Pion Production, Current Algebra, and Coupling-Constant Relations

A. N. MITRA AND R. NATH
Department of Physics and Astrophysics, University of Delhi, Delhi, India
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The techniques of current algebra are applied to the process $\pi N \to \pi\pi N$, with a view of obtaining certain coupling-constant sum rules. The emphasis is on the evaluation of the limit when all three pions become soft. While the matrix elements for such a process all vanish in this limit, it is possible to use a suitable limiting procedure to obtain nontrivial relations involving various strong coupling constants and certain low-energy π-π parameters. The entire formalism is based on using an extensive pole model for the evaluation of the $\pi N \to \pi\pi N$ matrix elements. The problem of ambiguous terms which have a more complicated structure in such processes is treated in some detail. One of the interesting results is an independent determination of the pion scattering lengths (in terms of the known coupling constants of hadrons), in almost exact agreement with Weinberg's original result.

1. INTRODUCTION

THE applications of current-algebra techniques to strong-interaction processes developed by various authors in recent times[1–6] are mainly characterized by extrapolation from the soft-pion limits to physical pions, assuming smooth behavior of the amplitudes. The extrapolation itself is expressed by expanding the amplitude in terms of the usual invariant parameters (s,t,u) of the process, and relating the coefficients in the expansion by current algebra. These techniques have provided highly successful predictions of the π-N scattering lengths, indicating that the π-N amplitudes do not vary appreciably between their physical threshold values with actual pion masses and the (unphysical) values when these masses are made to vanish. For the π-π amplitudes, however, the extrapolation is probably more ambiguous, though most of the above investigations agree on the "smallness" of the π-π scattering lengths obtained by these methods. Such small scattering lengths are, of course, an essential requirement for a consistent treatment of current-algebra techniques which would not make sense if the π-π interaction were so strong as to make the contributions of the unitarity cuts quite significant. While this result is at variance with the predictions of the older bootstrap calculations,[7] or the phenomenological evidences from π-N scattering,[8] the idea of "not-so-strong" interactions among pions and nucleons seems well worth pursuing, despite recent attempts to accommodate larger π-π scattering lengths via suitable unitarization procedures.[9,10] Indeed, a more comprehensive calculation by Khuri,[5] who kept the second-order terms in the expansion of the π-π amplitude, strongly indicated the relatively mild role of the unitarity cut, thus providing evidence against any particularly strong π-π interaction. Moderately strong interactions between hadrons, which represent an essential requirement for current-algebra applications, should also allow the use of "effective Lagrangians" that have been shown to yield the same results[11] as the former by the straightforward methods of perturbation theory for any particular process calculated in the lowest order.

The purpose of this paper is to extend the idea of moderately strong hadron interactions to the process $\pi N \to \pi\pi N$, assuming the amplitude to be dominated only by low-lying poles and extrapolating down to zero masses of all the pions, with a view to determining relations between certain coupling constants. In such an approach we must tacitly assume that the unitarity effects are not much more important than can be simulated by low-lying resonances in the intermediate states.[12] However, since our object is not to calculate the physical $\pi N \to \pi\pi N$ amplitude but merely the (unphysical) extrapolated one for zero pion 4-momenta, one may expect the unitarity effects to be less important than for the physical amplitude itself. Of course, this amplitude vanishes in the limit of zero pion 4-momenta, due to parity considerations. Yet it should be possible to obtain some useful results by keeping track of the terms of the lowest (first) order in these momenta. The idea is analogous to the evaluation of the forward-angle derivative in the dispersion relation for a spin-flip amplitude which by itself vanishes in the forward direction.

An important question in this investigation concerns the problem of ambiguous terms in the production amplitude. It is well known that the corresponding terms in scattering processes are exactly cancelled by

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[7] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 567 (1960); Nuovo Cimento **19**, 752 (1961).
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[10] K. Kang and T. Akiba, Phys. Rev. **164**, 1836 (1967).
[11] S. Weinberg, Phys. Rev. Letters **18**, 188 (1967); J. Schwinger, Phys. Letters **24B**, 473 (1967); Phys. Rev. Letters **18**, 923 (1967).
[12] Such an assumption is absolutely necessary for the treatment of production processes which, at the present (inadequate) stage of development in dispersion techniques, cannot reasonably be expected to admit of a full dispersion treatment with as many as five independent variables involved.

the "pole" terms arising from the interaction with the axial-vector field related to the pion field by partially conserved axial-vector current (PCAC).[13] In the present case of a production process, the structure of the ambiguous terms is more complicated, being roughly of the *second-order* type

$$(q_1/p_1\cdot q_1)(q_2/p_2\cdot q_2),$$

where q_1, q_2 are pion 4-momenta and p_1, p_2 are nucleon 4-momenta, rather than the *first-order* type $(q/q\cdot p)$ characteristic of scattering processes via nucleon poles. Thus the inclusion of the extra pole terms contributed by the axial-vector field (related to the pion field by the PCAC condition) in the present case of a production process is expected merely to reduce the ambiguity by one order, so that one would still be faced with the presence of first-order terms like $(q/q\cdot p)$, in addition, of course, to unambiguous terms. One of the purposes of the present investigation is to analyze such terms in the limit when all the three pion momenta are made to vanish. In particular, it will be shown that the number of varieties of such terms is only *1*, *independently* of how the pions become soft.

