Relativistic Lagrangian Field Theory for Composite Systems*

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It is shown that infinite-component field theories provide a useful alternative to the Bethe-Salpeter equation as a fully relativistic treatment of composite systems. Scattering amplitudes obtained by models of this type satisfy both Mandelstam analyticity and conspiracy requirements. Current algebras are saturated with a combination of discrete and continuous spectra. Most of the paper is devoted to a special example, for which it is found that the mass spectrum has a discrete part (bound states) and a continuous part (scattering states); that the metric in physical Hilbert space is positive definite; and that vertex functions and scattering amplitudes are analytic functions of s and t, with singularities at the same locations as in local field theory. The role of "spacelike solutions" is studied in detail, with some surprising results.

I. INTRODUCTION

THE real motivation for this work is to demonstrate the usefulness of infinite-multiplet techniques in the context of recent developments in elementary particle physics. For example, it is easy to find exact representations of current algebra, with either one-particle intermediary states, two-particle states, or both. Regge poles and Lorentz poles fit in exceedingly well, and conspiracy theory is generalized to nonzero momentum transfer and unequal masses. The same simple Feynman rules that predict physically reasonable form factors and scattering amplitudes in the Born approximation, also apply to the coupling between Reggeons or between conspiracies. The amplitudes satisfy both Mandelstam analyticity and l-plane analyticity, which allows one to study the analytic properties in s and t of a scattering amplitude with Regge poles or conspiracies. The possibility of incorporating a relativistic version of $SU(6)$ is always open.

However, the ultimate relevance of all this remains doubtful if the theory is plagued by any of the so-called diseases of infinite-component field theory. Consequently, most of this paper is perforce devoted to an investigation of pathologies, real and imagined.

Two common attitudes are: that the diseases are so malign that all infinite-component field theories are going to die; or that the so-called diseases are irrelevant because the theory is used only to construct representations of current algebra, and one does not have to take the whole formal machinery of field theory seriously. This paper argues against the first point of view on the grounds that all the diseases are gradually being cured.[1] It argues against the second viewpoint because some of the diseases are far from being merely formal; for example, the positivity of the metric of the physical Hilbert space is closely related to the reality of coupling constants and to the positivity of the imaginary part of the forward scattering amplitude, and spacelike solutions tend to produce unphysical left-hand cuts in the scattering amplitudes.

Summary

Section II is a brief review of the main ideas on which this paper is based, for the convenience of readers who are unfamiliar with our previous papers.

In Sec. III, we study a particular infinite-component field equation. The mass spectrum (Sec. III B) consists of a hydrogenlike spectrum of single-particle states, a right-hand continuum of two-particle scattering states, and a left-hand continuum of "exchange states." We argue that a two-particle interpretation is possible, and determine the masses m_1 and m_2 of the two constituents. It turns out that most equations previously discussed in the literature are special limiting cases, for example, $m_1=m_2=\infty$, which explains why those equations led to form factors with unorthodox singularities. In the general case, with arbitrary constituent masses, all the singularities of the form factors—in the first Born approximation—are recognized as being completely conventional (Sec. III C). The nonrelativistic limit is investigated in Sec. III D. The metric in Hilbert space is calculated and found to be positive for the bound states as well as for the scattering states (Sec. III E).

Section IV is an investigation of the analytic properties of amplitudes in the first Born approximation. The vertex function is studied as an analytic function of two complex variables (Sec. IV A). All singularities are the conventional ones that are found in ordinary field theory, associated with the ordinary Feynman diagram of Fig. 5. This, of course, is strong support for

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[1] When our methods are confronted with other approaches, it is only fair to remember that the so-called diseases of infinite-component field theory come into evidence just because the theory is technically manageable. Some of the more conventional approaches to current algebra, for example, can be said to avoid problems by concerning themselves with a small set of matrix elements of the commutators. Similarly, amplitudes represented by means of Regge representations are not usually rejected on the basis that they fail to satisfy a Mandelstam representation. The Bethe-Salpeter equation in the ladder approximation is an example of an infinite-component field equation—and a very sick one at that, if the conclusions of Predazzi and others are correct [see E. Predazzi, Nuovo Cimento 40, 913 (1965), and other references quoted therein]—and yet it is used as a guide, not only near threshold where it derives legitimacy from being a good approximation to the Schrödinger equation, but also near $s=0$, where it surely has no relevance to any real physical problem.

the two-particle interpretation, including the values of the constituent masses. Results for "Compton" scattering are similar (Sec. III B). The singularities are the familiar 2-point, 3-point, and 4-point Landau singularities found in ordinary field theory, associated with the diagrams of Fig. 9. The distinguished boundary is found to coincide with the support of the Mandelstam double-spectral function for the box diagram, except for an extra, disjoint piece. This extra contribution to the Mandelstam representation contributes a fixed left-hand cut in s in the amplitude $A(s,t)$. It is associated with the spacelike solutions, but not so intimately as expected (see below).

In Sec. V, we express the scattering amplitude by means of an integral representation of the type that occurs in conspiracy theory. This allows a more detailed investigation of the connection between the fixed left-hand cut and the spacelike solutions (Sec. V A). A very surprising result is that the left-hand cut persists in some models that have no spacelike solutions (Sec. V B). It is easy to construct models without spacelike solutions, but it may turn out to be harder to expurgate the left-hand cut. The connection with conspiracy theory, generalized to nonzero momentum transfer and unequal masses, is pointed out (Sec. V C). The theory also includes a scheme for couplings between several Reggeons or conspiracies.

In Sec. VI A it is pointed out that most of the so-called diseases of infinite-component field theory are peculiar to specific models, and, for the most part, curable. The only difficulty that remains, the left-hand cut, is discussed in Sec. VI C. Finally, we point out that all models of this type provide exact representations of current algebras, with both one- and two-particle states.

II. SYNOPSIS

We present here a concise summary of the ideas on which our recent work is based. By means of the present paper we hope to demonstrate their physical content.

(i) Physical states are described by a subset of a set of wave functions $\psi_\sigma(x)$, where the argument is the spacetime coordinate, and σ stands for one or more discrete indices. The theory is trivial unless σ includes the spin indices j and j_z, and it is conventional if j is bounded. Our main interest is in the case when j is unbounded and σ takes an infinite set of values. The discrete index can be replaced by a set of continuous variables, and these are susceptible to interpretation as internal variables.[2] It is not necessary, in principle, for any group other than the Poincaré group to be introduced; however, it is technically convenient to let $\psi_\sigma(x)$ span a Hilbert space $\mathcal{H}_m$ on which is realized an irreducible representation of a group S that includes the spin part of the homogeneous Lorentz group as a subgroup. It is not necessary that this representation be unitary, but it turns out that the physical interpretation requires it to be "almost" unitary—this is to ensure that the physical Poincaré group be unitarily represented.[3] It needs to be emphasized that the larger group S is not an invariance group, though a subgroup of S may play that role.

(ii) Out of the Hilbert space $\mathcal{H}_m$ of wave functions $\psi_\sigma(x)$ a subset is selected by means of a wave equation. The most obvious content of the wave equation is the establishment of a mass spectrum, and this must be required to be reasonable. It is possible that the wave equation has the specification of a mass spectrum as its sole purpose and that the theory should be developed in a phenomenological direction.[4] Even so, it is useful to consider its place in a Lagrangian field theory, temporarily at least, in order that we may profit from the concise statements of fundamental physical insight that are contained in the Lagrangian formulation. In this context it is natural to require that the wave equation be a differential equation of low order.[5] Other physical requirements must also be met; some obvious ones are discussed in this paper.

(iii) If one chooses, as we do in this paper, to develop the Lagrangian approach, then the following directions are possible: *First*, one notes that the physical metric, with respect to which the stationary states are orthogonal and the energy is a Hermitian operator, is uniquely given. This metric must be used to define physical probability and may be called the probability metric. The physical states form a Hilbert space $\mathcal{H}_p$ that is distinct from the mathematical Hilbert space $\mathcal{H}_m$. To every physical operator O_p in $\mathcal{H}_p$ there corresponds an operator O_m in $\mathcal{H}_m$, so that the matrix elements of O_p in the physical metric are equal to the matrix elements of O_m in the mathematical metric. (As an example, we recall that if O_p is the dipole operator of the nonrelativistic hydrogen atom, then O_m is a generator of the group S [6]; this fact affords a great simplification in practical calculations[7] and tends to unify the methods of atomic and elementary particle physics.) *Second*, Feynman rules may be developed to calculate amplitudes for scattering of the physical states represented by the solutions of the wave equation by external sources, or by each other. This gives rise to Feynman rules for the coupling of Reggeons.[8] *Third*, second quan-

[2] This interpretation has been developed by T. Takabayasi in a series of papers. See, for example, the review article in Progr. Theoret. Phys. (Kyoto) **34**, 124 (1965).

[3] In the case of fermions it may be convenient to start with a pair of conjugate, nonunitary representations. See C. Fronsdal and R. White, Phys. Rev. **163**, 1835 (1967).

[4] This point of view has been stressed by A. O. Barut. See, for example, *Lectures in Theoretical Physics* (Gordon and Breach Science Publishers, Inc., New York, 1968), Vol. X B, and further references given therein.

[5] We first became interested in wave equations because they provide a method for assigning different masses to the states of infinite multiplets without running afoul of gauge invariance. From this point of view one must insist that the wave equation contain a finite number of derivatives only.

