand side of (3.41). If RPA is valid either for $m|pi\rangle$ or for $m^\dagger|pf\rangle$, or both, collective effects included in the $O(\epsilon_f)$ and $O(\epsilon_i)$ terms can be easily estimated by (3.41).

(b) If $\epsilon_i=0$ or $qm^\dagger|pf\rangle=0$, the third term of the right-hand side of (3.41) vanishes, and the formula (3.41) becomes (3.44), and similarly for $\epsilon_f=0$.

(c) If $\bar{\kappa}_i/\bar{\delta}_i$ and $\bar{\kappa}_f/\bar{\delta}_f=O(1)$ in (3.49), the correction terms in (3.44) are small. If these quantities, the upper

[19] Explicitly, $E_f+\Delta_q'=\langle pf|mH_{qq}m^\dagger|pf\rangle/p\langle f|mm^\dagger|pf\rangle$.

limits of (3.47), are much larger than 1, $\langle qf|R|pi\rangle$ and $\langle pf|R|qi\rangle$ must be carefully examined case by case. However, it seems unlikely that $|qi\rangle$ includes the collective component $R^\dagger|pf\rangle$ significantly, and similarly for $|qf\rangle$.

(d) It must be remembered that the arguments of Sec. 2 are based on the assumption, $\epsilon_i=0$. If both $|\delta i\rangle$ and $|\delta f\rangle$ include collective components $m^\dagger|f\rangle^0$ and $m|i\rangle^0$, respectively, the formula (3.41) seems to be better than (3.44).

4. DISCUSSION OF RESULTS

In this section we discuss the results in Sec. 3 and examine their relations to other theories.

A. Effective Operators

The projection operator p in (3.2) can be chosen to be the one projecting out the lowest configurations predicted by the shell or pairing model for initial and final nuclear states. In such a case, the expressions (3.33) and (3.41) define the "effective operators":

$$\langle f|m|f\rangle=\langle pf|m_{\rm eff}|pf\rangle, \tag{4.1a}$$

where

$$m_{\rm eff}=m+\frac{Hqm+mqH}{E_f-\langle H_{qq}\rangle}; \tag{4.1b}$$

$$\langle f|m|i\rangle=\langle pf|m_{\rm eff}|pi\rangle, \tag{4.2a}$$

where

$$m_{\rm eff}=m+\frac{Hqm}{E_f-E_i-\Delta_q}+\frac{mqH}{E_i-E_f-\Delta_q'}, \tag{4.2b}$$

or

$$m_{\rm eff}=\frac{[H,m]-m\Delta_q}{E_f-E_i-\Delta_q}, \tag{4.2c}$$

if $qm^\dagger|pf\rangle=0$. In (4.1) and (4.2), $\langle H_{qq}\rangle$ and Δ_q are given by (3.29b) and (3.38), respectively.

The formula (4.2) has been extensively applied to the study of β-decay matrix elements.[6,11,12,14,15] The formula (4.1a) is closely related to previous treatments of the effective coupling constant due to core polarizations.[20,21] If we use $m_{\rm eff}$ instead of m, collective higher-order effects can be taken into account, as shown in Sec. 3.

B. Configuration Mixing Method

The projection operator p in (3.1) can be chosen in order to project out the states corresponding to few lowest levels of a harmonic-oscillator potential, as commonly assumed in the conventional configuration mixing treatments.[22]

Then the correction terms, in (4.1b) and (4.2b), give estimates on the contributions of q components of wave functions, for which detailed calculations are not carried out. The magnitudes of such contributions can be easily estimated by the sum rule values,

$$\langle pf|mqH|pf\rangle,\quad \langle pf|Hqm|pi\rangle,\quad \text{etc.}$$

It seems to be desirable to check these sum rules when a configuration mixing calculation is done. It is interesting to note that, if $qm^\dagger|pf\rangle=0$ or $qm|pi\rangle=0$ or $qH|pf\rangle=0$, the above sum rule values vanish. In such a case the model can be considered as "self-contained."

For β decays the configuration mixing calculations for which $qm|pi\rangle=qm^\dagger|pf\rangle=0$ have been carried out.[6,23–25] The calculations are not only complicated, but also sensitive to the assumed models.

