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Continuum Theory of Thomson Scattering*

B. SAMUEL TANENBAUM†
Cornell University Center for Radiophysics and Space Research, Arecibo, Puerto Rico
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The theory of incoherent (Thomson) scattering of electromagnetic waves by thermal fluctuations in a plasma is rederived using continuum equations instead of kinetic theory. Because of the inherent simplicity of this approach, it is possible to extend the theory by including the effect of density fluctuations of the neutral molecules upon the scattering. The following results are obtained from the continuum theory: (1) The total predicted scattered power is exactly the same as in the previous kinetic theories. (2) The frequency spectrum of the scattered signal when neutral fluctuations are neglected agrees very well with the spectrum obtained using kinetic theory (with the Bhatnagar-Gross-Krook model for collisions) whenever λ_{in}, the ion-neutral mean free path, is smaller than λ_0, the wave number of the incident signal. (3) The major difference in the spectrum when the neutral fluctuations are included is the addition of two resonances shifted from the signal frequency by $\pm kU_n$, where $k=4\pi/\lambda_0$, and U_n is the neutral sound speed. For a weakly ionized gas, these resonances are found to be significant whenever $\lambda_{in}<r^{1/2}\lambda_0$, with r the ratio of the electron density to the neutral-atom density.

I. INTRODUCTION

IN this paper the theory of incoherent (Thomson) scattering by thermal fluctuations in a plasma is rederived using continuum equations instead of kinetic theory. Aside from the fact that continuum equations are inherently simpler than kinetic equations, the principal motivation for this work is the fact that in a partly ionized gas with no magnetic field there can be three types of waves involving density fluctuations of the electrons. The scattering from two of these, the high-frequency electron and the low-frequency ion acoustic waves, are adequately treated by prior theories[1–17] which consider fluctuations of the electrons and ions but assume a constant neutral molecule density. However, scattering from the third wave (which, though called a neutral acoustic wave, still involves electron-density fluctuations) can only be treated by using a three-fluid model. Such a calculation, while extremely difficult in the kinetic-theory approach, can be made fairly easily using continuum (or hydrodynamic) equations for the plasma.

Very simple collisionless hydrodynamic equations have been used previously by Cohen[14] to calculate the Thomson-scattering spectrum from the electron and ion acoustic waves. Since no damping mechanism is included in his equations, the frequency spectrum of the scattered signal in his work is a sum of δ functions; but his expressions for the scattered power (assuming isothermal motion) are in exact agreement with those found using kinetic theory. In Sec. II we present a more complete two-fluid theory (appropriate for a weakly ionized gas) and show that (1) the total predicted scattered power is exactly the same (without the need for an isothermal assumption) as in the previous kinetic theories; (2) the predicted frequency spectrum agrees very well with results obtained by Dougherty and Farley[12,18] using the Boltzmann equation with the Bhatnagar-Gross-Krook (BGK) model[19] for collisions with neutrals provided that the ion-neutral mean free path λ_{in} is less than λ_0, the incident wavelength. (For larger mean free paths, where Landau damping is important, the continuum equations are no longer valid, so the disagreement is not surprising.)

In Sec. III we explicitly include the effect upon the Thomson scattering of density fluctuations of the neutral molecules by using a three-fluid theory. From the analysis we find that in comparison with the two-fluid results (1) the total scattered power is unchanged; (2) the major difference in the spectrum is the addition of two resonances shifted from the signal frequency by $\pm kU_n$, where $k=4\pi/\lambda_0$ and U_n is the neutral sound speed. This resonant scattering from the neutral acoustic wave is found to be a large effect provided that λ_0 satisfies the two conditions

$$\lambda_0\gg\lambda_D \quad \text{and} \quad \lambda_0\gg r^{-1/2}\lambda_{in}, \tag{1.1}$$

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† Permanent address: Case Western Reserve University, Cleveland, Ohio.

1 J. P. Dougherty and D. T. Farley, Proc. Roy. Soc. (London) **A259**, 79 (1960).
2 E. E. Salpeter, Phys. Rev. **120**, 1528 (1960).
3 J. A. Fejer, Can. J. Phys. **38**, 1114 (1960).
4 J. Renau, J. Geophys. Res. **65**, 3631 (1960).
5 T. Laaspere, J. Geophys. Res. **65**, 3955 (1960).
6 D. T. Farley, J. P. Dougherty, and D. W. Barron, Proc. Roy. Soc. (London) **A263**, 283 (1961).
7 E. E. Salpeter, Phys. Rev. **122**, 1663 (1961).
8 J. A. Fejer, Can. J. Phys. **39**, 716 (1961).
9 T. Hagfors, J. Geophys. Res. **66**, 1699 (1961).
10 J. Renau, H. Camnitz, and W. Flood, J. Geophys. Res. **66**, 2703 (1961).
11 M. N. Rosenbluth and N. Rostoker, Phys. Fluids **5**, 776 (1962).
12 J. P. Dougherty and D. T. Farley, J. Geophys. Res. **68**, 5473 (1963).
13 E. E. Salpeter, J. Geophys. Res. **68**, 1321 (1963).
14 M. H. Cohen, J. Geophys. Res. **68**, 5675 (1963).
15 F. Perkins and E. E. Salpeter, Phys. Rev. **139**, A55 (1965).
16 P. Waldteufel, Ann. Geophys. **21**, 106 (1965).
17 D. T. Farley, J. Geophys. Res. **71**, 4091 (1966).
18 J. P. Dougherty, J. Fluid Mech. **16**, 126 (1963).
19 P. L. Bhatnagar, E. P. Gross, and M. Krook, Phys. Rev. **94**, 511 (1954).