A second object of this investigation is to make an independent determination of the π-π scattering length through the envisaged coupling-constant relations discussed in the previous paragraph. More specifically, we shall obtain a direct estimate of the parameter A of Ref. 1 in terms of the (known) coupling constants involving N, N^*, ρ, π and show that this estimate, together with the Weinberg relation[1]

$$B-C=8\pi L/\mu \tag{1.1}$$

and the Adler self-consistency condition[1]

$$A=-\mu^2(2B+C), \tag{1.2}$$

yields

$$a_0\approx 0.18\mu^{-1},\quad a_2\approx -0.06\mu^{-1}, \tag{1.3}$$

in almost exact agreement with the result of Ref. 1. We shall also obtain certain other relations involving the coupling constants of the so-called σ field with π and N and discuss their significance.

In Sec. 2 we briefly outline the reduction procedure for the $\pi N\to\pi\pi N$ amplitude in the soft-pion limit and give a simple recipe in a covariant form for the inclusion of the pole contributions arising from the axial-vector counterpart of the pion field. In Sec. 3 we discuss the relevant pole diagrams for the $\pi N\to\pi\pi N$ process and also give the values of the various coupling constants that are involved. Section 4 describes the evaluation of the different matrix elements when the soft-pion limit is carried out and the derivation of the coupling-constant relations. Section 5 summarizes the main conclusions of this investigation.

2. REDUCTION OF THE $\pi N\to\pi\pi N$ AMPLITUDE

In the standard notation and normalization, the $\pi N\to\pi\pi N$ amplitude M_3 on the mass shell is defined in terms of the (third-order) S-matrix element by[1]

$$\langle f;\beta q_2,\gamma q_3|S_3|i;\alpha q_1\rangle = -i(2\pi)^4\delta^4(P)(2\pi)^{-15/2}(m^2/8E_iE_fq_{10}q_{20}q_{30})^{1/2} \times\langle f;\beta q_2,\gamma q_3|M_3|i;\alpha q_1\rangle_{(\text{on shell})}, \tag{2.1}$$

$$P\equiv p_1+q_1-p_2-q_2-q_3=0. \tag{2.2}$$

The corresponding amplitude M_3 off the mass shell is expressible by Lehmann-Symanzik-Zimmermann (LSZ) techniques as

$$\langle f;\beta q_2,\gamma q_3|M_3|i;\alpha q_1\rangle(-1)(2\pi)^4\delta^4(P) \times\{\prod_{i=1}^{3}(\mu^2+q_i^2)^{-1}\}(2\pi)^{-3}(m^2/E_iE_f)^{1/2} = \int d^4x_1d^4x_2d^4x_3\exp(iq_2\cdot x_2+iq_3\cdot x_3-iq_1\cdot x_1) \times\langle f|T(\phi^\alpha(x_1)\phi^\beta(x_2)\phi^\gamma(x_3))|i\rangle. \tag{2.3}$$

Here p_1 and p_2 are the initial (i) and final (f) 4-momenta of the nucleon, q_1 and (q_2,q_3) are the 4-momenta of the initial (one) and final (two) pions, respectively, and α, β, γ are the isospin labels for the pions. The metric is such that $A_4=iA_0$ and $A\cdot B=A_\mu B_\mu=A_1B_1+A_2B_2+A_3B_3-A_0B_0$. The normalization of the pion field $\phi^\alpha(x)$ is expressed by

$$\langle 0|\phi^\alpha(x)|\pi_\beta q\rangle=(2\pi)^{-3/2}(2q_0)^{-1/2}\delta^{\alpha\beta}e^{iq\cdot x}, \tag{2.4}$$

and the PCAC relation for the corresponding axial-vector field is given by[1]

$$\partial_\mu A_\mu{}^\alpha=\mu^2F_\pi\phi^\alpha(x)\equiv C_\pi\phi^\alpha(x), \tag{2.5}$$

where

$$F_\pi=2mg_A/Gg_V \tag{2.6}$$

and μ, m, G are the pion mass, nucleon mass, and pion-nucleon coupling constant, respectively. Using (2.5) for the field in (2.3) makes the right-hand side of the latter equal to

$$ic_\pi{}^{-1}q_{1\mu}\int d^4x_1d^4x_2d^4x_3\exp(iq_2\cdot x_2+iq_3\cdot x_3-iq_1\cdot x_1)\langle N_f|T(A_\mu{}^\alpha(x_1)\phi^\beta(x_2)\phi^\gamma(x_3))|N_i\rangle - c_\pi{}^{-1}\int d^4x_1d^4x_2d^4x_3\exp(iq_2\cdot x_2+iq_3\cdot x_3-iq_1\cdot x_1)\langle N_f|T(\delta(x_{10}-x_{20})[A_0{}^\alpha(x_1),\phi^\beta(x_2)]\phi^\gamma(x_3) + \delta(x_{10}-x_{30})[A_0{}^\alpha(x_1),\phi^\gamma(x_2)]\phi^\beta(x_2))|N_i\rangle. \tag{2.7}$$

[13] V. Alessandrini, M. Bég, and L. Brown, Phys. Rev. 144, B1137 (1966).