C. Comparison with Solvable Models

In order to make the physical meaning of above arguments clearer, a solvable model is treated in the framework of the present formalism. Suppose that

$$H=H^0+H^1, \tag{4.3}$$

where

$$H^0=E_0\sum_{s=1}^{N}|f_s\rangle\langle f_s|+(E_0-\delta)|f_0\rangle\langle f_0| \tag{4.4a}$$

and

$$H^1=G(\sum_{s=0}^{N}|f_s\rangle)(\sum_{s'=0}^{N}\langle f_{s'}|), \tag{4.4b}$$

the sth unperturbed state being denoted as $|f_s\rangle$. If we put $\delta=0$, (4.3) agrees with the degenerate model proposed by Brown and Bolsterli.[4] The case of $\delta\neq0$ has been discussed[15,21] in connection with the hindrance factor for β decays.

Let us introduce the projection operators p and q prescribing the model space; for $s=0,1,\cdots,N$,

$$p|f_s\rangle=\delta_{s,0}|f_s\rangle \tag{4.5a}$$

and

$$q|f_s\rangle=(1-\delta_{s,0})|f_s\rangle, \tag{4.5b}$$

or more explicitly

$$p=|f_0\rangle\langle f_0| \tag{4.6a}$$

and

$$q=\sum_{s=1}^{N}|f_s\rangle\langle f_s|. \tag{4.6b}$$

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Now we introduce the model transition operator

$$m=\sum_{s=0}^{N}|f_s\rangle\langle i| \tag{4.7}$$

and the corresponding projection operator

$$P=1-Q=[1/(N+1)](\sum_{s=0}^{N}|f_s\rangle)(\sum_{s'=0}^{N}\langle f_{s'}|). \tag{4.8}$$

Using these projection operators, H in (4.3) can be rewritten as

$$H^0=E_0q+(E_0-\delta_0)p \tag{4.9a}$$

and

$$H^1=(N+1)GP, \tag{4.9b}$$

in which the part of H^0 connected with $|i\rangle$ is omitted for simplicity. It is easily recognized that

$$Pp-pP=[1/(N+1)]\times[\sum_{s=1}^{N}|f_s\rangle\langle f_0|-|f_0\rangle\sum_{s'=1}^{N}\langle f_{s'}|] \tag{4.10}$$

is not zero in general.

The following expressions are easily derived:

$$H_{pp}=(E_0-\delta_0)|f_0\rangle\langle f_0|+G|f_0\rangle\langle f_0|, \tag{4.11a}$$

$$H_{qq}=E_0\sum_{s=1}^{N}|f_s\rangle\langle f_s|+G(\sum_{s=1}^{N}|f_s\rangle)(\sum_{s'=1}^{N}\langle f_{s'}|) \tag{4.11b}$$

and

$$H_{pq}=(H_{qp})^{\dagger}=G|f_0\rangle\sum_{s=1}^{N}\langle f_s|; \tag{4.12}$$

$$H_{PP}=P(E_i+\Delta), \tag{4.13a}$$

where

$$E_i+\Delta=E_0-\delta_0/(N+1)+(N+1)G; \tag{4.13b}$$

$$H_{Pq,qP}=P(E_i+\Delta_q), \tag{4.14a}$$

where

$$E_i+\Delta_q=(E_0+GN)N/(N+1). \tag{4.14b}$$

In Refs. 15 and 21 the exact solutions are obtained in this model, and for $N\gg1$ we have

$$\langle f|m|i\rangle=\delta_0/(NG+\delta_0). \tag{4.15}$$

Since we can see that $\Delta=\Delta_q+O(N^{-1})$ and $E_f=E_0-\delta_0[1-O(N^{-1})]$, it can be proved that

$$\frac{\langle pf|[H,m]-m\Delta_q|i\rangle}{E_f-E_i-\Delta_q}=\frac{\delta_0}{NG+\delta_0} \tag{4.16}$$

for $N\gg1$ in this model. Comparison of (4.15) with (4.16) shows the validity of (4.2b).