[6] C. Fronsdal, Phys. Rev. **156**, 1665 (1967).

[7] A. Barut and H. Kleinert, Phys. Rev. **160**, 1149 (1967).

[8] C. Fronsdal, Phys. Rev. **168**, 1845 (1968).

tization may be carried out. Notwithstanding, the (possibly) formal nature of the canonical commutation rules; this provides very nontrivial solutions of current algebra. In the case of a purely discrete mass spectrum the algebra is saturated with one-particle states, but many-particle contributions are included in general.

(iv) Couplings of the physical states to external sources are taken to be "local."[9] That means that the scalar, vector, $\cdots$ densities that are coupled to the external fields are of the form $\psi_\sigma^\dagger(x)O_{\sigma\sigma'}\psi_{\sigma'}(x)$, $\psi_\sigma^\dagger(x)O_{\sigma\sigma'}{}^\mu\psi_{\sigma'}(x)$, $\cdots$, where the operators O, O^μ, $\cdots$ are independent of x and contain at most a finite number of derivatives with respect to x. The form factors given by local couplings have been shown to have a qualitatively correct momentum dependence.[10–12] In the case of the nonrelativistic hydrogen atom the exact interaction (including all multipoles) with a transverse external electromagnetic field is of this form.[6] It should be noted that the physical application of local couplings requires rules for the evaluation of matrix elements. That is, one has to specify the physical states *and* the physical metric with respect to which these are orthogonal. This can be done without invoking the Lagrangian formalism, but in any case, it must be done. The physical metric is not the same as the mathematical metric, unless the mass spectrum has infinite degeneracy.[6,13]

(v) The identification of the "physical states"—the solutions of the wave equation—with actual physical systems, depends on the nature of the mass spectrum and on other details of the model. It is obvious that if the positive part of the spectrum of (mass)^2 contains a continuum, then this must be associated with states of two or more particles. The nonrelativistic hydrogen atom is a good example of the economy that may be achieved by including discrete states and continua in a single irreducible representation.[6] The mass spectrum of a two-particle system contains information about the mass of each of the two constituents, and their electromagnetic properties can be inferred from the form factors of the bound states.

III. SOLUBLE TWO-PARTICLE DYNAMICS

A. Model

The example chosen for detailed study is, we believe, one of the simplest ones that is susceptible to detailed and realistic physical interpretation. The space $\mathcal{H}_m$ of wave functions $\psi_\sigma(x)$ carries a unitary irreducible representation of the group $SO(4,1)$. The representation is the same that was used for treating the nonrelativistic hydrogen atom.[14,15,6] It is the analog for $SO(4,1)$, of the representation of $SO(3,1)$ used by Majorana in his field equation of 1932.[16] In addition to the ten generators S_{ab}, a, $b=0$, 1, 2, 3, 4, there exists, in this particular representation, a set of covariant matrices, Γ_a, $a=0$, 1, 2, 3, 4, that transform among themselves like the components of a five-vector. The complete set of 15 constant operators generate a unitary irreducible representation of $SO(4,2)$.[17]

The most general second-order differential equation that is linear in the generators of $SO(4,2)$ can be reduced to the form

$$L(i\partial/\partial x)\psi(x)=0\,, \tag{III1}$$

$$L(p)=\alpha p\Gamma+(p^2-\beta)\Gamma_4+\tfrac{1}{2}e^2(p^2-\gamma)\,. \tag{III2}$$

Here α, β, γ, and e^2 are constants, $p\Gamma$ is the Lorentz-invariant vector product, and Γ_4 is Lorentz invariant. This equation is asymmetrical with respect to the sign of the energy, so we hasten to remedy that by doubling the representation space[18] and interpreting α as a two-by-two matrix:

$$\alpha\rightarrow|\alpha|\tau_3\,. \tag{III3}$$

Special cases of this equation are Nambu's first equation,[19] the Schrödinger equation for the nonrelativistic hydrogen atom,[6] and several examples studied by Barut and Kleinert,[20] Takabayasi,[21] and others.

B. Mass Spectrum

Equation (III1) is easily solved by Nambu's method.[19] Introduce the unit vector—spurion[22]—λ_a and the operator n by

$$\lambda_a=Q^{-1/2}\{|\alpha|p_\mu,p^2-\beta\}\,, \tag{III4}$$

$$Q=\alpha^2p^2-(p^2-\beta)^2\,,\quad n=\lambda_a\Gamma_a\,. \tag{III5}$$

[9] C. Fronsdal, Phys. Rev. **156**, 1653 (1967).
[10] C. Fronsdal, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy*, edited by B. Kursunoglu, A. Perlmutter, and I. Sakmar (W. H. Freeman and Co., San Francisco, Calif., 1966), p. 48.
[11] G. Cocho, C. Fronsdal, Harun Ar-Rashid, and R. White, Phys. Rev. Letters **17**, 275 (1966).
[12] A. O. Barut and H. Kleinert, Phys. Rev. **156**, 1546 (1967).
[13] E. Abers, I. T. Grodsky, and R. E. Norton, Phys. Rev. **159**, 1222 (1967).
[14] I. A. Malkin and V. I. Man'ko, Zh. Eksperim. i Teor. Fiz. Pis'ma v Redaktsiyu **2**, 230 (1965) [English transl.: Soviet Phys.—JETP Letters **2**, 146 (1965)].
[15] A. O. Barut and H. Kleinert, Phys. Rev. **156**, 1541 (1967).
[16] E. Majorana, Nuovo Cimento **9**, 335 (1932).
[17] The matrix elements of all the generators have been given, e.g., in Ref. 6. The fact that this irreducible representation of $SO(4,1)$ can be extended to $SO(4,2)$ was pointed out in Ref. 14. The first suggestion that $SO(4,2)$ could be useful was made by Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Letters **17**, 148 (1965).
[18] The complete separation of the representation space into two disjoint halves is artificial. It would be better to choose an irreducible representation of $SO(4,2)$ in which the spectrum of Γ_0 is symmetric about zero. Another way to avoid the doubling is introduced in Sec. VI C.
[19] Y. Nambu, Progr. Theoret. Phys. (Kyoto) Suppl. **37**, 368 (1966); **38**, 368 (1966).
[20] A. O. Barut and H. Kleinert, Phys. Rev. **157**, 1180 (1967).
[21] T. Takabayasi, University of Nagoya Report, 1967 (unpublished).
[22] The spurion is analogous to the "kinetic spurion" introduced into broken $SU(6)$ theory by several people. For every value of the four-vector p_μ, λ_a is a direction in 5-space. The existence of a distinguished direction represents breaking of $O(4,1)$ symmetry. The degeneracy group is that subgroup of $O(4,1)$ that leaves λ_a invariant; it is isomorphic to $O(4)$ if λ_a is "timelike" and to $O(3,1)$ if λ_a is "spacelike." See also Ref. 6, last section.

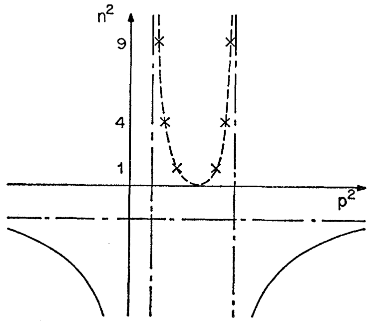

FIG. 1. The (unphysical) spectrum given by Eq. (III7). The discrete states are at $n=1, 2, 3, \cdots$; the solid lines are the two continua.

If $Q>0$, then the five-vector λ_a is "timelike," $\lambda_a{}^2=+1$; in this case the eigenvalues of the operator n are the same as those of Γ_0 (the positive integers) or $-\Gamma_0$, according to the sign of p_0. If $Q<0$, then the spurion is "spacelike" and purely imaginary; the spectrum of n is the same as that of $i\Gamma_4$, the entire imaginary axis. The wave equation in momentum space now takes the form

$$[nQ^{1/2}\tau_3+\tfrac{1}{2}e^2(p^2-\gamma)]\psi=0 \qquad \text{(III6)}$$

or

$$n^2=\tfrac{1}{4}e^4\frac{(p^2-\gamma)^2}{(\alpha p)^2-(p^2-\beta)^2}. \qquad \text{(III7)}$$

This spectrum is plotted in Figs. 1 and 2.

In Fig. 1 it is assumed that the values of α and β are such that the denominator has real zeros—otherwise no physical interpretation seems possible. The discrete states are indicated by crosses on the dashed curve, the continua by solid curves. We note that the zeros of the denominator are accumulation points of discrete states. The appearance of an accumulation point on the high side is normal in the presence of long-range forces, but the other accumulation point is unphysical. We therefore choose the values of α, β, and γ in such a way that $p^2=\gamma$ is a zero of the denominator.

Putting

$$\gamma=m_-{}^2, \quad \beta=m_-m_+, \qquad \text{(III8)}$$

$$\alpha^2=(\gamma-\beta)^2/\gamma=(m_+-m_-)^2, \qquad \text{(III9)}$$

we obtain the simpler formula

$$n^2=-\tfrac{1}{4}e^4\frac{p^2-m_-{}^2}{p^2-m_+{}^2}. \qquad \text{(III10)}$$

In the limit $-m_-{}^2\to\infty$ this reduces to the spectrum of Nambu's first equation,[19] illustrated in Fig. 2. In order to obtain an ascending mass spectrum it is necessary to take $m_-{}^2<m_+{}^2$. This case is illustrated in Fig. 3. We attempt to show that this mass spectrum is reasonable by giving a coherent physical interpretation of each of its parts.