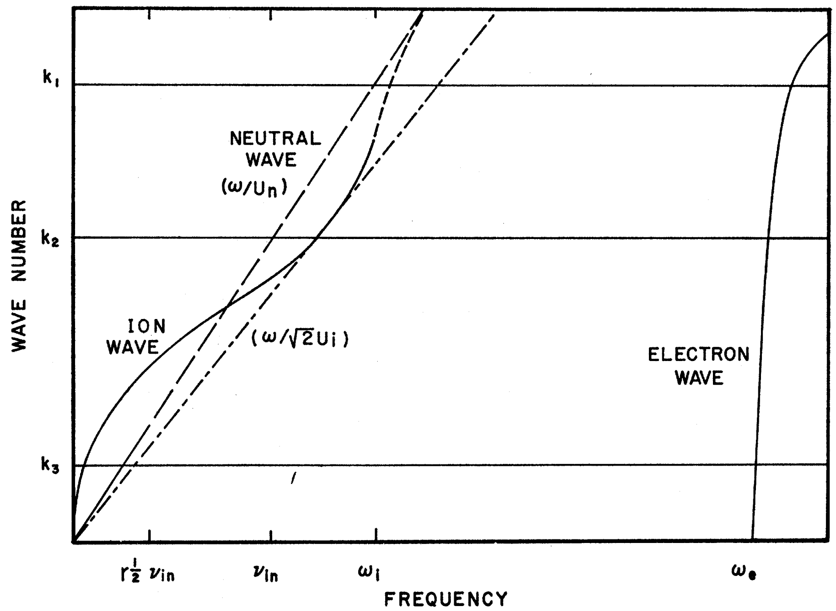

FIG. 1. Wave number versus frequency for electron, ion, and neutral acoustic waves in a plasma. The curves are solid over the frequency range where they involve electron density fluctuations.

where λ_D is the Debye length, and $r=N_{0e}/N_{0n}$ is the ratio of the electron and neutral number densities.

The physical basis for this result is that the scattering is produced only by fluctuations of the *electron*-number density. However, the first condition in (1.1) assures that the electrons follow any fluctuations in the ion density, while the second is needed for the ions to follow any fluctuations in the neutral density.[20] Hence, when the first condition is violated, scattering is produced just by the electron acoustic waves; when *only* the first condition is satisfied, scattering is from the ion and the electron acoustic waves; and when both conditions are met, scattering is from all three waves. (Note that we assume in this discussion that $\lambda_D < r^{-1/2}\lambda_{in}$ as is typically the case for a weakly ionized gas.)

Our results can also be understood in terms of the wave-number-versus-frequency plot for the electron, ion, and neutral acoustic waves shown in Fig. 1. The curves in the figure are *solid* lines for frequencies where the waves involve fluctuations of the electron density and dashed lines where they do not. The conditions used in this plot are the frequency analog to Eq. (1.1); that is, electron-density fluctuations are coupled to those of the ions for $\omega<\omega_i$ and ion-density fluctuations are coupled to those of the neutrals for $\omega<r^{1/2}\nu_{in}$, where ν_{in} is the ion-neutral collision frequency for momentum transfer and ω_i is the ion plasma frequency.

Thomson scattering is produced mainly by those electron-density fluctuations with wave numbers of order $k=4\pi/\lambda_0$. Hence we can tell whether any wave produces Thomson scattering simply by observing whether its wave-number plot is solid in the vicinity of k. For large k's, like k_1 in the figure, scattering is only from the electron wave; for smaller values like k_2, the ion waves also produce scattering; while for very small values like k_3, scattering is produced by all three waves.

The shape of the spectrum can also be inferred to a large extent from the figure, since radiation scattered by an electron-density fluctuation is concentrated at the frequencies $\omega_0\pm\omega$, where ω, the Doppler shift of the scattering, equals the frequency of the acoustic wave producing the scattering. The degree of attenuation is also of importance,[1] however, since an undamped wave produces a sharp resonance, while a damped wave produces a smeared out spectrum. Hence from the figure it is clear that scattering from the electron wave has a Doppler shift of order ω_e, with ω_e the electron plasma frequency; and similarly, the Doppler shift when there is scattering from the neutral acoustic wave is at the frequency kU_n. The Doppler shift in the scattering from the ion wave is not quite so simple, but one can easily see that for values of k near k_2 it is of order $\sqrt{2}kU_i$, while for smaller values of k or larger values of ν_{in} the shift is smaller. (In fact, for $kU_i/\nu_{in}\ll 1$, the Doppler shift can be shown to be of order $4k^2U_i^2/\nu_{in}$.)

II. TWO-FLUID THEORY

The method that we use to calculate the scattering the electron-density fluctuations follows that of from Dougherty and Farley,[1] who show from the Born scattering formula[21] and Nyquist's theorem[22] that the differential cross section for backscattering with no

[20] See, for example, B. S. Tanenbaum and D. Mintzer, Phys. Fluids 5, 1226 (1962).

