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Optical-Model Analysis of Pion-Nucleus Scattering. II. Pion Form Factor*

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The sensitivity of elastic $\pi^{\pm}$-α scattering to the pion form factor is calculated, using the velocity-dependent Kisslinger optical model. At 60 MeV an accuracy of 0.01 mb/sr should give the pion rms charge radius r_π to a few tenths of a fermi; additional form-factor details are not readily measurable. The use of the deuteron and other targets is examined.

INTRODUCTION

RECENTLY, we presented[1] an analysis of pion-nucleus scattering based upon the velocity-dependent optical model proposed by Kisslinger.[2] We obtained good fits to all the available data, using nuclear densities which agreed with those derived from electron scattering, and optical parameters which were in most cases reasonably close to those predicted from pion-nucleon phase shifts. We have now applied this model to the problem of determining the pion-charge form factor from $\pi^{\pm}$-α elastic scattering, as proposed earlier.[3]

Measurement of the nucleon form factors[4] led to the prediction of the existence of neutral vector mesons.[5] The measurement of the pion form factor, or even its rms charge radius r_π, would be a useful further test of the theory of form factors. Its determination from $\pi^{\pm}$-α scattering involves, as one might expect, looking for relatively small effects and making calculations with specific dynamical models. The incentive for pursuing this program is provided by the serious difficulties associated with the other methods attempted, π-e scattering[6] and π^+ electroproduction.[7]

If one assumes that the $\pi^{\pm}$-α elastic scattering amplitude is given by $f^{(\pm)}=f_N\pm f_c$, i.e., a sum of nuclear and Coulomb terms, then $D\equiv d\sigma^{(-)}/d\Omega-d\sigma^{(+)}/d\Omega=-4\,\mathrm{Re}(f_c^*f_N)$. This difference is a large fraction of the average $A=\frac{1}{2}(d\sigma^{(-)}/d\Omega+d\sigma^{(+)}/d\Omega)$ near the minimum at about 75° and is sensitive there to deviations from the point Coulomb amplitude.[3]

However, in analyzing experimental data, one must also include[8,9] in $f^{(\pm)}$ a distortion amplitude $f_D^{(\pm)}$ arising from the Coulomb distortion of the incident wave on which the nuclear forces operate. Including this term in the analysis of 24-MeV $\pi^{\pm}$-α elastic scattering[10] is sufficient to reduce the result[1,11] for r_π from 1.8 ± 0.8 F to $r_\pi\leq2.0$ F (two standard deviations). Solving a wave equation containing both the Coulomb and optical potentials automatically produces amplitudes $f^{(\pm)}$ containing a distortion correction.

The optical well-depth parameters and r_π can be varied to obtain a fit directly to the experimental data. Alternatively, distortion amplitudes obtained from the optical-model fit can be used with a phase-shift expansion for f_N and an analytic Coulomb amplitude. These distortion amplitudes are relatively insensitive to uncertainties in the optical parameters.[1] The question of their dependence on the particular dynamical model used, i.e., the Kisslinger optical model, is more subtle and is still under investigation.

In the following sections, we examine the sensitivity of $\pi^{\pm}$-α scattering to r_π as a function of energy, the possibility of determining additional form-factor de-

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[11] G. B. West, Phys. Rev. **162**, 1677 (1967); J. Math. Phys. **8**, 942 (1967); M. Block, Phys. Letters **25B**, 604 (1967). These papers estimate f_D using wave functions given by a wave equation containing only the optical potential. Their results are quite close to those of Ref. 1, found by including also the Coulomb potential in the wave equation. The simple denstiy-proportional optical potential used by Block is inadequate for fitting low-energy π-nucleus scattering in general, although it apparently works here. The quoted error for r_π is smaller than that of Ref. 1; this appears to be due to neglect of the dependence of χ^2 on variables other than r_π, i.e., the optical parameters.

TABLE I. Best-fit optical parameters and densities.

Nucleus	Energy (MeV)	Reb_1 (F^3)	Imb_1 (F^3)	Reb_0 (F^3)	Imb_0 (F^3)	Nuclear density[a]	Charge density[a]
^{4}He	20[b]	6.0	0.2	−5.0	1.0	$\rho_0 \exp(-r^2/1.22^2)$	$\rho_0 \exp[-3r^2/(2r_c^2)]$
	24	6.06	0.13	−4.4	0.77	$\rho_0 \exp(-r^2/1.22^2)$	$\rho_0 \exp[-3r^2/(2r_c^2)]$
	40[b]	6.0	0.20	−3.2	0.1	$\rho_0 \exp(-r^2/1.22^2)$	$\rho_0 \exp[-3r^2/(2r_c^2)]$
	48	5.9	0.24	−2.8	0.1	$\rho_0 \exp(-r^2/1.22^2)$	$\rho_0 \exp[-3r^2/(2r_c^2)]$
	60[b]	5.7	0.2	−2.6	0.1	$\rho_0 \exp(-r^2/1.22^2)$	$\rho_0 \exp[-3r^2/(2r_c^2)]$
	66	5.5	0.2	−2.6	0.1	$\rho_0 \exp(-r^2/1.22^2)$	$\rho_0 \exp[-3r^2/(2r_c^2)]$
	80[b]	5.7	0.2	−2.3	0.05	$\rho_0 \exp(-r^2/1.22^2)$	$\rho_0 \exp[-3r^2/(2r_c^2)]$
	92	5.6	0.13	−2.1	0.04	$\rho_0 \exp(-r^2/1.22^2)$	$\rho_0 \exp[-3r^2/(2r_c^2)]$
	100[b]	5.6	0.2	−2.0	0.04	$\rho_0 \exp(-r^2/1.22^2)$	$\rho_0 \exp[-3r^2/(2r_c^2)]$
^{2}H	61	4.6	1.4	−2.7	0.20	$\rho_0[\exp(-r/2.15)$	Uniform sphere
	85	4.2	1.6	−2.35	0.20	$-\exp(-5r/2.15)]^2/r^2$	Uniform sphere
C	80	5.9	1.15	−1.0	0.24	$\rho_0[1+\frac{1}{3}(Z-2)r^2/1.48^2]$ $\times\exp(-r^2/1.48^2)$	$\rho_0[1+\frac{1}{2}(Z-2)r^2/r_c^2]$ $\times\exp(-3r^2/(2r_c^2))$