We must now consider the limit $q_{1\mu}\to 0$ for the above expression before any other operations are performed, so that after this stage the 4-momentum conservation condition would reduce to

$$p_1=p_2+q_2+q_3. \tag{2.8}$$

In general, the "pole" term of (2.7) arising from the $A_\mu{}^\alpha(x_1)$ field will contribute in the limit $q_{1\mu}\to 0$. We transfer this term to the left-hand side of (2.7). Now the net contribution of the left-hand side can be evaluated, as in Ref. 13, so as to cancel the effect of the ambiguous terms. However, we use below the following equivalent recipe for taking account of the effect of this term in a simple way. Thus, if the replacement of the pion field by its axial-vector counterpart is made at a vertex where a *free* nucleon line, say, p_1, meets, the net effect of combining the "pole" term with the corresponding matrix element for the pion interaction is to make the replacement

$$\gamma_5 u(p_1)\to\gamma_5(1\mp i\mathbf{q}/2m)u(p_1) \tag{2.9}$$

when the pion momentum q is absorbed or emitted, respectively. Now, since the reduction (2.7) is made with respect to the absorbed momentum q_1, we must have

$$\gamma_5 u(p_1)\to\gamma_5(1-i\mathbf{q}_1/2m)u(p_1) \tag{2.10}$$

and

$$\bar{u}(p_2)\gamma_5\to\bar{u}(p_2)(1+i\mathbf{q}_1/2m)\gamma_5 \tag{2.11}$$

when the absorption occurs at the vertex of the incoming or the outgoing nucleon, respectively. Even if the absorption occurs at a vertex where both nucleon lines are internal, the modification $\gamma_5(1-i\mathbf{q}_1/2m)$ or $(1+i\mathbf{q}_1/2m)\gamma_5$ of γ_5 still holds.

The above prescription follows from the relation

$$2Mg_A=\mu^2F_\pi G, \tag{2.12}$$

where G is the coupling constant for the πNN vertex

$$H_\phi=iG\bar{\psi}\gamma_5\tau_\alpha\psi\phi^\alpha(x) \tag{2.13}$$

and g_A is the coupling constant for the axial-vector current

$$A_\mu{}^\alpha(x)=ig_A\bar{\psi}\gamma_\mu\gamma_5\tau_\alpha\psi. \tag{2.14}$$

Similar relations can also be written for the coupling constants of the pion field and those of the axial-vector current associated with other combinations like $(\bar{N},N^*)$, etc. However, since the masses of such particles which would be involved in the intermediate states are different from the nucleon mass, such "pole" terms [and hence the vertex modifications corresponding to (2.9)] would not be of physical interest.

As for the second term in (2.7), one must use the the equal-time current-commutation relations[1,5]

$$[Q^\alpha(t),a_\pi\phi^\beta(x)]=i\delta^{\alpha\beta}\sigma(x), \tag{2.15}$$

$$[Q^\alpha(t),\sigma(x)]=-i\delta^{\alpha\beta}a_\pi\phi^\beta(x), \tag{2.16}$$

where

$$Q^\alpha(t)=\int d\mathbf{x}\,A_0{}^\alpha(x) \tag{2.17}$$

and a_π is a constant which one can determine (in principle) from the $\sigma\pi\pi$ vertex function f_σ defined by[5]

$$f_\sigma(\mu^2,0,\mu^2)=-a_\pi c_\pi{}^{-1} \tag{2.18}$$

in the limit when one of the pions becomes soft. This reduces the second term of (2.7), in the limit $q_{1\mu}\to 0$, to

$$-c_\pi{}^{-1}a_\pi{}^{-1}i\int d^4x_2d^4x_3\exp(iq_2\cdot x_2+iq_3\cdot x_3)$$
$$\times[\delta^{\alpha\beta}\langle p_2|T(\sigma(x_2)\phi^\gamma(x_3))|p_1\rangle+\delta^{\alpha\gamma}\langle p_2|T(\sigma(x_3)\phi^\beta(x_2))|p_1\rangle]. \tag{2.19}$$

Further reduction of (2.19), which consists in once again making the replacement (2.5) for the ϕ^γ and ϕ^β fields, leads to a "pole term" as well as a commutator term which, respectively, simplify to

$$\text{Pole}=-a_\pi{}^{-1}c_\pi{}^{-2}\int d^4x_2d^4x_3\exp(iq_2\cdot x_2+iq_3\cdot x_3)$$
$$\times[q_{3\nu}\delta^{\alpha\beta}\langle p_2|T(\sigma(x_2)A_\nu{}^\gamma(x_3))|p_1\rangle+q_{2\lambda}\delta^{\alpha\gamma}\langle p_2|T(\sigma(x_3)A_\lambda{}^\beta(x_2))|p_1\rangle] \tag{2.20}$$

and

$$\text{Comm}=c_\pi{}^{-2}\int d^4x\exp[ix\cdot(q_2+q_3)]$$
$$\times\langle p_1|\phi^\delta(x)|p_2\rangle(\delta^{\alpha\beta}\delta^{\gamma\delta}+\delta^{\alpha\gamma}\delta^{\beta\delta}), \tag{2.21}$$

where we have made use of Eq. (2.16). The last term in (2.21) has simply the structure of a π-N vertex and can be calculated by S-matrix methods by replacing the field $\phi^\delta(x)$ with the quantity

$$(\mu^2-\Box^2)^{-1}j_\phi(x)\approx\mu^{-2}j_\phi(x)\equiv\mu^{-2}iG\bar{\psi}\gamma_5\tau_\delta\psi. \tag{2.22}$$

Similarly, (2.20), which physically corresponds to the $\pi N\to\sigma N$ process via the nucleon pole,[14] can be evaluated by making the substitution (2.14) for $A_\mu{}^\alpha(x)$ and the value $m_\sigma{}^{-2}g_{NN\sigma}\bar{\psi}\psi$ for $\sigma(x)$, where $g_{NN\sigma}$ is the σNN coupling constant.