The discrete states are the most obvious ones; they are one-particle states (bound states). The timelike continuum, represented by the solid lower right-hand curve in Fig. 3, can only be interpreted as many-particle states. We attempt a two-particle interpretation. This identification of the timelike continuum with two-particle scattering states is justified by the analogous formulation of the Schrödinger theory of the nonrelativistic hydrogen atom.[6]

If our system consists of two particles, in bound or scattering states, then the wave equation is a kind of Bethe-Salpeter equation,[23] and the wave operator $L(p)$ is the inverse of the two-particle Green's function. The latter is related to the two-particle scattering amplitude. The solutions of the wave equation are related to the singularities of the scattering amplitude; thus the discrete states correspond to poles, and the timelike continuum corresponds to the right-hand physical region. The two-particle scattering amplitudes of conventional field theory have a left-hand physical region as well, and this provides an interpretation of the left-hand continuum. In fact, consider the two scattering diagrams of Fig. 4, where the two internal lines represent particles with masses m_1 and m_2. The first diagram illustrates the right-hand physical region, $s>(m_1+m_2)^2$, of the two-particle Green's function; it is the domain of s covered when the momenta of the intermediary particles are real and on the mass shell. The second diagram illustrates, in the same way, the left-hand physical region $s<(m_1-m_2)^2$.[24]

These arguments are not meant to justify the existence of the spacelike solutions—that is a question that must be decided by detailed examination of the scattering amplitudes (Secs. IV and V). The only conclusion that we wish to draw here is that the endpoints of the two continua are to be identified with the true and the false thresholds; that is, the two constituent masses are

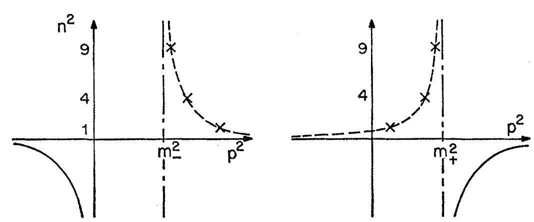

FIG. 2. Limiting cases of Fig. 1. On the left, the spectrum of Nambu's first equation (Ref. 19); on the right, the spectrum of the nonrelativistic hydrogen atom. The Nambu equation describes finite-energy bound states of two infinitely heavy quarks.

[23] It has recently been shown that the Bethe-Salpeter equation for two scalar particles interacting through the exchange of a massless scalar particle, in the ladder approximation, is equivalent to an infinite-component field equation not much more complicated than the one studied here. E. Kyrakopoulos (private communication); C. Fronsdal and Y.-C. Yang (to be published).

[24] The Bethe-Salpeter equation, in the limit of zero coupling, between the two particles is $(p_1{}^2-m_1{}^2)(p_2{}^2-m_2{}^2)\psi(p_1p_2)=0$. It has solutions with $E_1E_2>0$, $(p_1+p_2)^2>(m_1+m_2)^2$ and other solutions with $E_1E_2<0$, $(p_1+p_2)^2<(m_1-m_2)^2$.

determined by

$$m_+ = m_1 + m_2, \quad m_- = m_1 - m_2. \tag{III11}$$

Strong support for this is found in the structure of the form factors.

C. Form Factors

Let $A(x)$ be an external unquantized scalar field and let the interaction density be the simplest one possible:

$$g\psi^\dagger(x)\psi(x)A(x). \tag{III12}$$

The form factors can easily be calculated by the same method as in the case of nonrelativistic hydrogen.[6–8] In particular, the form factor of the lowest state is[6]

$$F_{11}(t) = 2(\lambda_1\lambda+1)^{-1}, \quad t=(p_1-p)^2 \tag{III13}$$

where $\lambda_1\lambda$ is the hyperbolic cosine defined by the spurions of the initial and the final states:

$$\lambda_1\lambda = \lambda_a(p_1)\lambda_a(p). \tag{III14}$$

Thus

$$F_{11}(t) = (1-t/M^2)^{-1}, \quad M = 8m_1e^2/(4+e^4). \tag{III15}$$

As usual, one obtains form factors that fall off with increasing momentum transfer. More important than this is their singularity structure.

The function $F_{11}(t)$ is singular at $\lambda_1\lambda=-1$. It may easily be verified that all the transition form factors have the same property. Let p_1^2 and p^2 be the masses of the initial and final states and introduce

$$\begin{aligned} x &= (p^2-m_1^2-m_2^2)/2m_1m_2, \\ k_1 &= (p_1^2-m_1^2-m_2^2)/2m_1m_2, \\ k_2 &= (t-2m_1^2)/2m_1^2. \end{aligned} \tag{III16}$$

Then the point $\lambda_1\lambda=-1$ corresponds to

$$k_2 = -xk_1 + (1-k_1^2)^{1/2}(1-x^2)^{1/2}. \tag{III17}$$

The form factor $F_{n_1,n}(t)$ is singular at this point, and at this point only.

Consider the ordinary Feynman diagram of Fig. 5. The corresponding Feynman amplitude has an anoma-

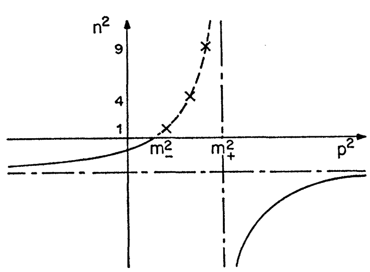

FIG. 3. On the left, the spectrum of the model that is the principal object of this investigation. The limiting case studied by Takabayasi (Ref. 21) $m_1=m_2\rightarrow\infty$, is a straight line through the origin.

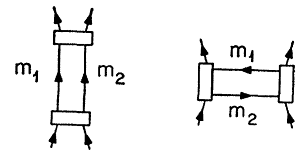

FIG. 4. Illustration of the right-hand and the left-hand continuous physical regions of the two-particle Green's function. If the intermediary particles are on the mass shell, then $s>(m_1+m_2)^2$ in the first configuration and $s<(m_1-m_2)^2$ in the second.

lous threshold singularity at precisely the value of t given by Eq. (III17).[25] There is a double infinity of transition form factors, each one singular for a different value of t, and all of them match; that is, the simple "Born term" defined by (III12) places all the singularities exactly where they should be according to ordinary field theory or S-matrix theory.[26]

We feel that this result confirms the correctness of two basic postulates:

(i) the identification of the right-hand and left-hand continua in terms of two-particle states, because it is this identification that furnishes the values of the two masses m_1 and m_2, and shows the relevance of the Feynman diagram of Fig. 5; and

(ii) the choice of local interactions, as exemplified by (III12), without which the form factors could not have been predicted.

One small puzzle remains: The singularity of the companion diagram to Fig. 5, in which the roles of m_1 and m_2 are exchanged, does not appear in the model. Thus it appears that only one of the two constituent particles interacts with the external field—the heaviest or the lightest, according to the sign of the parameter β. This is not unphysical, it merely means that only one particle is charged, but we should have liked to be able to treat the case of two charged particles as well.

D. H-Atom

It is instructive to investigate the nonrelativistic limit, that is, the solutions of Eq. (III1) in the neighborhood of the ionization point. Putting $p^2=(m_++E)^2$ and $\mathbf{p}=0$, one gets, to lowest order in E,

$$-\tau_3[E(\Gamma_0-\Gamma_4)-(\alpha^2/2\mu)(\Gamma_0+\Gamma_4)-\alpha ee']\psi=0, \tag{III18}$$

where $\mu^{-1}=m_1^{-1}+m_2^{-1}$ and $e'=\tau_3 e$. This is the Schrödinger equation for a system of two particles with charges e and e', and reduced mass μ, interacting through a Coulomb potential.[6] The physical interpretation of our system is thus

(i) $p_0\approx m_+$, $\tau_3=-1$: hydrogen, electron-proton scattering states;

[25] R. Karplus, C. M. Sommerfield, and E. H. Wickmann, Phys. Rev. **111**, 1187 (1958); C. Fronsdal and R. E. Norton, J. Math. Phys. **5**, 100 (1964).

[26] Our model, like potential theory, lacks the normal threshold singularity at $t=4m_1^2$; more about this below.

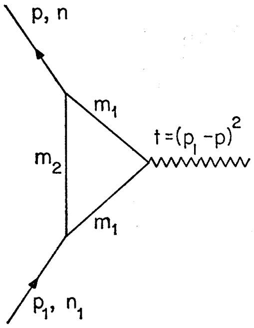

FIG. 5. One of the diagrams that contribute to the vertex function in ordinary scalar field theory. The external straight lines represent the lowest bound state of the composite system, and the external lines represent the constituent particles of mass m_1 and m_2.

(ii) $p_0 \approx m_+$, $\tau_3=+1$: positron-proton scattering states[27];

(iii) $p_0 \approx -m_+$, $\tau_3=-1$: electron-antiproton scattering states;

(iv) $p_0 \approx -m_+$, $\tau_3=+1$: antihydrogen, antiproton-positron scattering.

E. Metric

Here we shall attempt to prove that the physical metric,

$$\eta=(\partial/\partial p_0)L(p), \qquad \text{(III19)}$$

has the sign of p_0, as is required in a field theory quantized with commutators. For the discrete states this is trivial; for the scattering states the proof is not complete.