D. Extension to the Continuum Region

Even if $|f^{(\pm)}\rangle$ belongs to the continuum region, the formula (2.12) is valid;

$$\langle f^{(\pm)}|m|i\rangle=\frac{\langle Qf^{(\pm)}|[H,m]|i\rangle}{E_f-E_i-\Delta}, \tag{4.17}$$

provided that $E_f\neq E_i+\Delta$. However, it is convenient to express the right-hand side of (4.17) in terms of $|Q\tilde{f}^{(\pm)}\rangle$, which is a solution of

$$(H_{QQ}-E_f)|Q\tilde{f}^{(\pm)}\rangle=0. \tag{4.18}$$

It is to be noticed that $|Q\tilde{f}^{(\pm)}\rangle$ includes the states in open channels. The result is given by

$$\langle f^{(\pm)}|m|i\rangle=\frac{\langle Q\tilde{f}^{(\pm)}|[H,m]|i\rangle}{E_f-E_i-\Delta'\pm i\frac{1}{2}\Gamma}, \tag{4.19}$$

where

$$\Delta'\mp i\tfrac{1}{2}\Gamma=\Delta+\langle Pf^{(\pm)}|H_{PQ}(E_f-H_{QQ}\pm i\epsilon)^{-1}\times H_{QP}|Pf^{(\pm)}\rangle/\langle Pf^{(\pm)}|Pf^{(\pm)}\rangle, \tag{4.20a}$$

or

$$\Delta'-\Delta=\langle i|m^{\dagger}H_{PQ}\frac{\mathcal{P}}{E_f-H_{QQ}}H_{QP}m|i\rangle/\langle i|m^{\dagger}m|i\rangle, \tag{4.20b}$$

and

$$\Gamma=2\pi\langle i|m^{\dagger}H_{PQ}\delta(E_f-H_{QQ})H_{QP}m|i\rangle/\langle i|m^{\dagger}m|i\rangle, \tag{4.20c}$$

P being defined by (2.5) and $\mathcal{P}$ representing the principal value.

The formula of Breit-Wigner type (4.19) has been derived previously[8] and applied to electromagnetic transitions. For the sake of completeness, let us briefly sketch the proof of (4.19) here. Solving the simultaneous equations

$$(H_{QQ}-E_f)|Qf\rangle=-H_{QP}|Pf\rangle \tag{4.21a}$$

and

$$(H_{PP}-E_f)|Pf\rangle=-H_{PQ}|Qf\rangle, \tag{4.21b}$$

we obtain

$$|Pf^{(\pm)}\rangle=(E_f-H_{PP})^{-1}H_{PQ}|Qf^{(\pm)}\rangle \tag{4.22a}$$

and

$$|Qf^{(\pm)}\rangle=|Q\tilde{f}^{(\pm)}\rangle+(E_f-H_{QQ}\pm i\epsilon)^{-1}H_{QP}|Pf^{(\pm)}\rangle \tag{4.22b}$$

$$=|Q\tilde{f}^{(\pm)}\rangle+(E_f-H_{QQ}\pm i\epsilon)^{-1}H_{QP}\times(E_f-H_{PP})^{-1}H_{PQ}|Qf^{(\pm)}\rangle. \tag{4.22c}$$

Then multiplying $\langle Pf^{(\pm)}|H_{PQ}$ from the left of (4.22c) we obtain

$$\langle Pf^{(\pm)}|H_{PQ}|Qf^{(\pm)}\rangle=\langle Pf^{(\pm)}|H_{PQ}|Q\tilde{f}^{(\pm)}\rangle+\langle Pf^{(\pm)}|H_{PQ}(E_f-H_{QQ}\pm i\epsilon)^{-1}H_{QP}|Pf^{(\pm)}\rangle\times\frac{\langle Pf^{(\pm)}|H_{PQ}|Qf^{(\pm)}\rangle}{\langle Pf^{(\pm)}|Pf^{(\pm)}\rangle}, \tag{4.23}$$

which leads to (4.19) straightforwardly.

The formula (4.19) shows that the transition amplitude $\langle f^{(\pm)}|m|i\rangle$ looks like a giant resonance which has the peak energy $E_i+\Delta'$ and total width Γ, and the fine structure should be given by calculating the numerator.