3. VERTICES AND GRAPHS FOR VARIOUS AMPLITUDES

The matrix elements for the process $\pi N\to\pi\pi N$, can be classified according to the diagrams listed in Figs. 1–3. Figure 1 is the traditional pion-pole diagram for pion production, one end of the pion line being tied to an effective ϕ^4 vertex, where the full amplitude off the energy shell is operative, and the other end to the $NN\pi$

[14] The pion-pole term will contribute a propagator like $[(q_2+q_3)^2+\mu^2]^{-1}$ which, however, remains finite in the limit $q_2,q_3\to 0$.

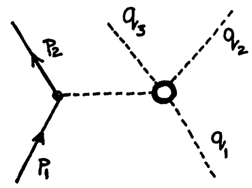

FIG. 1. Pion-pole diagram for $\pi N \to \pi\pi N$.

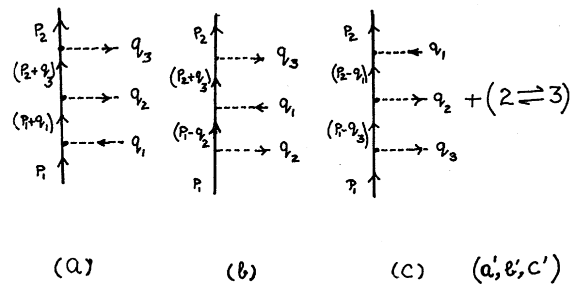

FIG. 2. Pole diagrams for $\pi N \to \pi\pi N$, where the internal line represents either N or N^*. The symbol $(2 \rightleftarrows 3)$ indicates the interchange of pions *2* and *3* in the respective diagrams preceding this symbol.

vertex, where the renormalized coupling constant G is present. Such an effective π-π vertex also incorporates π-π scattering diagrams through σ, ρ, etc., poles in s, u, or t channels. Figure 2 is a collection of diagrams with various baryons as internal lines, the possible pairs being NN, NN^* (or N^*N), and N^*N^*. Corresponding to each such pair, there are six subdiagrams, shown as (a,a'), (b,b'), and (c,c'), which, by an extension of the usual terminology of s and u channels for two-body processes, may be conveniently classified as (s,s), (s,u) [or (u,s)], and (u,u) channels. Here each letter in the channel classification indicates whether the corresponding baryon pole is s type (direct line) or u type (exchanged line). These diagrams in Fig. 2 involve the couplings between the $NN\pi$, $NN^*\pi$, and $N^*N^*\pi$ fields. Finally, in Fig. 3, we list the $\pi N \to \pi\pi N$ diagrams where one of the internal lines is a ρ or σ, while the other internal line is an N or N^*. These diagrams involve the couplings

$$NN\rho,\ NN^*\rho,\ NN\sigma. \tag{3.1}$$

In Fig. 3(d) we give the diagram for $N\pi \to N\sigma$, where the initial pion is replaced by its axial-vector current (2.14) since this process is related to the evaluation of (2.20).

The basic interaction Lagrangians necessary for the evaluation of the above diagrams are the following:

$$\mathcal{L}_{NN\pi} = -iG\bar{\psi}\tau_\alpha\gamma_5\psi\phi^\alpha(x)\,, \tag{3.2}$$

$$\mathcal{L}_{NN\rho} = -ig_{NN\rho}\tfrac{1}{2}\bar{\psi}\gamma_\mu\tau_\alpha\psi\rho_\mu{}^\alpha(x)\,, \tag{3.3}$$

$$\mathcal{L}_{NN\sigma} = -g_{NN\sigma}\bar{\psi}\psi\sigma(x)\,, \tag{3.4}$$

$$\mathcal{L}_{\pi\pi\rho} = -g_{\rho\pi\pi}\varepsilon_{abc}\rho_\mu{}^a\pi^b\overleftrightarrow{\partial}_\mu\pi^c\,, \tag{3.5}$$

$$\mathcal{L}_{\pi\pi\sigma} = -g_\sigma\phi^\alpha(x)\phi^\alpha(x)\sigma(x)\,, \tag{3.6}$$

$$\mathcal{L}_{N^*N\pi} = -G_1 m^{-1}\bar{\psi}\overleftrightarrow{\partial}_\mu\phi^\alpha(x)\Psi_\mu{}^{(\alpha)} + \text{H.c.}\,, \tag{3.7}$$

$$\mathcal{L}_{N^*N\rho} = -ig_{N^*N\rho}\bar{\psi}\rho_\mu{}^\alpha(x)\gamma_5\Psi_\mu{}^{(\alpha)} + \text{H.c.}\,, \tag{3.8}$$

$$\mathcal{L}_{N^*N^*\pi} = -iG_D\bar{\Psi}_\mu{}^{(\beta)}\gamma_5\phi^\alpha(x)\tau_\alpha\Psi_\mu{}^{(\beta)}\,. \tag{3.9}$$