[21] F. Villars and V. F. Weisskopf, Proc. IRE 43, 1232 (1955).

[22] H. Nyquist, Phys. Rev. 32, 110 (1952); see also H. B. Callen and T. A. Welton, *ibid.* 83, 34 (1951); and H. B. Callen and R. F. Green, *ibid.* 86, 702 (1952).

magnetic field is given by

$$\sigma_b(\omega_0+\omega)d\omega = \mathrm{Re}(N_0 r_e^2 y'/\pi\omega)d\omega. \quad (2.1)$$

In this equation σ_b is the power scattered through 180° per unit incident power per unit volume per unit frequency, r_e is the classical electron radius ($e^2/4\pi\epsilon_0 mc^2$ in mks units), N_0 is the ambient electron-number density, and y', the normalized response of the electron-fluid velocity $\mathbf{u}_e$ to a longitudinal oscillating force $\mathbf{F}_e$ applied to the electrons, is given by

$$y' = k^2\kappa T_0 u_e/\omega F_e, \quad (2.2)$$

where u_e and F_e are both assumed proportional to $e^{i(\omega t-kx)}$, $k=2\omega_0/c$ is twice the incident wave number, κ is Boltzmann's constant, and T_0 is the ambient temperature (assumed equal for all species).

To find y', Dougherty and Farley use kinetic equations for the electrons and ions plus the Maxwell equation (for $\nabla\times\mathbf{H}=0$)

$$0 = \partial\mathbf{D}/\partial t + \mathbf{J}, \quad (2.3)$$

with $\mathbf{D}$ the displacement current and $\mathbf{J}$ the current density. However, when the gas density is sufficiently high, one can obtain a much simpler expression for y' by replacing the kinetic equations with a set of mass, momentum, and energy equations for each species of charged particles. Such transport equations have been derived recently for a binary mixture by Goldman and Sirovich.[23] These authors use the inherently simpler form for the species transport equations that one obtains by defining each species' pressure tensor, temperature, and heat-flow vector relative to $\mathbf{u}_s$ rather than to the mean velocity of the gas as a whole, and neglecting external forces as well as certain quadratic terms in the calculation of the traceless pressure tensors, heat-flow vectors, and collision integrals (for non-Maxwell molecules). As they point out, one result of this is the loss of the thermodiffusion effect; but this is small except under extreme conditions. Granting that there may be small errors due to these approximations, if we nevertheless apply their equations to small-amplitude disturbances of electrons or singly charged ions (with equal ambient-number density) in a weakly ionized gas with $\mathbf{u}_n=0$ and $T_n=T_0$, we find, to first order in the fluctuating quantities,

$$\dot\rho_s + \rho_s\nabla\cdot\mathbf{u}_s = 0, \quad (2.4)$$

$$\dot{\mathbf{u}}_s - (q_s/m_s)\mathbf{E} + (\nabla p_s)/\rho_s - \mathbf{F}_s/m_s - (\eta_{0s}/\rho_s) \times[\nabla^2\mathbf{u}_s + \tfrac{1}{3}\nabla(\nabla\cdot\mathbf{u}_s)] = -\nu_{sn}\mathbf{u}_s, \quad (2.5)$$

$$\dot p_s - (5/3)(p_s/\rho_s)\dot\rho_s - \tfrac{2}{3}\lambda_{0s}\nabla^2 T_s = -N_0\kappa\nu_{sn}'(T_s - T_0). \quad (2.6)$$

The terms on the left here are similar to the linearized Navier-Stokes equations, while the extra terms on the right are the contributions due to momentum and energy transfer by collisions between the charged particles and the neutrals. Note that s takes on the values e (for electrons) and i (for ions), while n denotes neutrals. Also, ρ_s, q_s, m_s, p_s, and T_s are the mass density, charge, mass, pressure, and temperature, respectively, for species s; $\mathbf{E}$ is the electric-field strength; $\mathbf{F}_s$ is the force whose response we are calculating ($\mathbf{F}_i=0$ here); ν_{sn} and $\nu_{sn}'=2m_s\nu_{sn}/(m_s+m_n)$ are, respectively, the effective collision frequencies for momentum and energy transfer with the neutrals; and η_{0s} and λ_{0s}, the viscosity and thermal conductivity for charged particles in a weakly ionized gas, are given by

$$\eta_{0s} = \frac{p_s}{d_s\nu_{sn}}, \quad \lambda_{0s} = \frac{15\kappa p_s}{4m_s c_s \nu_{sn}}, \quad (2.7)$$

with d_s and c_s constants (of order 1 or 2) which depend on the interparticle force law and the mass ratio between the electrons or ions and the neutrals. For interparticle forces of the form K/r^p with all species temperatures equal and $m_i=m_n$, we find[24] from Ref. 23

$$d_i = 1+R, \quad d_e = 2R, \quad c_i = R+\tfrac{3}{2}, \quad c_e = \tfrac{3}{2}, \quad (2.8)$$

with $n=-4/(p-1)$, $R=3(n+6)A_2(p)/20A_1(p)$, and $A_l(p)$ a nondimensional cross section which has been tabulated for a number of cases.[25] Values of these constants for some typical force laws are shown in Table I.