For the discrete states we shall use the virial theorem. In the nonrelativistic limit this means that the physical expectation value of $E+q^2/2\mu$ vanishes. The nonrelativistic metric is $-\tau_3(\Gamma_0-\Gamma_4)$, and we have[6]

$$\begin{aligned}\psi^\dagger(\Gamma_0-\Gamma_4)(E+q^2/2\mu)\psi \\ =\psi^\dagger[E(\Gamma_0-\Gamma_4)+(\alpha^2/2\mu)(\Gamma_0+\Gamma_4)]\psi=0 \qquad \text{(III20)}\end{aligned}$$

if ψ is a solution of (III18) with negative energy E.

Consider the relativistic wave equation in the form (III6). In the case of the discrete states the vector λ_a defined by (III4) is "timelike." The diagonal matrix elements of Γ_4 vanish in a basis where Γ_0 is diagonal; more generally, if $\lambda\lambda'=0$, then $\lambda_a'\Gamma_a$ has no diagonal matrix elements in the basis where $\Lambda_a\Gamma_a$ is diagonal. Thus, in the frame $\mathbf{p}=0$,

$$\psi^\dagger[\alpha p_0\Gamma_4+(p^2-\beta)\Gamma_0]\psi=0 \qquad \text{(III21)}$$

if ψ is a solution of the wave equation with energy p_0. This is the relativistic generalization of the virial theorem. From (III2) we have

$$\begin{aligned}H\equiv p_0\eta=p_0(\partial/\partial_0)L(p) \\ =\tau_3[\alpha p_0\Gamma_0+2p_0^2\Gamma_4]+e^2p_0^2. \qquad \text{(III22)}\end{aligned}$$

[27] For this suggestion we are indebted to Y. Nambu (private communication).

Using (III1) and (III21) we obtain

$$\frac{\psi^\dagger H\psi}{\psi^\dagger\psi}=\frac{2m_1m_2e^2p^2}{m_+^2-p^2}>0. \qquad \text{(III23)}$$

Hence all the bound states have positive metric.

In the case of scattering states there is no virial theorem, but a similar statement holds; namely, the expectation value of the potential vanishes. That is, in the nonrelativistic theory,

$$\frac{\psi^\dagger(\Gamma_0-\Gamma_4)r^{-1}\psi}{\psi^\dagger(\Gamma_0-\Gamma_4)\psi}=\frac{\psi^\dagger\psi}{\psi^\dagger(\Gamma_0-\Gamma_4)\psi} \qquad \text{(III24)}$$

tends to zero in the limit as ψ tends to a stationary scattering state. It seems almost obvious that, in the relativistic case, the corresponding statement is that

$$\frac{\psi^\dagger\psi}{\psi^\dagger\eta\psi}\to 0, \qquad \text{(III25)}$$

as ψ tends to a stationary scattering state. This has not yet been verified by direct calculation, and for that reason our proof of the positivity of the metric in the case of scattering states is incomplete. From (III22), with the help of (III1) and (III25), we get

$$\psi^\dagger H\psi\to -\tau_3p_0\alpha\frac{p^2+\beta}{p^2-\beta}\psi^\dagger\Gamma_0\psi.$$

In the nomenclature of the nonrelativistic H-atom interpretation, the energy density is positive for particle-particle (electron-proton) states and for antiparticle-antiparticle states, while it is negative for particle-antiparticle states.

Thus it is seen that quantization with commutators is possible. A state with $p_0>0$ and $\tau_3=-1$ is a "neutral" boson, and a corresponding state with $p_0<0$, $\tau_3=+1$ is its antiparticle; the relative sign of the energy density for these states is positive. A state with $p_0>0$ and $\tau_3=+1$ is a "charged" boson, and a corresponding state with $p_0<0$, $\tau_3=-1$ is its antiparticle; again, the relative sign of the energy density is positive. The energy density for "charged" states is opposite in sign from that of "neutral" states, which means that a superselection rule must be operating between positive-energy states with $\tau_3=+1$ and $\tau_3=-1$, respectively. This superselection rule is closely related to time-reversal invariance. If one considers the case $m_1\gg m_2$ and the states $p_0>0$ (proton-electron and proton-positron states), then the alternation of the sign of the energy density is the same as in the Dirac theory of hydrogen. This suggests that our model comes close to describing a system of two fermions, rather than two bosons.

IV. ANALYTIC PROPERTIES OF AMPLITUDES

A. Vertex Functions

Here we study the analytic properties of the form factor as a function of two variables, p^2 and $t=(p_1-p)^2$.

Consider the transition between the ground state and an arbitrary discrete state, in a scalar external field. The vertex function has the form[8]

$$F_{1,n}(t)\lambda_{a_1}(p_1)\cdots\lambda_{a_{n-1}}(p_1)\chi^{a_1\cdots a_{n-1}}(p)\,, \qquad \text{(IV1)}$$

where $\lambda_a(p_1)$ is the spurion [see Eq. (III4)] for the ground state and χ is the finite tensor wave function for the final state. The form factor is a simple generalization of (III15)[28]:

$$F_{1,n}(t)=-[-1+(k_2+xk_1)\times(1-k_1^2)^{-1/2}(1-x^2)^{-1/2}]^{-n}. \qquad \text{(IV2)}$$

The k_i and x were defined by (III16); p_1^2 is the mass of the ground state, hence k_1 is a fixed parameter while k_2 and x will be considered as complex variables. To interpolate between the integer values of n, we use Eq. (III10), or

$$n=\tfrac{1}{2}e^2[(1+x)/(1-x)]^{1/2}. \qquad \text{(IV3)}$$

Equations (IV2) and (IV3) define an analytic function of k_2 and x.

As a function of t, i.e., of k_2, the form factor is singular at the point (III17) only. This is a logarithmic branch point except for those values of x that correspond to integer values of n. This anomalous threshold singularity occurs in field theory in the case of weak binding; that is, when $k_2+x>0$. In field theory the form factor has a branch point at the point $k_2=1$ as well; this is the normal threshold singularity at $t=4m_1^2$. As the masses are varied, and k_2+x turns negative (strong binding), the anomalous singularity moves to an unphysical sheet, through the normal cut that starts at $k_2=1$.[25] Our model, like potential theory, lacks this cut, and the anomalous threshold singularity has no escape. The absence of the normal cut is consistent with the fact that states of two particles, both with mass m_1, are ignored by the model—precisely the existence of such states accounts for the normal branch cut in ordinary field theory; see Fig. 5.

Next, consider the form factor—defined by (IV2) and (IV3)—as a function of p^2, or x. To begin with, we suppose that k_1^2 and k_2^2 are less than 1, and $k_1+k_2>0$. This is true if the binding is weak and t is small—the domain in which the physical relevance of the model is clearest. The form factor has four singularities in the complex x plane, at $x=\pm 1$ and at

$$x=L_x^{\pm}\equiv -k_1k_2\pm(1-k_1^2)^{1/2}(1-k_2^2)^{1/2}. \qquad \text{(IV4)}$$

(a) $x=+1$: This is the normal threshold, at the beginning of the right-hand continuum, $p^2=(m_1+m_2)^2$.

(b) $x=-1$: This is the end of the left-hand continuum, $p^2=(m_1-m_2)^2$. It occurs in field theory as well, but not on the physical sheet.

(c) $x=L_x^+$: We define all the square roots so that their real parts are positive in a plane cut from $-\infty$ to -1 and from $+1$ to $+\infty$. Then L_x^+ is a singular point on the first sheet, but not on the second sheet. This is, once again, the anomalous threshold singularity. As the parameters are varied such that k_1+k_2 turns negative, the point L_x^+ loops around the normal branch point at $x=+1$, and the singularity moves on to the second sheet, exactly as in ordinary local field theory.[25]

(d) $x=L_x^-$: This singularity moves from one sheet to another, through the left-hand cut, when k_1-k_2 changes sign. The same happens in ordinary local field theory.

To summarize: With respect to the variable t, the discrepancy between our model and ordinary local field theory is the absence of a branch point at $4m_1^2$. With respect to the variable p^2, the only difference is the fact that the left-hand branch point occurs on a different Riemann sheet. The left-hand branch point is closely related to the spacelike solutions, and the fact that it occurs on the physical sheet is an indication that these solutions may represent acausal behavior. However the question cannot be decided by an examination of the form factors, because the singularity in question occurs far from the physical region. We therefore study a scattering amplitude where the spacelike solutions contribute to the sum of "intermediary" states.

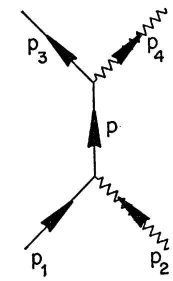

FIG. 6. The infinite-component diagram associated with the scattering amplitude $A(s,t)$. In contrast to Fig. 5. the internal line here represents the composite system. The propagator is the Green's function (IV5), and the vertices are given by (III12).