Here for the N^* particles we have used the Rarita-Schwinger representation[15] for both the Dirac as well as the isospin indices. Thus, if $\Psi_\mu{}^{(\alpha)}$ is the wave function for this particle, then

$$\gamma_\mu\Psi_\mu{}^{(\alpha)} = 0\,, \tag{3.10}$$

$$\tau_\alpha\Psi_\mu{}^{(\alpha)} = 0\,. \tag{3.11}$$

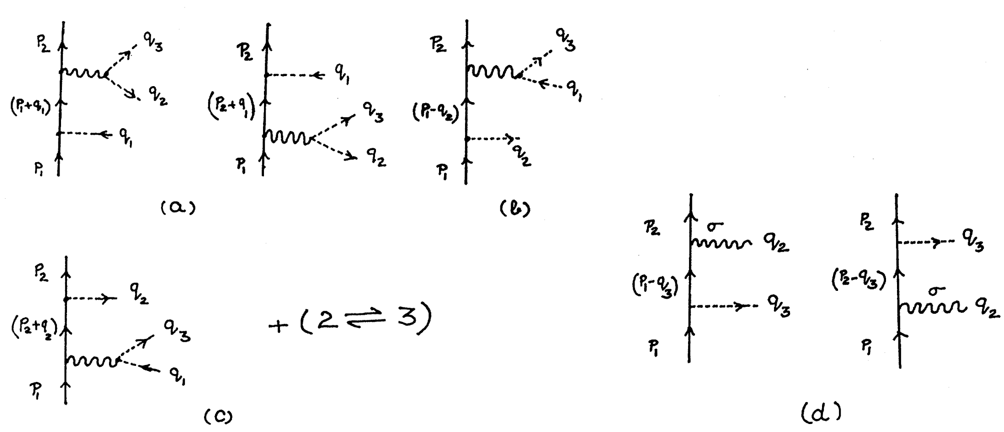

FIG. 3. (a)–(c) Pole diagrams for $\pi N \to \pi\pi N$, where one of the internal lines is an N or N^*, and the other a ρ meson or a σ meson. (d) Pole diagram for $\pi N \to \sigma N$. The symbol $(2 \rightleftarrows 3)$ indicates the interchange of pions *2* and *3* in the respective diagrams preceding this symbol.

[15] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941).

For completeness, we also list the propagator for the N^* particle (of mass M) in the form

$$\tfrac{1}{2}D_{F\mu\nu}{}^{\alpha\beta}(p)=-i(2\pi)^{-4}(p^2+M^2)^{-1}\times P_{\mu\nu}(\delta^{\alpha\beta}-\tfrac{1}{3}\tau_\alpha\tau_\beta)\,, \quad (3.12)$$

$$P_{\mu\nu}=[\delta_{\mu\nu}-\tfrac{1}{3}\gamma_\mu\gamma_\nu+\tfrac{4}{3}M^{-2}p_\mu p_\nu-\tfrac{1}{3}M^{-2}(\gamma_\mu\gamma\cdot p p_\nu+p_\mu\gamma\cdot p\gamma_\nu)](M-i\gamma\cdot p) \quad (3.13)$$

and that of the ρ meson as

$$\tfrac{1}{2}V_{F\mu\nu}{}^{\alpha\beta}(p)=-i(2\pi)^{-4}(m_\rho{}^2+p^2)^{-1}\times(\delta_{\mu\nu}+p_\mu p_\nu m_\rho{}^{-2})\delta^{\alpha\beta}. \quad (3.14)$$

In general, the different coupling constants involved in Eqs. (3.2)–(3.9) are too numerous to be of physical interest. However, we can use the universality of isospin interaction to relate $g_{\pi\pi\rho}$ and $g_{NN\rho}$ according to[16]

$$g_{\pi\pi\rho}{}^2/4\pi=g_{NN\rho}{}^2/4\pi\equiv g_\rho{}^2/4\pi\approx 2.1\,. \quad (3.15)$$

Similarly, the $N^*N\pi$ coupling G_1m^{-1} can be determined from the width of $N^*\to N\pi$ according to the formula

$$\Gamma_{N^*\to N\pi}=\tfrac{1}{3}G_1{}^2(E_N+m)p_\pi{}^3M^{-1}m^{-2}\pi^{-1}\approx 120\text{ MeV}\,, \quad (3.16)$$

where E_N is the nucleon (N) energy and p_π the pion momentum. This gives

$$G_1{}^2/4\pi\approx 4.0\,. \quad (3.17)$$

The quark-model prediction for G_1 works out as

$$G_1{}^2\colon G^2=8:25\,, \quad (3.18)$$

which is not too different from (3.17) when we use the familiar value

$$G^2/4\pi\approx 15.5\,. \quad (3.19)$$

As for the $N^*N^*\pi$ coupling constant G_D, a simple possibility is again to use $SU(4)$ symmetry (via the quark model), which yields

$$G_D=(9/5)G\,. \quad (3.20)$$

Similarly, the coupling constant $g_{N^*N\rho}$ can be related to $g_{NN\rho}$ ($\equiv g_\rho$) by $SU(4)$ symmetry. Now in the nonrelativistic quark model, the basic coupling $\bar{Q}\rho_\mu{}^\alpha\gamma_\mu\tau_\alpha Q$ is non-spin-flip so that it cannot connect the spin-$\frac{3}{2}$ state of N^* with the spin-$\frac{1}{2}$ state of N. Hence the nonrelativistic quark model predicts