TABLE I. Viscosity and thermal-conductivity constants.

Interparticle force	n	R	d_i	d_e	c_i	c_e	c_n
Hard spheres	0	0.6	1.6	1.2	2.10	1.50	1.2
Maxwell, K/r^5	−1	0.775	1.78	1.55	2.28	1.50	1.55
Coulomb, K/r^2	−4	0.6	1.6	1.2	2.10	1.50	1.2

We now assume that $\rho_s=\rho_{s0}+\rho_s'$, $p_s=p_{s0}+p_s'$, and $T_s=T_0+T_s'$, with the fluctuating quantities ρ_s', p_s', and T_s' (along with $\mathbf{F}_e$, $\mathbf{u}_s$, and $\mathbf{E}$) all proportional to $e^{i(\omega t-kx)}$. In addition, since we have specified longitudinal disturbances, $\mathbf{F}_e$, $\mathbf{u}_s$, and $\mathbf{E}$ are all in the x direction. [Transverse fluctuations, from Eq. (2.4), involve no density changes and therefore no scattering.] With these assumptions, we find from Eqs. (2.3) and (2.4) that

$$E = iN_0 e(u_i - u_e)/\omega\epsilon_0 \quad (2.9)$$

and

$$\rho_s' = \rho_{s0}u_s k/\omega. \quad (2.10)$$

Similarly, from the energy equation (2.6) we find, after

[23] E. Goldman and L. Sirovich, Phys. Fluids **10**, 1928 (1967).

[24] Slightly different values for c_i and c_e are given in E. L. Walker, Ph. D. thesis, Case Institute of Technology, 1967 (unpublished).

[25] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, London, 1952), 2nd ed., p. 172. Values of $A_l(p)$ for $p=-2$ are from B. S. Tanenbaum, *Plasma Physics* (McGraw-Hill Book Co., New York, 1967), pp. 332–335.

using the identity $p_s=\rho_s\kappa T_s/m_s$, that

$$p_s'=\rho_s' V_s^2\Delta_s, \qquad (2.11)$$

where $V_s=(\kappa T_0/m_s)^{1/2}$ is the thermal velocity, and

$$\Delta_s=\frac{1+i(5\omega/3\sigma_s)}{1+i(\omega/\sigma_s)}, \quad \sigma_s=\nu_{sn}'+\frac{5k^2V_s^2}{2c_s\nu_{sn}}. \qquad (2.12)$$

Note as an aside that for Thomson scattering σ_s is constant, so that, as ω approaches 0, $\Delta_s=1$ and Eq. (2.11) reduces to the isothermal law, while as $\omega\to\infty$, $\Delta_s=5/3$ and Eq. (2.11) reduces to the adiabatic law [for particles with no internal degrees of freedom, as is assumed in the derivation of Eq. (2.6)].

Equations (2.9)–(2.11) enable us to eliminate the pressure and **E** field terms in Eq. (2.5), so that (after multiplication by $i\omega/k^2V_s^2$) the x components of these two momentum equations reduce to

$$\begin{aligned} u_e(\alpha^2+z_e)-\alpha^2u_i&=iF_e\omega/m_ek^2V_e^2,\\ -\alpha^2u_e+u_i(\alpha^2+z_i)&=0, \end{aligned} \qquad (2.13)$$

where $\alpha=(k\lambda_D)^{-1}$, the Debye length $\lambda_D=(\kappa T_0\epsilon_0/N_0e^2)^{1/2}$, and

$$z_s=\Delta_s+2i\theta_s[\psi_s+2(3d_s\psi_s)^{-1}]-2\theta_s^2, \qquad (2.14)$$

with $\theta_s=\omega/\sqrt{2}kV_s$ and $\psi_s=\nu_{sn}/\sqrt{2}kV_s$. Note that, because of the large mass difference, $\theta_i\gg\theta_e$, although $\psi_i\sim\psi_e$. From Eq. (2.13) one can easily find the ratio u_e/F_e, and from that ratio and Eqs. (2.1) and (2.2), we find that the cross section is given by

$$\sigma_b(\omega_0+\omega)d\omega=-\mathrm{Im}\left[\frac{N_0r_e^2(\alpha^2+z_i)}{\pi\omega[\alpha^2(z_i+z_e)+z_iz_e]}\right]d\omega. \qquad (2.15)$$

Equation (2.15), while relatively complicated, can be used very easily to calculate the spectra shown in Fig. 2. These are seen to be in very close agreement with the calculations obtained from the Boltzmann equation (with the BGK model for collisions) by Dougherty and Farley[12] over the range of conditions ($\psi_i\geq 1$) where the continuum equations have some validity, but would tend to diverge markedly for lower values of ψ_i. (Note that ψ_i is about $\frac{1}{5}$ times the ratio of the wavelength producing the scattering to the mean free path for an ion.) In these curves, we let $\psi_e=\frac{1}{10}\psi_i$ as is done in the calculations in Ref. 12. However, calculations using $\psi_e=\psi_i/2.8$ (the value which one obtains by assuming hardsphere collisions[26]) show no noticeable difference except at very high frequencies where the power is very small. In addition, we let $\alpha=12.7$, $c_e=d_e=1$, $c_i=d_i=2$, and $m_i/m_e=31\times 1836$; but, again, as long as $\alpha^2\gg 1$, the results are not sensitive to the exact values of these constants.