B. Compton Scattering

We consider an amplitude that is a close analog of elastic Compton scattering from hydrogen. The Feynman diagram is shown in Fig. 6; the wavy lines represent the external, conventional, scalar field $A(x)$, and the solid lines are states of our model. The external lines will be taken to be in the lowest discrete state. The vertices are given by the local interaction density (III12), and for propagator we take the inverse of the wave operator (III2). When the total energy is below the ionization point, $s<m_+^2$, the amplitude can be written as a sum over the quantum number n of the intermediary state; this is a kind of generalized partial-wave expansion, in which the operator $L^{-1}(p)$ is replaced by its eigenvalues $L_n^{-1}(s)$. The latter are given by the diagonalized form (III6) of the wave operator; thus

$$L_n^{-1}(s)=[-nQ^{1/2}(s)+\tfrac{1}{2}e^2(s-\gamma)]^{-1}. \qquad \text{(IV5)}$$

[28] The following expression for $F_{1,n}(t)$ is obtained from Eq. (A6) of Ref. 8 by putting $N=-2$ and $f=4$.

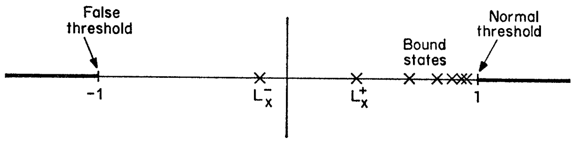

FIG. 7. Fixed singularities and cuts in the x plane. The points $L_x^\pm$ are not singular if k_1 and k_2 are in the range (IV16).

The calculation of the scattering amplitude, for the more general case of an arbitrary spin-independent propagator, has already been reported.[8] The result is[29]

$$A=\sum_n nP_{n-1,4}(\cos\phi)f_n(s)\,,$$

$$f_n(s)=n!(n-1)!L_n^{-1}(s)\times P_{N,5}{}^{1-n}(\lambda\lambda_1)P_{N,5}{}^{1-n}(\lambda\lambda_3)\,. \qquad \text{(IV6)}$$

The P's are four- and five-dimensional hyperspherical functions; λ_1, λ, and λ_3 are the spurions of the initial, intermediary; and final states; and[30]

$$\cos\phi=[(\lambda\lambda_1)(\lambda\lambda_3)-(\lambda_1\lambda_3)][(\lambda\lambda_1)^2-1]^{-1/2}\times[(\lambda\lambda_3)^2-1]^{-1/2}. \qquad \text{(IV7)}$$

The number N is related to the Casimir operator of $SO(4,1)$. The choice of the Majorana representation fixes N at the value -2. In this case the hyperspherical functions reduce to elementary functions:

$$n!P_{-2,5}{}^{1-n}(\rho)=(\rho^2-1)^{-1/2}\left(\frac{\rho-1}{\rho+1}\right)^{n/2},$$

$$P_{n-1,4}(\cos\phi)=\sin n\phi/\sin\phi\,, \qquad \text{(IV8)}$$

and this happy circumstance allows us to sum the series. To simplify the expressions, we put the masses of the two scalar field quanta equal, $p_2^2=p_4^2$; then $\lambda_1\lambda=\lambda_3\lambda$, and

$$A=c\sum_n L_n^{-1}\left(\frac{\lambda\lambda_1-1}{\lambda\lambda_1+1}\right)^n \sin n\phi\,, \qquad \text{(IV9)}$$

with

$$c=[\sin\phi(\lambda\lambda_1-1)(\lambda\lambda_1+1)]^{-1}$$

or

$$A=c[ie^2(s-m_-^2)]^{-1}[F(1,-b;-b+1;u)-F(1,-b;-b+1;v)], \qquad \text{(IV10)}$$

where

$$u=e^{i\phi}\frac{\lambda\lambda_1-1}{\lambda\lambda_1+1}\,,\quad v=e^{-i\phi}\frac{\lambda\lambda_1-1}{\lambda\lambda_1+1}\,, \qquad \text{(IV11)}$$

$$b=\tfrac{1}{2}e^2[(s-m_-^2)/(m_+^2-s)]^{1/2}. \qquad \text{(IV12)}$$

To simplify the analysis, we introduce the parameters

$$k_1=(p_1^2-m_1^2-m_2^2)/2m_1m_2\,,$$
$$k_2=(p_2^2-2m_1^2)/2m_1^2,$$

and the variables

$$x=(s-m_1^2-m_2^2)/2m_1m_2\,,\quad s=p^2=(p_1+p_2)^2\,,$$
$$y=(t-2m_1^2)/2m_1^2\,,\qquad t=(p_1-p_3)^2.$$

Then

$$\lambda_1\lambda_3=1-(y+1)/(1-k_1^2)=-1-(y-L_y^+)/(1-k_1^2)\,, \qquad \text{(IV13)}$$

$$\lambda\lambda_1=\lambda\lambda_3=-(xk_1+k_2)(1-x^2)^{-1/2}(1-k_1^2)^{-1/2}\,,$$

$$\cos\phi=1+(y+1)(1-x^2)/(x-L_x^+)(x-L_x^-)=-1+(y-y^+)(1-x^2)/(x-L_x^+)(x-L_x^+)\,, \qquad \text{(IV14)}$$

where

$$L_y^+=1-2k_1^2\,,$$
$$L_x^\pm=-k_1k_2\pm(1-k_1^2)^{1/2}(1-k_1^2)^{1/2}, \qquad \text{(IV15)}$$
$$y^+=1-2(k_1^2+k_2^2+2xk_1k_2)/(1-x^2)\,.$$

We begin by fixing the parameters as follows:

$$0<k_1<1\,,\quad -1<k_2<-k_1. \qquad \text{(IV16)}$$

In order that there be no doubt about the convergence of the series expansion (IV9) it is sufficient that x and y lie in the ranges

$$L_x^+<x<1\,,\quad y^+<y<-1. \qquad \text{(IV17)}$$

The positions of the fixed points $L_x^\pm$ and L_y^+ are indicated in Figs. 7 and 8, together with the location of the point y^+ when x lies in (IV17).

With x fixed in (IV17) we explore the singularities in the complex y plane. When y lies in (IV17) the variable ϕ is real, and the arguments of the F functions remain inside the unit circle, because

$$0<\frac{\lambda_1\lambda-1}{\lambda_1\lambda+1}<1\,, \qquad \text{(IV18)}$$

when x lies in (IV17). The points y^+ and -1, at which ϕ has square-root branch points, are not singular when (IV18) is satisfied, because (IV9) is an even function of ϕ in the region of convergence of the series expansion. The only singularities in the y plane, when x lies in (IV17), are at infinity and at the points where the hypergeometric functions are singular, i.e., at

$$e^{\pm i\phi}\frac{\lambda_1\lambda-1}{\lambda_1\lambda+1}=+1. \qquad \text{(IV19)}$$

This is satisfied only at $y=L_y^+$. This point corresponds

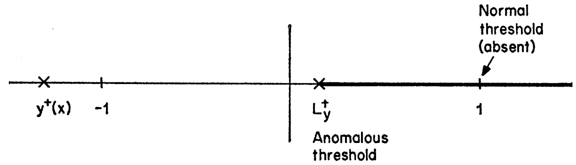

FIG. 8. Singularities in the y plane. When x lies in the range (IV17) only L_y^+ is a singular point.

[29] See Ref. 8, Eqs. (II43) and (II44). Note the change in notation, $n\to n-1$.

[30] Let μ_1 and μ_2 be the projections of λ_1 and λ_2 into the four-space orthogonal to λ; then ϕ is the angle between μ_1 and μ_2.

to $\lambda_1\lambda_3=-1$ and is our old friend, the anomalous threshold singularity. It is a vertex singularity that occurs in ordinary field theory in the scattering amplitudes that correspond to the Feynman diagrams of Fig. 9.

We introduce a real cut in the y plane, from L_y^+ to $+\infty$, and evaluate the discontinuity of (IV10) across the cut. We fix the sign,

$$i\sin\phi>0,\quad y>-1$$

by convention; then only the first hypergeometric function is singular at $y=L_y^+$. When $|u|>1$ and $|\arg(-u)|<\pi$,[31]

$$F(1,-b;1-b;u)=-F(1,b;b+1;1/u)+1 +(b\pi/\sin b\pi)(-u)^b. \quad \text{(IV20)}$$

On top of the cut, $-u=e^{-i\pi}u$; below, the phase is $+\pi$. Thus

$$\operatorname{disc}(y)F=-2\pi i u^b$$

and

$$\operatorname{disc}(y)A=\pi Q^{-1/2}ce^{i\phi b}\left(\frac{\lambda\lambda_1-1}{\lambda\lambda_1+1}\right)^b. \quad \text{(IV21)}$$

If $b<1$—that is, in the unphysical region were the total energy is below the mass of the ground state—this function decreases as $y\to\infty$, and we have a fixed-x dispersion relation:

$$A(x,y)\sim\int_{L_y^+}^{\infty}dy'(y-y')^{-1}\operatorname{disc}(y)A(x,y').$$

When b is increased to 1, the integral diverges because of the fixed-x bound-state pole. A subtraction can be made and a dispersion relation that is valid up to $b=2$ obtained. An infinite number of subtractions must be made in order for a fixed-x dispersion relation to hold in the neighborhood of the ionization point $x=1$. This is obviously directly related to the existence of an infinite number of bound states—the Coulomb problem. We shall not attempt to derive a Mandelstam representation under such circumstances, as it seems more appropriate and instructive to do so in a model with a finite number of bound states. Nevertheless, it is useful to study the analytic properties of the function $\operatorname{disc}(y)A(x,y')$ in the variable x.