$$g_{N^*N\rho}\approx 0\,. \quad (3.21)$$

Finally, we are left with the σ couplings of pions and nucleons. These are the most uncertain quantities in the present theory since, apart from the experimental uncertainties concerning the σ particle, even the formal connection of the σ field used in these current-algebra techniques to the σ particle of experimental interest is not at all clear. Assuming that the σ field of current algebra is the same as the experimental σ particle, one possibility is to estimate $g_{\sigma NN}$ from the σ contribution to the nucleon-nucleon potential. Thus the phase N-N shift analysis of Bryan and Scott[17] suggests that

$$g_{NN\sigma}{}^2/4\pi\approx 9.0\,. \quad (3.22)$$

As for the $\sigma\pi\pi$ coupling constant g_σ, even less is known, but a very crude estimate based on $\sigma\to\pi\pi$ width $\Gamma_\sigma\approx\mu$ at a mass $m_\sigma\approx 4\mu$ gives

$$g_\sigma{}^2/4\pi=\tfrac{2}{3}\Gamma_\sigma m_\sigma{}^2(m_\sigma{}^2-4\mu^2)^{-1/2}\approx 3\mu^2\,. \quad (3.23)$$

Since it would be futile to take these estimates (3.22) and (3.23) seriously, we shall not make any quantitative use of them, but merely use them as guides for comparison with the corresponding estimates from the present formalism.

Finally, the $\pi\pi\to\pi\pi$ vertex which is operative in Fig. 1 can be represented by an invariant off-shell $\pi\pi$ amplitude which for small (s,t,u) is[1]

$$T(q_1\alpha,q_2\beta\to q_3\gamma,q_4\delta)=\delta^{\alpha\beta}\delta^{\gamma\delta}[A+B(u+t)+Cs+\cdots]+\delta^{\beta\gamma}\delta^{\alpha\delta}[A+B(s+t)+Cu+\cdots]+\delta^{\alpha\gamma}\delta^{\beta\delta}[A+B(u+s)+Ct+\cdots]. \quad (3.24)$$

4. EVALUATION OF MATRIX ELEMENTS

Using the results of the last two sections, we now write the matrix elements corresponding to the two sides of the equality expressed by (2.3)–(2.7) and (2.19)–(2.21), in the soft-pion limit. To illustrate the manner in which the limiting procedure is carried out, we describe in some detail the contribution to the off-shell matrix element M_3 of Eq. (2.3), arising from the (N,N) intermediate states of Fig. 2, keeping separate track of the (s,s), (s,u), and (u,u) channels. Since, in the procedure described in Sec. 2, the 4-momentum $q_{1\mu}$ first tends to zero, the "pole" contribution to this matrix element [i.e., the first term of (2.7)] is incorporated by the modification (2.10) or (2.11) at the vertex where the pion $(q_1\alpha)$ is absorbed. This gives for the matrix elements from Fig. 2 the following expression:

$$M_{\text{II}}=M_{\text{II}}{}^{(1)}+M_{\text{II}}{}^{(2)}+M_{\text{II}}{}^{(3)}\,, \quad (4.1)$$

where $M_{\text{II}}{}^{(i)}$ ($i=1,2,3$) are the respective contributions of Figs. 2 (a,a'), 2 (b,b'), and 2 (c,c'). Thus

$$M_{\text{II}}{}^{(1)}=-2iG^3\bar{u}(p_2)\tau_\alpha\delta^{\beta\gamma}\left[\frac{-iq_3}{2p_2\cdot q_3+q_3{}^2}+\frac{-iq_2}{2p_2\cdot q_2+q_2{}^2}\right]\times\frac{ip_1+iq_1-m}{2p_1\cdot q_1+q_1{}^2}\gamma_5\left(1-\frac{i}{2m}q_1\right)u(p_1)\,, \quad (4.2)$$

which simplifies, in the limit $q_{1\mu}\to 0$, to

$$M_{\text{II}}{}^{(1)}=-iG^3\bar{u}(p_2)\tau_\alpha\delta^{\beta\gamma}\times\left[\frac{iq_3}{2p_2\cdot q_3+q_3{}^2}+\frac{iq_2}{2p_2\cdot q_2+q_2{}^2}\right]\gamma_5u(p_1)\,. \quad (4.3)$$

[16] J. J. Sakurai, Phys. Rev. Letters **17**, 1021 (1966).

[17] R. Bryan and B. L. Scott, Phys. Rev. **135**, B434 (1964).

TABLE I. Contributions of the various pole diagrams to the elements of the M matrix defined by Eq. (2.1), in the limit when all tend to zero. The factor $\bar{u}(p_2)i\gamma_5 u(p_1)$ is suppressed throughout.

Poles	$\delta^{\beta\gamma}\tau_\alpha$	$\delta^{\alpha\beta}\tau_\gamma+\delta^{\alpha\gamma}\tau_\beta$
(N,N)	0	$-2G^3D(q_2q_3)$
(N,N^*)	$-(16/9)GG_1^2(3M-2m)m^{-1}M^{-2}$	$-(16/9)GG_1^2m(M+m)M^{-2}D(q_2q_3)$
(N^*,N^*)	$\frac{4}{9}G_DG_1^2\left[\frac{13}{9}-\frac{2}{3}\frac{m}{M}\right]\frac{M+m}{M^3}$	$\frac{4}{9}G_DG_1^2\left[\frac{13}{9}-\frac{2}{3}\frac{m}{M}\right]\frac{M+m}{M^3}$
(N,ρ)	$-4Gg_\rho^2/m_\rho^2$	$2GG_\rho^2/m_\rho^2$
(N^*,ρ)	0	0
(N,σ)	$-2Gg_\sigma g_{\sigma NN}m^{-1}m_\sigma^{-2}$	$-2g_\sigma g_{\sigma NN}Gmm_\sigma^{-2}D(q_2q_3)$
π pole	$-GA\mu^{-2}$	$-GA\mu^{-2}$