The total scattered power is easily found by integrating Eq. (2.15) over a contour closed by a semicircle around the lower half-plane. All the poles of the integrand should be in the upper half-plane[27] except the one at $\omega=0$, and the integrand for large ω is proportional to ω^{-3}, so that there is no contribution from the infinite semicircle. Hence, from the theory of residues we have

$$\begin{aligned}\int_{-\infty}^{\infty}\sigma_b(\omega_0+\omega)d\omega&=\mathrm{Im}\,\pi i\,\mathrm{Res}(\omega=0)\\ &=N_0r_e^2(\alpha^2+1)/(2\alpha^2+1) \end{aligned} \qquad (2.16)$$

in exact agreement with the kinetic-theory result.

There are other calculations which also agree very closely with Dougherty and Farley's results. For example, when the collision frequency is high, in the limit of very *small* Debye length ($\alpha\to\infty$), Eq. (2.15) reduces to

$$\sigma_b(\omega_0+\omega)d\omega=-\mathrm{Im}[N_0r_e^2d\theta_i/\pi\theta_i(z_i+z_e)]\to \frac{N_0r_e^2\psi_id\theta_i}{2\pi(1+\theta_i^2\psi_i^2+\{\theta_i^4\})}, \quad \psi_i\gg 1. \qquad (2.17)$$

Conversely, in the limit of very large Debye length ($\alpha\to 0$), the spectrum reduces to

$$\sigma_b(\omega_0+\omega)d\omega=-\mathrm{Im}[N_0r_e^2d\theta_e/\pi\theta_ez_e]\to \frac{2N_0r_e^2\psi_ed\theta_e}{\pi(1+4\theta_e^2\psi_e^2+\{4\theta_e^4\})}, \quad \psi_e\gg 1. \qquad (2.18)$$

Except for the terms in braces here (which merely reduce the already negligibly small cross sections for very large θ's), these spectra agree exactly with the kinetic-theory results. Note that in the limit as $\psi\to\infty$, these spectra are δ functions centered at ω_0. This is because the characteristic Doppler shift for the scattering from a wave with phase velocity V_{ph} is of order kV_{ph}, but for a collision-damped electron or ion acoustic wave, $V_{\mathrm{ph}}\to 0$.

To calculate the value of the spectrum at zero Doppler shift, note that for small ω, $z_s=1+i\omega\beta_s+O(\omega^2)$, with

$$\beta_s=(2/3\sigma_s)+(\sqrt{2}/kV_s)[\psi_s+2(3d_s\psi_s)^{-1}]. \qquad (2.19)$$

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[26] B. S. Tanenbaum, *Plasma Physics* (McGraw-Hill Book Co., New York, 1967), p. 253.

[27] For arbitrary α we assume this to be true on physical grounds, since any other singularity arises only if $\alpha^2(z_i+z_e)+z_iz_e=0$. This condition is the dispersion equation for longitudinal waves (with k real) in the plasma. Hence, it is satisfied by a complex ω in the lower half-plane only if there is a plasma wave that grows exponentially in time (a result which we do not expect here). Moreover, in the limits when α approaches 0 or ∞, we can show that the conditions for another singularity ($z_e=0$ for $\alpha=0$ and $z_e+z_i=0$ for $\alpha=\infty$) cannot be met by an ω in the lower half-plane. The proof is simple. For $\omega=x-iy$ (with x and y real and y positive) the conditions lead to an equation of the form

$$P_1-2x^2+iP_2x=0,$$

with P_1 and P_2 both real and positive definite. Since this equation cannot be satisfied by any real x, there are no solutions in the lower half-plane.

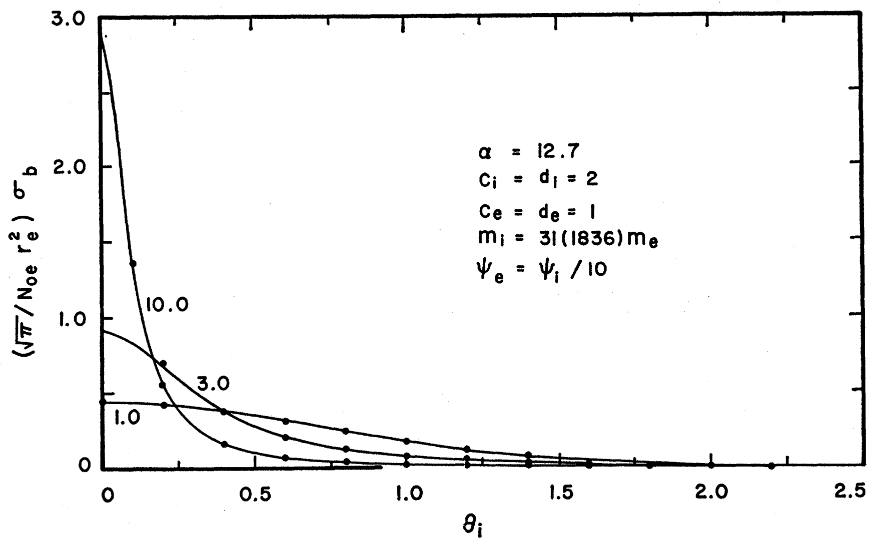

FIG. 2. Normalized Thomson-scatter cross sections versus frequency ($\theta_i=\omega/\sqrt{2}kV_i$) for the two-fluid continuum theory (solid curves) and for the kinetic theory of Ref. 12 (circles). The curves are for $\psi_i=10$, 3, and 1, with $\psi_i=\nu_{\text{in}}/\sqrt{2}kV_i$.