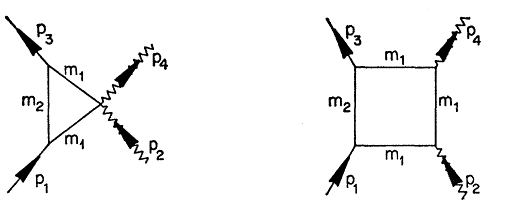

FIG. 9. Two of the diagrams that contribute to the scattering amplitude in ordinary scalar field theory. The interpretation is the same as in Fig. 5.

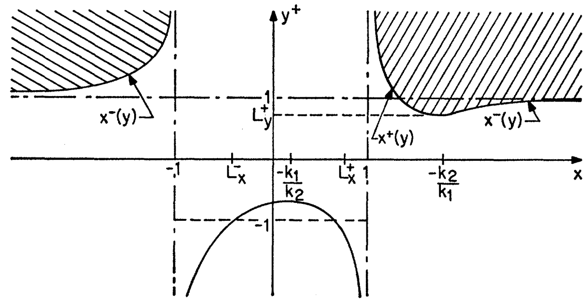

FIG. 10. The shaded domain is the distinguished boundary of the analytic function $A(s,t)$. The right-hand portion agrees with the support of the Mandelstam double spectral function calculated from the box diagram of Fig. 9.

Close inspection of (IV21) reveals that $\operatorname{disc}(y)A$ is one-valued near $x=\pm1$. The only branch points are those at which $\phi=0$; that is, at $y'=y^+(x)$, or

$$x=x^\pm(y')=\frac{2k_1k_2}{y'-1}\left[1\mp\left(1-\frac{1-y'}{2k_1^2}\right)^{1/2}\left(1-\frac{1-y'}{2k_2^2}\right)^{1/2}\right].$$

When $L_y^+<y'<1$, then $x^\pm(y')$ are both on the real line above 1, and $\operatorname{disc}(y)A(x,y')$ is one-valued in the plane cut from $x^+(y')$ to $x^-(y')$. When $y'>1$, $x^+(y')$ remains larger than 1, while $x^-(y')<-1$; the x-plane must now be cut from $x^+(y')$ to $+\infty$ and from $-\infty$ to $x^-(y')$. The two-dimensional manifold given by $L_y^+<y'<\infty$ and x on these cuts is shown shaded in Fig. 10, it is the distinguished boundary of $A(s,t)$.

If we could have justified a Mandelstam representation,[32] then this domain would have been the support of the double spectral function. The portion of this domain that lies in the upper right-hand quadrant is precisely the support of the Mandelstam double spectral function associated with the Feynman box diagram shown in Fig. 9.[33] The other portion of the support of the double spectral function, in the upper left-hand quadrant of Fig. 10, has no counterpart in conventional local field theory. It corresponds to the "spacelike solutions," that is, to the left-hand continuum in the mass spectrum, and shows that the scattering amplitude has a fixed left-hand cut. This presumably corresponds to the propagation of a signal with velocity greater than that of light, and hence a breakdown of macrocausality. The precise reason for the appearance of the extra domain of the double spectral function is not hard to find. As noted above, in connection with the vertex function, our amplitudes lack a normal threshold in t. In ordinary field theory the point $x=x^-(y')$ is a singularity of $\operatorname{disc}(y)A(x,y')$

[31] *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Co., New York, 1953), p. 108.

[32] We repeat: The reason we cannot derive a Mandelstam representation is that we do not know how to handle the essential singularity at $x=1$; this is due to the infinite range of the potential.

[33] C. Fronsdal, R. E. Norton, and K. T. Mahanthappa, J. Math. Phys. 4, 859 (1963).

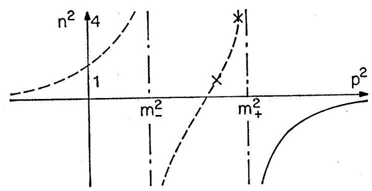

FIG. 11. The spectrum of Eq. (V7), the second model.

when $L_y<y'<1$, just as in our model, but not when $y'>1$. This is possible precisely because disc$(y)A(x,y')$ has the normal branch point at $y'=1$. Since our model lacks this branch point, there is no way for the singularity at $x=x^-(y')$ to disappear into another sheet as y' passes 1.

The relationship between the extra portion of spectral function support, and the absence of the normal cut in t, is very intimate. For if we discard the unorthodox contribution to the Mandelstam representation, then this introduces a normal threshold in t, with a discontinuity that is precisely of the right type; now, the anomalous threshold singularity vanishes into the second sheet as k_1 turns negative (strong-binding case). This follows from the fact that the dip in the boundary of the double spectral domain disappears from the right-hand side and reappears on the left. (The minimum occurs that $x=-k_2/k_1$.)

V. SOMMERFELD-WATSON TRANSFORMATION

A. The Problem

The preceding discussion has shown that the "Compton" scattering amplitude has a left-hand cut. It is not clear, however, whether this is due to the spacelike solutions alone, to the left-hand cut of the vertex function, or to a combination of both. Here we shall clarify this question by an alternative approach. At the same time, the relevance of our work to conspiracy theory will be explained.

The sum in (IV9) can be converted into an integral over a hairpin contour, and this contour can be deformed till it essentially coincides with the imaginary axis in the complex n plane. This involves no difficulties or subtleties so long as x and y lie in (IV17); that is, if s is below the ionization point and t is small and negative. The result is

$$A=\tfrac{1}{4}c\int dn\,\csc(\pi n)L_n^{-1}(s)[(-u^n)-(-v)^n], \quad \text{(V1)}$$

where u and v were defined by (IV11). The contour of integration crosses the real axis between -1 and $+1$, and passes to the right of the poles of $L_n^{-1}(s)$.

Let s be increased to above the normal threshold m_+^2, and let us calculate the values of A on the two sides of the cut. The quantities c, $\lambda\lambda_1$, and ϕ all have square-root branch points at the threshold; let c, $\lambda\lambda_1$, ϕ denote their values on top of the cut, then their values below the cut are $-c$, $-\lambda\lambda_1$, $-\phi$. Thus

$$A(s+i\epsilon)=\tfrac{1}{4}c\int dn\,\csc(n\pi)L_n^{-1}(s+i0)\times[(-u)^n-(-v)^n], \quad \text{(V2)}$$

$$\begin{aligned}A(s-i\epsilon)&=-\tfrac{1}{4}c\int dn\,\csc(n\pi)L_n^{-1}(s-i0)\times[(-1/u)^n-(-1/v)^n]\\&=\tfrac{1}{4}c\int dn\,\csc(n\pi)L_{-n}^{-1}(s-i0)\times[(-u)^n-(-v)^n]. \quad \text{(V3)}\end{aligned}$$

The last expression was obtained by a change of integration variable, $n\to -n$; hence the contour in (V3) goes to the left of the poles of $L_{-n}^{-1}(s-i\epsilon)$.

In the model studied above we have $L_{-n}^{-1}(s-i0)=L_n^{-1}(s+i0)$. Hence

$$\text{disc}(x)A=\tfrac{1}{4}c\int dn\,\csc(n\pi)L_n^{-1}(s+i0)\times[(-u)^n-(-v)^n], \quad \text{(V4)}$$

where the contour is the difference between the contours of (V2) and (V3); that is, it is a closed contour circling the pole of $L_n^{-1}(s)$. Thus

$$\text{disc}(x)A=\pi Q^{-1/2}c\frac{\sin b(\phi+\pi)}{\sin b\pi}\left(\frac{\lambda\lambda_1-1}{\lambda\lambda_1+1}\right)^b. \quad \text{(V5)}$$

It is easy to verify that the double spectral function disc(y)disc$(x)A$ given by (V5) agrees with disc(x)disc$(y)A$ calculated from (IV21).

If we examine the left-hand cut, $s<m_-^2$, in the same way, we find an almost identical result. Notice that the discontinuities of the form factors are not enough to produce cuts in $A(s,t)$. In fact, the symmetry property of our special model,

$$L_{-n}^{-1}(s-i0)=L_n^{-1}(s+i0)\,, \quad \text{(V6)}$$

guarantees that the effects of the branch cuts of the vertex singularities cancel exactly. Consequently, the sole cause of the two cuts in $A(s,t)$ is the fact that $L_n^{-1}(s)$ has a pole. Let us return to (V1). When s varies from m_+^2 to $+\infty$ on the right-hand cut, the pole in $L_n^{-1}(s)$ moves along the imaginary axis from $n=\pm i\infty$ to $n=\pm\tfrac{1}{2}ie^2$. When s varies from m_-^2 to $-\infty$ on the left-hand cut, the pole in $L_n^{-1}(s)$ varies from 0 to $\pm\tfrac{1}{2}ie^2$. Hence (V1) is very similar to a dispersion representation, in which the integration domains $|in|>\tfrac{1}{2}e^2$ correspond to the right-hand continuum and $|in|<\tfrac{1}{2}e^2$ corresponds to the left-hand continuum.

B. The Cure—?

We have just noted the intimate relationship between the left-hand discontinuity of the scattering amplitude and the poles of the propagator—more precisely, the zeros of the wave operator $L_n(s)$ for $s<m_-^2$. Next let us see what happens in a theory where there are no solutions for $s\leq m_-^2$; that is, in a model in which there are no spacelike solutions.