It may be seen that while the ambiguity in q_1 has now disappeared from this expression, the ones in q_2 and q_3 are still present. Similar expressions can be written for the other two members of (4.1), and when all the three are added, one obtains

$$M_{\rm II}=-iG^3m^{-1}\bar{u}(p_2)(\delta^{\alpha\beta}\tau_\gamma+\delta^{\alpha\gamma}\tau_\beta)A\gamma_5u(p_1)\,, \quad (4.4)$$

where

$$A=\frac{iq_2+iq_3}{2p_2\cdot q_2+q_2^2}+\frac{iq_2+iq_3}{2p_2\cdot q_3+q_3^2} \quad (4.5)$$

and use has been made of the (reduced) conservation condition

$$p_1=p_2+q_2+q_3 \quad (4.6)$$

to make substitutions like

$$-2p_1\cdot q_3+q_3^2=2p_2\cdot q_2+q_2^2,\quad \text{etc.,} \quad (4.7)$$

in certain energy denominators. Use of the Dirac equation simplifies A to

$$A=2mD(q_2,q_3)\,, \quad (4.8)$$

where the single function

$$D(q_2,q_3)=(2p_2\cdot q_2+q_2^2)^{-1}+(2p_2\cdot q_3+q_3^2)^{-1} \quad (4.9)$$

represents the structure of the ambiguity in terms of q_2 and q_3.

In a similar way, we can obtain the contributions from the (N,N^*) and (N^*,N^*) intermediate states of Fig. 2 as well as the various pole diagrams of Figs. 1 and 2. We list all these contributions to the matrix M_3 [defined by (2.1) in terms of the S-matrix elements] in a two-dimensional table (Table I), where the two columns are classified according to their proportionality to the following two possible isospin operators:

$$I^{\alpha\beta\gamma}\equiv\delta^{\beta\gamma}\tau_\alpha\,,\quad J^{\alpha\beta\gamma}\equiv\delta^{\alpha\beta}\tau_\gamma+\delta^{\alpha\gamma}\tau_\beta\,, \quad (4.10)$$

and the rows are designated by the appropriate pairs of intermediate states or poles. One notices from this table that there is only a *single* type of ambiguous term, represented by the function $D(q_2,q_3)$ of Eq. (4.9), *irrespective* of how q_2 and/or q_3 tend to zero.

Finally, the contributions of the terms (2.20) and (2.21), the procedure for whose evaluation is described at the end of Sec. 3, are, respectively,[18]

$$\text{Pole}=-a_\pi^{-1}c_\pi^{-1}Gg_{\sigma NN}m_\sigma^{-2}mD(q_2,q_3) \quad (4.11)$$

and

$$\text{Comm}=c_\pi^{-2}\mu^{-2}G\,, \quad (4.12)$$

where, in both the terms, we have suppressed the common factor

$$i\bar{u}(p_2)\gamma_5J^{\alpha\beta\gamma}u(p_1)\,. \quad (4.13)$$

Substitution of all these results in the basic equations (2.3)–(2.7) between certain quantities all of which are proportional to the (vanishing) factor $i\bar{u}(p_2)\gamma_5u(p_1)$. The isospins are involved only in the two independent symmetrical combinations (4.10), so that, equating their coefficients separately, we have two independent relations between various coupling constants and the $\pi\pi$ scattering parameter A. The coefficients of $I^{\alpha\beta\gamma}$, which do not involve any ambiguous terms, satisfy the relation

$$\left(\frac{A}{4\pi}=-4\frac{g_\rho^2}{4\pi}\right)\left(\frac{\mu}{m_\rho}\right)^2-\frac{16}{9}\left(\frac{G_1^2}{4\pi}\right)\left(\frac{\mu}{M}\right)^2\frac{3M-2m}{m}$$
$$-\frac{2g_\sigma g_{\sigma NN}\mu^2}{4\pi mm_\sigma^2}+\frac{4}{5}\left(\frac{G_1^2}{4\pi}\right)\left(\frac{\mu}{M}\right)^2\frac{m+M}{M}\left(\frac{13}{9}-\frac{2}{3}\frac{m}{M}\right), \quad (4.14)$$

where we have used the relations (3.20) and (3.21). The coefficients of the other isospin operator $J^{\alpha\beta\gamma}$ are, however, vitiated by the presence of the ambiguous terms [proportional to $D(q_2,q_3)$]. Physically, it is a manifestation of the fact that while, for the limit $q_1\to 0$, there are counter terms available for cancellation of "q_1 ambiguities" in the matrix elements, the reduction

[18] A word of explanation is necessary on the appearance of the factor D in (4.11). The actual evaluation of this matrix element gives the essential expression

$$i\bar{u}(p_2)D(q_2q_3)(iq_2\gamma_5\delta^{\alpha\beta}\tau_\gamma+iq_3\gamma_5\delta^{\alpha\gamma}\tau_\beta)u(p_1),$$

which, in general, is a sum of two independent isospin combinations $\delta^{\alpha\beta}\tau_\gamma\pm\delta^{\alpha\gamma}\tau_\beta$ that, respectively, multiply the momentum factors $iq_2\pm iq_3$. While the term with the symmetrical combination reduces to (4.11) via the Dirac equation, the antisymmetrical combination (corresponding to an $I=1$ $\pi\pi$ state) involves the momentum combination iq_2-iq_3. This last term, which would lead to a further ambiguous structure, can be made zero only with the further assumption of $|q_2-q_3|\ll|q_2+q_3|$, which implies that π_2 and π_3 become soft in a *symmetrical* fashion.