Hence, after a bit of algebra, one easily finds that

$$\sigma_b(\omega_0)d\omega=\frac{N_0r_e^2}{\pi(2+\alpha^{-2})^2}[\beta_i+(1+\alpha^{-2})^2\beta_e]d\omega. \quad (2.20)$$

Of the three terms in our expression for β_s the first comes from the damping terms in the energy equation, the second from the momentum-transfer term, and the last from the viscosity term. For high collision frequencies, when the momentum-transfer term dominates, and $\beta_s\cong\sqrt{2}\psi_s/k_sV_s$, Eq. (2.20) again agrees exactly with the kinetic-theory result[28] of Ref. 12.

In summary, we find that continuum equations suitable for electrons and ions in a weakly ionized gas can be used to calculate Thomson scattering. Moreover, for $\psi_i>1$ there is excellent agreement in all respects with a previous kinetic-theory calculation of the spectrum of the scattered radiation. Hence one has some confidence that the new results calculated in the next section with a three-fluid theory should also be valid when the mean free paths are relatively small.

III. THREE-FLUID THEORY

In order to find the effect of thermal fluctuations of the neutral-molecule density upon the Thomson scattering from a plasma, one must use a set of coupled equations for the electrons, ions, and neutrals. In a full three-fluid theory, one would have three sets of mass, momentum, and energy equations like Eqs. (2.4)–(2.6), but with additional heat-flow and pressure-tensor terms on the left and the collisional transfer terms for momentum and energy on the right given, respectively, by

$$-\sum_t \nu_{st}(\mathbf{u}_s-\mathbf{u}_t) \quad \text{and} \quad -\sum_t N_{0s}\kappa\nu_{st}'(T_s-T_t), \quad (3.1)$$

[28] Note that Eq. (19) of Ref. 12 has a misprint, and should have the factor $(1+h^2k^2)^2$, not $(1+h^2k^2)$.

with t summed over all species. However, when all these coupling terms are used, the resulting expression for y' becomes extremely complicated. Therefore, to simplify the analysis, while still retaining most of the physically important terms, we make the following assumptions: (1) that collisions between charged particles are so infrequent that they can be neglected; hence the collisional transfer terms for momentum for the charged particles reduce to $-\nu_{sn}(\mathbf{u}_s-\mathbf{u}_n)$; and (2) that the energy equation for each species can be replaced by an equation of the form

$$p_s'=V_s^2\rho_s'\Delta_s' \quad (3.2)$$

instead of the more general result $p_s'=\sum_t A_{st}\rho_t'$ that one obtains with no approximation. In (3.2) we also assume that the charged-particle density fluctuations causing the scattering are isothermal, so that $\Delta_e=\Delta_i=1$, and that the neutral-molecule density fluctuations do not depend significantly on the energy collisional transfer terms, so that [after an analysis similar to the one leading to Eqs. (2.11) and (2.12)] we find

$$\Delta_n'=\frac{1+i(5\omega/3\sigma_n)}{1+i(\omega/\sigma_n)}, \quad \sigma_n=\frac{5k^2V_i^2}{2c_n\nu_{in}}. \quad (3.3)$$

Note that c_n is a constant (again of order 1 or 2) that is given by $c_n=\rho_{0n}V_i^2/\eta_{0n}\nu_{in}$ and is tabulated for some typical force laws in Table I (assuming $V_i^2=V_n^2$ and, somewhat artificially, that the interparticle force between an ion and a neutral is the same as between two neutrals).

With these simplifications we find that when we consider the response of the plasma to an electron perturbation proportional to $e^{i(\omega t-kx)}$, the three momentum equations reduce [after using Eqs. (2.9) and (2.10) and multiplying by $i\omega/k^2V_s^2$] to

$$\begin{aligned} u_e(\alpha^2+z_e')-\alpha^2u_i-i\xi_eu_n&=i\omega F_e/m_ek^2V_e^2\,,\\ -\alpha^2u_e+u_i(\alpha^2+z_i')-i\xi_iu_n&=0\,,\\ -ir\xi_eu_e-ir\xi_iu_i+z_n'u_n&=0\,,\end{aligned}\tag{3.4}$$

where $r=N_{0e}/N_{0n}$, $\xi_s=2\theta_s\psi_s$, and z_s' is similar to z_s as given in Eq. (2.14) except that here, for the charged particles, $\Delta_s'=1$ because of the isothermal assumption, while for the neutrals,

$$z_n'=\Delta_n'+ir(\xi_i+\xi_e)+(4i\theta_i/3d_n\psi_i)-2\theta_n^2\,.\tag{3.5}$$

The constant d_n satisfies $d_n=p_{0n}/\eta_{0n}\nu_{in}=c_nV_n^2/V_i^2$; hence, generally d_n and c_n are equal.