Retaining all the notations and definitions of Sec. III, we write down the wave equation

$$\{[\alpha p\Gamma+(p^2-\beta)\Gamma_4]^2-m_1m_2e^4(p^2-\gamma)\}\psi=0. \quad \text{(V7)}$$

(Note the distinction between "writing down an equation" and constructing a physically interpretable model: We have not verified that this new equation yields a positive-definite physical metric, and it will only be used to study a particular aspect of the problem.) The form factors are completely determined by the parameters α and β and are exactly the same as in our first model. Since the two-particle interpretation, and the correct location of most of the singularities, resulted from the identification $\alpha=(m_+-m_-)^2$ and $\beta=m_+m_-$, we shall retain these values of α and β. The constant γ, which will appear in the propagator but not in the vertex functions, will now be allowed any value between m_-^2 and m_+^2.

Instead of the spectrum given by Eq. (III10) and illustrated in Fig. 3, we now have

$$n^2=m_1m_2e^4\frac{p^2-\gamma}{(m_+^2-p^2)(p^2-m_-^2)}. \quad \text{(V8)}$$

This is illustrated in Fig. 11. As before, the spectrum of the operator n^2 is discrete when $m_-^2<p^2<m_+^2$, and negative continuous when p^2 lies outside these limits. Consequently, only the crosses and the heavy line in Fig. 11 represent solutions of the wave equation: Equation (V7) has no spacelike solutions. The vertex functions are the same as before, and the "Compton" scattering amplitude is given by Eq. (V1), with

$$L_n(s)=n^2(p^2-m_-^2)(m_+^2-p^2)-m_1m_2e^4(p^2-\gamma). \quad \text{(V9)}$$

Although $L_n^{-1}(s)$ now has two poles in the complex n plane, the calculation of the right-hand discontinuity presents nothing new. As p^2 is increased past m_+^2 the poles move out on the real axis (in opposite directions), describe large quarter circles, and reach the imaginary axis. The discontinuity across the normal cut is the sum of the residues of the two poles.

If we now continue analytically in s, along the real axis and past the beginning of the left-hand continuum at $s=m_-^2$, the poles of L_n^{-1} move as shown in Fig. 12. Meanwhile, the contour of integration in (V1) always remains to the right of both poles. This means that we have something more serious than a left-hand cut; every time the pole of $L_n^{-1}(s)$ passes by a positive integer value of n, we have a pole of $A(s,t)$ for the corresponding value of s. The mechanism is exactly the same as that which produces the bound-state poles: The contour is pinched between the pole of $L_n^{-1}(s)$ and the poles of $\csc(n\pi)$, and this produces a ghost. Thus it is seen that, if the spectral curve passes through an integer value of n for some value s_n of n, then the analytic continuation of the scattering amplitude $A(s,t)$ has a pole at $s=s_n$, even if the wave equation has no solution at $p^2=s_n$. That is, avoidance of spacelike solutions does not automatically cure the difficulties with which spacelike solutions are intuitively associated.

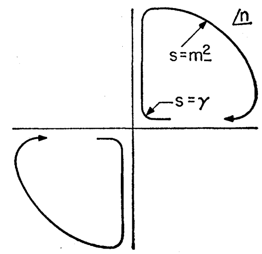

FIG. 12. The migration of the poles of the propagator of Eq. (V7) as s moves to the left past $s=\gamma$ and past $s=m_-^2$. (Second model.)

Since we do not like ghosts, let us take a special value of γ that exorcises them, namely, $\gamma=m_-^2$. In this case (V8) reduces to

$$n^2=\frac{m_1m_2e^4}{m_+^2-p^2}. \quad \text{(V10)}$$

The spectrum is the same as that of the nonrelativistic hydrogen atom, and the spectral curve is illustrated in the right-hand part of Fig. 2. The point $p^2=m_-^2$ is no longer of any special significance as far as this curve is concerned. It is, nevertheless, important, because it is one of the two points where the spurion is "lightlike." The vertex functions, as well as the location of singularities of the scattering amplitude, are the same as before.

With this special value of γ let us once again consider the continuation of $A(s,t)$, defined by (V1), below the point $s=m_-^2$. The propagator has two poles, both on the real n axis, symmetrically placed with respect to the origin. The point $s=m_-^2$ is not a branch point of $L_n^{-1}(s)$, but a discontinuity develops nevertheless, because of the left-hand branch point singularity of the vertex function. To isolate the singular part of $A(s,t)$ we may write (V1) as follows:

$$A(s,t)=\tfrac{1}{4}c\int_{-i\infty}^{+i\infty}dn\,\csc(\pi n)L_n^{-1}(s)[(-u)^n-(-v)^n]$$
$$+\frac{\pi ic}{b'\sin\pi b'}\frac{(-u)^{b'}-(-v)^{b'}}{(s-m_-^2)(m_+^2-s)}, \quad \text{(V11)}$$

where the integral contour follows the imaginary axis and the extra term is the "pole term" that results from that part of the original contour that looped around the pole at

$$n=b'=+[m_1m_2e^4/(m_+^2-p^2)]^{1/2}.$$

The "background" integral has no branch point at $s=m_-^2$, but the pole term does. If "u" is the value of

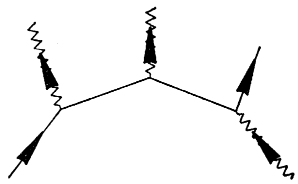

FIG. 13. A production process with double conspiracy exchange. The interpretation is the same as in Fig. 6.

this variable on top of the real axis below $s=m_-^2$, then $1/``u"$ is its value below the real axis.

To summarize, we have found examples of three different behaviors of the scattering amplitude continued below $s=m_-^2$:

(a) In the original model, which has spacelike solutions, the corresponding poles of the propagator are the sole cause of a left-hand cut.

(b) In the second model, which has no spacelike solutions, poles nevertheless appear at those points where the spectral curve passes through integer values of n.

(c) In the last model, in which there are no spacelike solutions, and in which the spectral curve is confined between $n=0$ and $n=1$ when $s<m_-^2$, the left-hand branch point singularity of the vertex function gives rise to a left-hand cut of the amplitude.

Some of the possible implications of these results are discussed in the last section. Meanwhile, we shall comment on the relevance of our Eq. (V11) to conspiracy theory.

C. Conspiracy

The integral formulas for the "Compton" scattering amplitude derived above are of the type developed by Toller[34] and others,[35] and are applied in conspiracy theory.[36] Strictly, this theory applies to the point $s=0$ only, and to the case of pair-wise equal masses (which in our case means four equal masses, $p_1^2=p_2^2=p_3^2=p_4^2$), though several extensions have been suggested.[37–39] The similarity is not superficial. Expression (IV9) is a decomposition of the amplitude according to the $O(4)$ degeneracy group of the mass spectrum. When s is outside the interval from m_-^2 to m_+^2, this group is not $O(4)$ but $O(3,1)$, and the sum must be replaced by an integral that represents the decomposition of the amplitude according to $O(3,1)$. Conspiracy theory makes use of the observation that when the masses are equal and $s=0$, $O(3,1)$ invariance is a consequence of Poincaré invariance. Extension to unequal masses or to $s\neq 0$ requires, first of all, that the action of $O(3,1)$ as a transformation group on the external physical states be defined. The work of Bali, Ball, Chew, and Pignotti,[39] who showed that this can be done when the external legs of the scattering diagram stand for multiparticle states, is very close to the two-particle model considered here. We have only added the observation that one-particle states may be considered as isolated points in a two-particle spectrum.

When $s=0$, and $p_1^2=p_2^2$, we may take $p_1+p_2=0$, since the model is Lorentz invariant. This gives $\lambda=(0,0,0,0,1)$, so that the degeneracy group is just the physical Lorentz group. In this case $\cos\phi$ reduces to the hyperbolic cosine defined by the four-vectors p_1 and p_3, and our integral formulas reduce to those of conspiracy theory. If either $s\neq 0$ or $p_1^2\neq p_2^2$, our model is a generalization of conspiracy theory. We do not pursue this point here, since the results are obviously strongly model-dependent.

It has been proposed that certain production processes may be dominated by double Regge poles, or even double conspiracies.[40] This has raised the question of how to construct a Yukawa coupling between two conspiracies and an ordinary particle. The infinite-component field theories obviously incorporate such couplings. The amplitude for the process of Fig. 13, for example, can be calculated in the same way as the elastic scattering amplitude and expressed as a double Sommerfeld-Watson integral. The high-energy dominant contribution is obtained by isolating the pole terms in both integrals, and one obtains a closed expression for the double-conspiracy contribution.

VI. ADDITIONAL COMMENTS

A. Diseases

Most of the so-called diseases of infinite component field theories have turned out to be specific to the early examples.