technique employed in the steps from (2.3) to (2.7) has no place for corresponding counter terms to cancel the "q_2 and q_3 ambiguities." Therefore the only reasonable assumption, for the coefficients of $J^{\alpha\beta\gamma}$, would be to equate them to zero *separately* for the ambiguous and unambiguous terms. This then gives *two* additional relations

$$\frac{A}{4\pi}=-\frac{1}{4\pi}\left(\frac{\mu}{F_\pi}\right)^2+\frac{2\mu^2}{m_\rho{}^2}\left(\frac{g_\rho{}^2}{4\pi}\right)+\frac{4}{5}\left(\frac{G_1{}^2}{4\pi}\right)\left(\frac{\mu}{M}\right)^2\frac{m+M}{M}\left(\frac{13}{9}-\frac{2}{3}\frac{m}{M}\right) \quad (4.15)$$

and

$$\frac{g_{\sigma NN}\mu^2}{a_\pi F_\pi m_\sigma{}^2}=\frac{2G^2}{m}+\frac{16}{9}G_1{}^2\frac{M+m}{M^2}+\frac{2g_\sigma g_{\sigma NN}}{m_\sigma{}^2}. \quad (4.16)$$

Equation (4.16) may be regarded as the defining relation for the pion parameter a_π, as provided by the present model. Equation (4.15) is more interesting because it gave a direct estimate of the A parameter of Ref. 1 in terms of entirely *known* coupling constants. Indeed, a substitution of the values listed in Sec. 3 yields

$$A/4\pi\approx1.45L\mu=0.16. \quad (4.17)$$

As already mentioned at the end of Sec. 1, this value of A, together with the Weinberg relation $B-C=8\pi L\mu^{-1}$, as well as the Adler self-consistency condition $A=-\mu^2(2B+C)$, yields an *independent* estimate of the pion scattering lengths, viz., $a_0=0.18\mu^{-1}$, $a_2=-0.06\mu^{-1}$, in almost exact agreement with Weinberg's alternative derivation through the use of $A=-\mu^2(B+C)$ instead of (4.17). This result indicates an internal consistency in these small values of a_0 and a_2 with the values of the strong coupling constants g_ρ, G, etc., known from entirely different sources. This result also strengthens our belief that perhaps the strong interactions are in practice "not strong" after all.

Another interesting result that is obtained by subtracting (4.15) from (4.14) is

$$\frac{g_\sigma g_{\sigma NN}\mu^2}{4\pi m m_\sigma{}^2}=-\frac{3\mu^2}{m_\rho{}^2}\left(\frac{g_\rho{}^2}{4\pi}\right)+\frac{\mu^2}{8\pi F_\pi{}^2}-\frac{8}{9}\left(\frac{G_1{}^2}{4\pi}\right)\frac{\mu^2}{M^2}\frac{3M-2m}{m}\approx-0.28, \quad (4.18)$$

which shows that the coupling constants $g_{\sigma NN}$ and g_σ are in *opposite phase*. If, further, we take the estimate of $g_{\sigma NN}$ from (3.22), we obtain for g_σ the value

$$g_\sigma{}^2/4\pi\approx6.8\mu^2, \quad \text{for} \quad m_\sigma\approx4\mu \quad (4.19)$$

which is, however, about double its estimate from Eq. (3.23).

5. SUMMARY AND CONCLUSIONS

We have tried to present here a somewhat new form of application of current algebra towards the determination of coupling-constant sum rules, taking as an example the process $\pi N\rightarrow\pi\pi N$. The emphasis in this approach is not on extrapolation to the physical region[19] but on certain consistency relations among the various matrix elements in the soft-pion limit itself. These elements vanish in the soft-pion limit, yet with the help of a suitable limiting process it is possible to extract useful relations among the coupling constants. The essential ingredients of the model are a fairly comprehensive set of pole diagrams involving the coupling constants of the familiar hadronic interactions. This kind of approach provides an independent estimate of the pion scattering length in very good agreement with the original determinations.[1–5] It is also capable of providing information on the phases of certain coupling constants. For example, the constants $g_{\sigma\pi\pi}$ and $g_{\sigma NN}$ are predicted to be of opposite signs, though their magnitudes are not in good agreement with other determinations. This may well be due to the elusive (and pathological) nature of the σ particle, which makes it extremely hard to know any of its properties with any confidence. Our calculations do tell us, however, that the presence of some such particle is essential for a consistency between the two determinations (4.14) and (4.15) of the parameter $A/4\pi$.

An approach like the present one, based on using an extensive set of pole diagrams for the matrix elements and taking the soft-pion limit, should, in principle, be applicable to other similar processes to obtain further relations between various coupling constants.

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[19] L. Chang [Phys. Rev. **162**, 1497 (1967)] has applied current algebra to the physical process $\pi N\rightarrow\pi\pi N$ to obtain the low-energy cross sections for the process by much the same kind of pole technique that has been used here. Unfortunately, we were not aware of the details of Chang's work until it appeared in the *Physical Review*.