Equation (3.4) is easily solved to find a new value for u_e/F_e, and from that ratio and Eqs. (2.1) and (2.2) we find that the three-fluid cross section for backscatter is

$$\sigma_b(\omega_0+\omega)d\omega=\frac{-\mathrm{Im}N_{0e}r_e^2d\omega}{\pi\omega}\left[\frac{\alpha^2+z_i'+T_1}{\alpha^2(z_e'+z_i')+z_e'z_i'+T_2}\right],\tag{3.6}$$

where

$$\begin{aligned} T_1&=r\xi_i^2/z_n'\,,\\ T_2&=r[\alpha^2(\xi_e+\xi_i)^2+\xi_e^2z_i'+\xi_i^2z_e']/z_n'\,.\end{aligned}\tag{3.7}$$

Before discussing the numerical calculations based upon Eq. (3.6), there are several points worth noting about this result.

(1) In the limit when $r\rightarrow0$ (very low ionization), T_1 and T_2 approach zero and (3.6) reduces to the two-fluid isothermal result [Eq. (2.15) with $\Delta_s=1$].

(2) As ω approaches zero, T_1 and T_2 are both proportional to ω^2. Hence, if we follow the same procedures used in the last section, we find that the total scattered power is again given by Eq. (2.16). In addition, $\sigma_b(\omega_0)$ is still given by an expression like Eq. (2.20) except that the values for β_e and β_i no longer include the (generally negligible) first term in Eq. (2.19), again due to the isothermal assumption in this section.

(3) The extra terms T_1 and T_2 are both *inversely* proportional to z_n'; hence they tend to become important at the neutral acoustic resonance when the real part of z_n' goes to zero. Simple analytic expressions for the cross section near this resonance are difficult to obtain in general, but can be found for some special cases. As an example (which is applicable for a typical back-scatter experiment from the lower ionosphere), we find that for $\alpha^2\gg1$, $\psi_i^2\gg1$, and $r\psi_i\ll1$, the spectrum is virtually identical to the two-fluid result except for frequencies very close to the resonance ($\theta_i^2\cong\frac{5}{6}$), where

$$\sigma_bd\omega\cong(N_{0e}r_e^2d\theta_i/\pi\theta_i\xi_i)[(r\xi_i^2+C_1)/C_1]\,,\tag{3.8}$$

with $C_1=(8\theta_i^2/3d_n)+(5/3c_n)$. For $r\xi_i^2\ll C_1$ (or $r\psi_i^2\ll1$) the term in parentheses equals 1, and (3.8) reduces to the two-fluid result [Eq. (2.17)]. However, for $r\xi_i^2\gg C_1$ (or $r\psi_i^2\gg1$), there is a large neutral acoustic resonance. This effect can be seen in Fig. 3, where the spectrum [calculated numerically from Eq. (3.5)] is plotted for $\psi_i=10$ and $r=10^{-3}$, 10^{-2}, and 10^{-1}, along with the two-fluid calculation. All constants are the same as in Fig. 2 and we also let $c_n=d_n=2$. When $r\psi_i^2=0.1$, the resonance is absent; for $r\psi_i^2=1$, it is clearly present; and for $r\psi_i^2=10$, it is very pronounced. In addition, the good agreement between these curves and the earlier two-fluid calculation away from the resonance suggests that the assumptions used to simplify the three-fluid theory do not affect the results very seriously.

IV. DISCUSSION

Aside from demonstrating that one can reproduce the Thomson-scattering spectrum for a plasma with $\psi_i>1$ by using continuum equations rather than kinetic

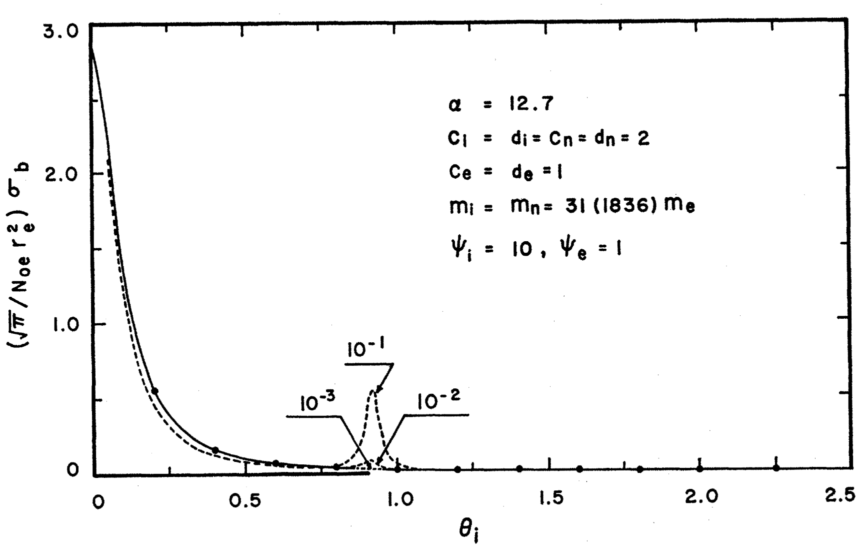

FIG. 3. Normalized Thomson-scatter cross sections versus frequency ($\theta_i=\omega/\sqrt{2}kV_i$) for the two-fluid continuum theory (solid curves), for the kinetic theory of Ref. 12 (circles), and for the three-fluid theory (dotted curves). The curves are for $r=10^{-1}$, 10^{-2}, and 10^{-3}, with $r=N_{0e}/N_{0n}$.