Descending Mass

The first equations that were proposed, by Majorana[16] and by Gel'fand, Yaglom, and Naimark,[41] had descending spectra with an accumulation point at zero. Nambu gave the first example of an ascending mass spectrum.[19]

Spacelike Solutions

All equations that had been proposed up to 1966, including Nambu's first equation, have solutions for spacelike momenta. Again Nambu was the first to invent an equation with no spacelike solutions.[42]

[34] M. Toller, Nuovo Cimento **37**, 631 (1965).
[35] R. Delbourgo, A. Salam, and J. Strathdee, Phys. Rev. **164**, 1981 (1967); G. Cosenza, A. Sciarrino, and M. Toller (unpublished).
[36] D. Freedman and J. Wang, Phys. Rev. Letters **18**, 863 (1967).
[37] R. Delbourgo, A. Salam, and J. Strathdee, Phys. Letters, **25B**, 230 (1967).
[38] G. Domokos and G. L. Tindle, Phys. Rev. **165**, 1906 (1968).
[39] N. F. Bali, J. S. Ball, G. F. Chew, and A. Pignotti, Phys. Rev. **161**, 1459 (1967).
[40] T. W. B. Kibble, Phys. Rev. **131**, 2282 (1963); N. F. Bali, G. F. Chew, and A. Pignotti, *ibid.* **163**, 1572 (1967).
[41] For a summary of the work of Gel'fand, Yaglom, and Naimark, see M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon Press, Ltd., London, 1964).
[42] Y. Nambu, Phys. Rev. **160**, 1171 (1967).

Complex Solutions

Most of the equations that have been considered in the literature, including the three examples discussed above, have no solutions for complex momenta.

TCP Invariance

In infinite-component field theories *TCP* invariance is an independent postulate; it is possible to violate it,[13] but just as easy not to. Therefore, this is not a problem.

Spin-Statistics

It has been claimed that the usual connection between spin and statistics is looser in infinite-component field theory than in conventional field theory. Even if this were true, it is difficult to see how this could create a problem; let us just take the usual correlation as an experimental fact and choose our method of quantization accordingly. The fact is, however, that models with unorthodox spin-statistics connection are very pathological.[9] In a given Lagrangian model there is at most one choice, commutators or anticommutators, that yields a positive energy density; if this turns out to be the "wrong" statistics, then the whole model must be rejected. (The claim that quantization with anticommutators is always—see particularly Feldman and Matthews[43]—impossible is based on the notion that the theory admit the spin group S as an exact invariance group. Causal quantization with anticommutators is obtained by the same authors under less restrictive assumptions in a subsequent paper.[44])

Metric

However, the metric in Hilbert space must satisfy a positivity condition. If the spin values are half-integral, then the metric must be positive definite. If the spins are integral, then the metric must have the same sign as p_0. Note that the metric η enters the canonical commutation relations through the definition of the canonical momentum:

$$\begin{gathered}\pi=\delta\mathcal{L}/\delta\dot{\psi}=\psi^*\eta\,,\\ [\psi(x,t),\,\pi(x',t)]_\pm=\delta^{(3)}(x-x')\,.\end{gathered}\tag{VI1}$$

In Sec. III we proved that the metric is positive definite in our first model. An example with half-integral spins has been treated previously.[3]

Analyticity

It is one of the conclusions of this paper that the analytic structure of scattering amplitudes is largely conventional. The difference between finite- and infinite-component field theories lies in the order of perturbation theory in which the various singularities are found. This is most dramatic with respect to bound-state poles, which do not occur in any finite order in ordinary field theory, but are incorporated into the first Born approximation of the infinite-component theories.

Crossing Symmetry

The idea that interactions are generated by local Lagrangian interaction densities gives rise to a type of crossing symmetry; that is, an intimate relationship between two physical amplitudes that are related to each other by a change of sign of the four-momenta of one or more external lines.[3] In the models that have been investigated up to now, this crossing symmetry has been found to differ from that of ordinary local field theory; it is not so intimately tied up with analytic continuation.[3] This is not necessarily objectionable, nor has it been shown to be uncurable. In ordinary local field theory crossing by analytic continuation is a result of invariance of the scattering amplitudes under the complex Lorentz group. This is due to the fact that every finite-dimensional representation of the spin group (i.e., the spin part of the homogeneous Lorentz group) can be extended to a representation of the complex Lorentz group. It has been suggested that ordinary crossing can be incorporated into infinite-component field theory by choosing the group S to include the complex Lorentz group. We plan to investigate this question soon.

B. Left-Hand Cut

All the well known difficulties seem to be avoidable, but a new one has been uncovered: the appearance of a left-hand cut in theories with no spacelike solutions. There are several possible ways out:

(1) The origin of the left-hand cut in the scattering amplitude is the left-hand cut of the vertex function. This in turn arises from the fact that the spurion is lightlike at the point $s=m_-^2$, the point where the spectrum of n changes from real discrete to imaginary continuous. The two-particle interpretation, and the determination of the constituent masses, depends crucially on this fact, as does the correct location of all the singularities. It is therefore necessary to insist on this significance of the point $s=m_-^2$. Nevertheless, it is conceivable that the discontinuity of the vertex function may contribute no discontinuity to the scattering amplitude if we choose a different representation in which the discrete spectrum of Γ_0 is symmetrical about the origin.

(2) A fixed left-hand cut in s is thought to be unphysical because it may imply propagation of signals faster than light. It is important to find out whether the signal propagates effectively over a large or a small distance. Pending an investigation of this point, we venture to guess that the signal propagates infinitely far in the model with spacelike solutions, but over a

[43] G. Feldman and P. T. Matthews, Phys. Rev. **151**, 1176 (1966).
[44] G. Feldman and P. T. Matthews, Phys. Rev. **154**, 1241 (1967).

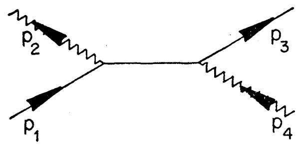

FIG. 14. The crossed amplitude. The interpretation is the same as in Fig. 6.

small distance related to the conspiracy parameter b' in the case of our last model (Sec. V). If this is the case there may be no real objection to a left-hand cut in the amplitude for scattering off a composite system.

(3) It has been implicitly assumed that the scattering amplitude $A(s,t)$ for the diagram of Fig. 6, analytically continued to negative s and positive u, is the scattering amplitude $A^x(s,t)$ for the diagram of Fig. 14. This is in accordance with conventional s-u crossing symmetry. In principle it is possible to make a direct evaluation of $A^x(s,t)$; then the important question is whether or not this function has a left-hand cut. Unfortunately, this turns out to be highly ambiguous in the models studied here. Without going into details, we mention the origin of the difficulty. The doubling of representation space (Sec. III A), which was carried out in order to make the energy spectrum symmetric about zero, is very artificial. The properties of the metric were found (Sec. III E) to require a superselection rule between states with $p_0\tau_3>0$ and states with $p_0\tau_3<0$. The local interaction density (III12) does not respect this superselection rule, and, in fact, no local interaction does. It is necessary to construct a model in which superselection rules of this type can be avoided before questions of crossing symmetry and left-hand cuts can be resolved.

C. Other Equations

The models presented here are exactly soluable; this is due to the fact that the wave operator is a function of a single spurion that picks out a single direction in 5-space. Another class of wave equations, though not exactly soluble, is amenable to a similar analysis. The vector spurion is replaced by a second-rank tensor; it picks out two distinguished directions in 5-space and leaves no accidental degeneracy. A simple example is

$$[(p\Gamma)^2+f(p^2)\Gamma_4{}^2+g(p^2)]\psi(p)=0,$$

where f and g are polynomials of low order in p^2. The spectrum of this equation is discrete if

$$p^2[p^2+f(p^2)]>0$$

and continuous otherwise. The Bethe-Salpeter-Wick-Cutkosky equation is essentially of this type.[23] The asymmetric top also obeys an equation similar to this one, the difference being that in that case Γ_4 is compact so that the spectrum is always discrete. According to Landau and Lifschitz[45] the eigenvalue problem cannot be solved exactly.

D. Current Algebras

Let the representation space be enlarged to allow the inclusion of some charge group with generators λ_i. From the canonical commutation relations (VI1) follows immediately that the time components of the currents,

$$J_\mu{}^i=\frac{\delta\mathfrak{L}}{\delta(\partial\psi/\partial x_\mu)}\lambda_i\psi+\psi^*\lambda_i\frac{\delta\mathfrak{L}}{\delta(\partial\psi^*/\partial x_\mu)},$$

satisfy equal-time commutation relations. This is a representation of current algebra by means of one- and two-particle intermediary states. However, these currents are no more and no less physical than the specific Lagrangian model they come from.[46]

Note added in proof. Grodsky and Streater[47] have investigated the structure of the commutation relations for unequal times. If one postulates that the time-ordered product of free fields is the Fourier transform of the propagator, then one obtains a field theory that is beyond the scope of their investigation, because this operator is not a finite covariant. Whether or not a formalism similar to ordinary field theory can be developed is still an open question. Here we have given a partial answer in exhibiting the extent to which interactions are local in theories with or without spacelike solutions. The intuitive argument (Sec. VI B), according to which our last model is only "weakly nonlocal," is supported by results of Todorov.[48]

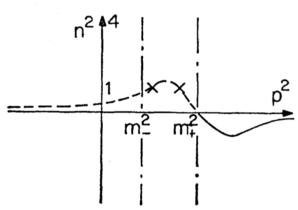

FIG. 15. The "ideal spectrum," with a finite number of bound states, a right-hand continuum, and conspiracy, might have this form.

[45] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Mass.), p. 374.

[46] A special example has been constructed by Leutwyler [H. Leutwyler, Universität Bern, Switzerland (unpublished report)]. See also M. Gell-Mann, D. Horn, and J. Weyers, Princeton Institute for Advanced Study Report, 1967 (unpublished).

[47] I. T. Grodsky and R. F. Streater, Phys. Rev. Letters **20**, 695 (1968).

[48] I. Todorov (private communication).