It should be pointed out that the formula (4.19) is useful as the basis of an approximation method in the sense of the arguments in Secs. 2 and 3. The main point is that, when the wave function $|Q\tilde{f}^{(\pm)}\rangle$ in (4.19) is replaced by a model wave function, we do not have to worry about the P component in the model wave function because of the presence of the Q operator in front of $|f^{(\pm)}\rangle$. The arguments in Sec. 3 can be easily extended to the continuum cases. It is shown that there is a close relationship between the validity of (4.19) and that of RPA. According to Feshbach's terminology our collective state $m|i\rangle$ corresponds to a "doorway state." Since the formula (4.19) has been already explored in the study of nuclear reactions,[8,13] we return to the discrete case in the next subsection.

E. Relations to Tomonaga Theory

Generally speaking, if we construct n collective coordinates from spatial $3A$ coordinates for A nucleons, the remaining degrees of freedom become $(3A-n)$. However, if $3A\gg n$, $(3A-n)$ can be regarded as approximately equal to the original degrees of freedom $3A$. In Tomonaga's theory,[1] the approximation in which $1/A$ is neglected compared with 1 plays a central role.

To the present treatment we applied this idea in a somewhat different way: In the p subspace we make use of the model wave functions obtained from the shell model or its variations, whereas the higher admixed states belonging to the q subspace are treated by adopting the RPA (corresponding to the $1/A$ approximation).

Therefore, the present procedure seems to be appropriate for the treatment of incomplete collective modes on the basis of conventional shell theories, and may be regarded as a new type of "unified theory."

F. Miscellaneous Remarks

(a) We can formulate the arguments in Sec. 3 by starting from the model wave function $|p\tilde{f}\rangle$ in (3.4). In this case we introduce the quantity $\tilde{\epsilon}_f$:

$$\langle\delta\tilde{f}|\delta\tilde{f}\rangle=\tilde{\epsilon}_f^{\,2}, \tag{4.24a}$$

where

$$|\delta\tilde{f}\rangle=|f\rangle-|p\tilde{f}\rangle. \tag{4.24b}$$

Then, the perturbation expansion can be written in the following way:

$$H=H_0+H_1, \tag{4.25a}$$

where

$$H_0=H_{pp}+H_{qq}; \tag{4.25b}$$

then

$$|f\rangle=\left\{1+\frac{P_0}{E_f-H}H_1\right\}|p\tilde{f}\rangle, \tag{4.26a}$$

where

$$\frac{P_0}{E_f-H}=\frac{P_0}{E_f-H_0}+\frac{P_0}{E_f-H_0}H_1\frac{P_0}{E_f-H}. \tag{4.26b}$$

In (4.26), the notation P_0 represents a projection operator $(1-|p\tilde{f}\rangle\langle p\tilde{f}|/\langle p\tilde{f}|p\tilde{f}\rangle)$, which commutes with H_0.

The remainder of discussion proceeds exactly in the same way as in Sec. 3. This type of formulation of nuclear models has been proposed previously.[26]

(b) Ichimura has shown[10,11] that the approximation method starting from the identity (2.12) corresponds to a sort of lowest-order perturbation approach.

It can be seen from (3.41b) that if $qm^\dagger|pf\rangle=0$ and $\Delta\cong\Delta_q'$,

$$\langle f|m|i\rangle\cong\langle pf|m|pi\rangle+\langle pf|\frac{H_{pq}m}{E_f-E_i-\Delta}|pi\rangle. \tag{4.27}$$

It is recognized that the difference between the both hands of (4.27) comes mainly from the higher-order effects of $H_{Pq,qQ}$.

5. CONCLUSIONS

In this paper, being motivated to reformulate the theory of collective motions due to Tomonaga in the projection operator formalism, we were led to the question of what is the best method for calculating $\langle f|m|i\rangle$, $\langle i|m|i\rangle$, and E when $|i\rangle$ and $|f\rangle$ are given with some error $|\delta i\rangle$ and $|\delta f\rangle$. In Sec. 2 it was pointed out that, if $|\delta f\rangle$ includes collective components $m|i\rangle$, the error can be quite large. In Sec. 3 it was concluded that such collective components can be taken out in terms of the collective energy and a type of sum rules. Therefore, the formulas obtained can be used also as a basis for semiphenomenological arguments based on experimental knowledge.[15] Mathematically speaking, the upper limits of errors were estimated by use of (2.4), essentially the Schwarz inequality. It was shown that validity of the equality sign in the Schwarz inequality is closely related to the existence of a collective mode.