theory, the major new result in this paper is the calculation of the neutral acoustic resonance in the spectrum for $r\psi_i^2>1$. If this effect can be observed experimentally, it would provide an additional diagnostic tool for studying plasmas. Since the resonance has a Doppler shift of kU_n, this provides a direct measure of the neutral sound speed. In addition, the relative amplitude of the resonance involves the percent ionization and the ion-neutral collision frequency. The latter can also (for $\psi_i\gg 1$) be determined from the value of the spectrum at ω_0. Hence, in principle at least, one can determine both parameters from the values of the spectrum at ω_0 and at $\omega_0\pm kU_n$.

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Variance of H in the Bijl-Feynman Description of an Elementary Excitation*

TOLLIE DAVISON†
Stephen F. Austin College, Nacogdoches, Texas

AND

EUGENE FEENBERG
Arthur H. Compton Laboratory of Physics, Washington University, St. Louis, Missouri

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The trial function $\rho_{\mathbf{k}}\psi_0$ is a useful approximation to the lowest eigenstate of H with momentum $\hbar\mathbf{k}$ in the theory of a uniform system of interacting bosons. The closeness of the approximation can be tested by studying the variance of H with respect to $\rho_{\mathbf{k}}\psi_0$. The leading term in the variance at long wavelengths ($k\ll 2\pi\rho^{1/3}$) is found to vanish when ψ_0 is given the asymptotic properties implied by the identification of the elementary excitations described by $\rho_{\mathbf{k}}\psi_0$ with the phonons in the quantized version of the classical sound field.

THE trial function $|\mathbf{k}\rangle=\rho_{\mathbf{k}}\psi_0[NS(k)]^{-1/2}$ is recognized as a useful approximation to the lowest eigenstate of H with momentum $\hbar\mathbf{k}$[1,2] in the theory of a uniform system of interacting bosons. The closeness of the approximation in the long-wavelength limit can be studied by computing the variance of H with respect to $|\mathbf{k}\rangle$ and examining the behavior of this quantity as a function of k. We use the following notation: ψ_0 is the exact normalized ground-state wave function of the N-particle system in a box of volume Ω (number density $\rho=N/\Omega$). The energy eigenvalue is E_0. $g(r)$ and $S(k)$ are the radial distribution function and the liquid structure function generated by ψ_0.[3] $\rho_{\mathbf{k}}=\sum_1^N e^{i\mathbf{k}\cdot\mathbf{r}_j}$ is the Fourier transform of the density operator. $\epsilon_0(k)=\hbar^2k^2/2mS(k)$ is the Bijl-Feynman formula for the energy of an elementary excitation. $S(\omega,k)$ is the dynamic form factor.[4]

Our problem is to extract physical information from the quantity

$$\begin{aligned}\langle\mathbf{k}|(H-E_0-\epsilon_0(k))^2|\mathbf{k}\rangle &=[1/NS(k)]\langle 0|\rho_{-\mathbf{k}}(H-E_0-\epsilon_0(k))^2\rho_{\mathbf{k}}|0\rangle \\ &=\frac{1}{S(k)}\int_0^\infty(\omega-\epsilon_0(k))^2S(\omega,k)d\omega.\end{aligned}\tag{1}$$

The exact relation

$$(H-E_0-\epsilon_0(k))\rho_{\mathbf{k}}\psi_0=(\hbar^2k^2/2m-\epsilon_0(k))\rho_{\mathbf{k}}\psi_0-\frac{\hbar^2}{m}\sum_j e^{i\mathbf{k}\cdot\mathbf{r}_j}i\mathbf{k}\cdot\nabla_j\psi_0\tag{2}$$

has the consequence

$$\begin{aligned}(1/N)\langle 0|\rho_{-\mathbf{k}}(H-E_0-\epsilon_0)^2\rho_{\mathbf{k}}|0\rangle=&-(\hbar^2k^2/2m)^2\\ &\times[\sqrt{S(k)}-1/\sqrt{S(k)}]^2+(\hbar^2/m)^2\int\mathbf{k}\cdot\nabla_1\psi_0\\ &\times[\mathbf{k}\cdot\nabla_1\psi_0+(N-1)e^{i\mathbf{k}\cdot\mathbf{r}_{12}}\mathbf{k}\cdot\nabla_2\psi_0]d\mathbf{r}_{1,2,\ldots N}.\end{aligned}\tag{3}$$

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[1] A. Bijl, Physica 7, 869 (1940).
[2] R. P. Feynman, Phys. Rev. 94, 262 (1954).
[3] *Lectures in Theoretical Physics*, edited by W. E. Brittin (University of Colorado Press, Boulder, Colo., 1965), pp. 160–174.
[4] K. Huang and A. Klein, Ann. Phys. (N. Y.) 30, 203 (1964).