The main results of this paper are given by (3.22), (3.33), or (4.1), (3.41) or (4.2), and (4.19) for energy, diagonal, nondiagonal (discrete), and nondiagonal (continuous) matrix elements, respectively. The correction terms were obtained in the lowest order of $H_{Pq,qQ}$. Therefore, if RPA is a good approximation for the relevant collective coordinate, these formulas give convenient estimates on the effects of remaining terms. If the correction terms are not small and RPA is not justified, these formulas give us warning that we should enlarge the definition of the subspace prescribed by p and treat the higher-order terms carefully.

Briefly, the argument of this paper is summarized: The validity and the limit of applicability of the previously proposed approximation procedure, which uses

[26] J. I. Fujita, J. Phys. Soc. Japan **19**, 1528 (1964).

a commutator of the nuclear Hamiltonian and the relevant transition operator, was clarified by repeated use of the Schwarz inequality, and the same basic idea was applied to the evaluation of diagonal matrix elements. This method is especially suitable for the treatment of a problem in which the model wave function is mostly shell-like but with small admixtures of collective effects.

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APPENDIX: SHORT-RANGE CORRELATIONS

In this Appendix are discussed effects of short-range correlations which were omitted in the body of the paper. It is well known that nuclear forces have strongly repulsive cores and the true wave functions must be quite different from the shell-type wave functions whenever two or more nucleons are close together. However, it is generally believed[27] that such short-range effects can be approximately removed through a proper unitary transformation U, $(U^\dagger U=1)$, satisfying

$$\langle f|m|i\rangle=\langle\tilde{f}|m|\tilde{\imath}\rangle, \tag{A1a}$$

where

$$[U,m]=0, \tag{A1b}$$

$$U|i\rangle=|\tilde{\imath}\rangle, \tag{A1c}$$

and

$$U|f\rangle=|\tilde{f}\rangle \tag{A1d}$$

for a single-body operator m. At the same time we have

$$(\tilde{H}-E)|\tilde{\imath}\rangle=0, \tag{A2a}$$

where

$$\tilde{H}=UHU^{-1}=U(T+V)U^{-1}=T+t. \tag{A2b}$$

In (A2), T and V represent the kinetic and potential energy parts, respectively, and t is assumed to be approximately expressed as a sum of effective two-body potentials.

In the present projection operator formalism, we can write

$$|i\rangle=U^{-1}|\tilde{\imath}\rangle=\frac{[1+(E_i-H_{qq})^{-1}H_{qp}]|pi\rangle}{[\langle pi|1+H_{pq}(E_i-H_{qq})^{-2}H_{qp}|pi\rangle]^{1/2}}, \tag{A3}$$

where $|pi\rangle/(\langle pi|pi\rangle)^{1/2}$ in (3.2) is identified with $|\tilde{\imath}\rangle$. If we assume that the approximate relations,

$$\langle pf|\left(1+\frac{1}{E_f-H_{qq}}H_{pq}\right)\left[m,\frac{1}{E_i-H_{qq}}H_{qp}\right]|pi\rangle=0 \tag{A4a}$$

and

$$|pi\rangle=\frac{[1+H_{pq}(E_f-H_{qq})^{-1}][1+(E_i-H_{qq})^{-1}H_{qp}]|pi\rangle}{[\langle pf|1+H_{pq}(E_f-H_{qq})^{-2}H_{qp}|pf\rangle\langle pi|1+H_{pq}(E_i-H_{qq})^{-2}H_{qp}|pi\rangle]^{1/2}}, \tag{A4b}$$

are satisfied independently of the value of E_i or E_f, we obtain the desired property (A1). A plausible argument in favor of (A4b) can be given as follows. First we notice that the operator (E_i-H_{qq}) in (A3) can be approximately replaced by a c number, $E_i-\langle H_{qq}\rangle\equiv\bar{E}$, independent of E_i and similarly for (E_f-H_{qq}), where $\bar{E}$ stands for a typical value of excitation energy corresponding to the short-range correlations. Then the dependence of (A4) on E_i or E_f can be neglected in first approximation.

[27] For instance, R. J. Eden and N. C. Francis, Phys. Rev. 97, 1366 (1955).