[a] Normalized to $\int \rho d^3r = 1$. Radii in fermis.
[b] Parameters interpolated or extrapolated.

tails, and the use of other targets. It will be assumed throughout that violations of charge symmetry in $\pi^{\pm}$-α scattering arise only from Coulomb forces.

CONTRIBUTION OF r_π TO $\pi^{\pm}$-α SCATTERING

Published π-α elastic scattering at laboratory kinetic energies T_π of 24,[10] 48,[12] 66,[12] and 153 [13] MeV and preliminary unpublished data[14] at 66 and 92 MeV have been fitted previously[1] by solving a Klein-Gordon equation

$$(-\nabla^2+\mu^2)\psi=[(E_\pi-V_c)^2-U]\psi. \quad (1)$$

Here we have retained only the term linear in the optical potential[2]

$$U\psi=-Ab_0p_0^2\rho\psi+Ab_1\nabla\cdot(\rho\nabla\psi), \quad (2)$$

where p_0 is the incident momentum and ρ is the nuclear density.

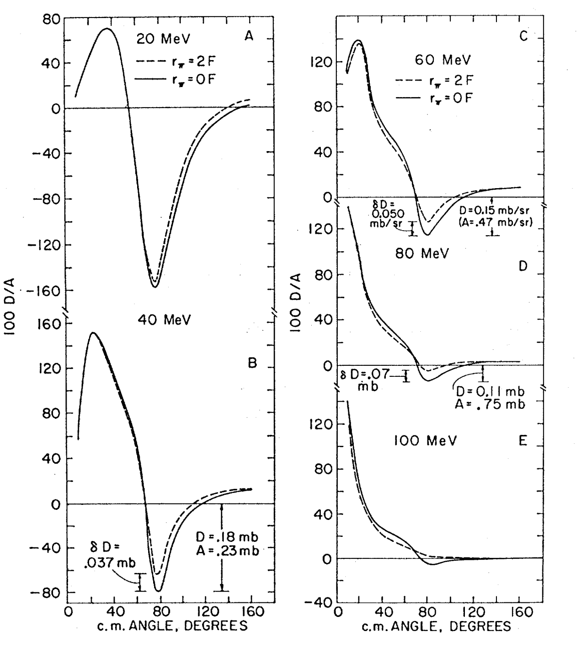

FIG. 1. Average A and difference to average ratio D/A for $\pi^{\pm}$-α scattering. See Table I for parameters used. Energies are laboratory kinetic energies.

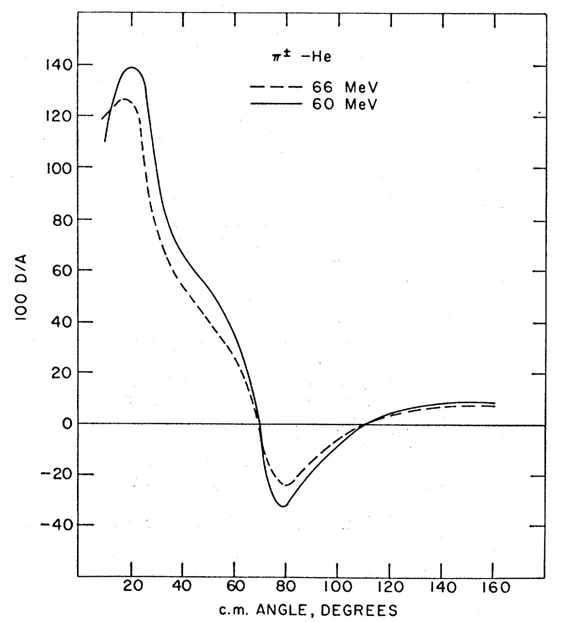

FIG. 2. D/A for $\pi^{\pm}$ scattering at 60 and 66 MeV.

[12] M. M. Bloch, I. Kenyon, J. Keren, D. Koetke, P. K. Malhotra, R. Walker, and H. Wenzeler, in Proceedings of the Williamsburg Conference on Intermediate Energy Physics, Williamsburg, Va., 1966 (unpublished), p. 447. [Later data appear in Phys. Rev. **169**, 1074 (1968).]

[13] Yu. A. Budagov, P. F. Ermolov, E. A. Kushnirenko, and V. I. Moskalev, Zh. Eksperim. i Teor. Fiz. **42**, 1191 (1961) [English transl.: Soviet Phys.—JETP **15**, 824 (1962)].

[14] K. Crowe (private communication).

If we use "theoretical" parameters b_0 and b_1 obtained from pion-nucleon phase shifts,[15] the computed differential cross sections are in fair qualitative agreement with the data. A very good fit was found[1] by varying the parameters. For all but the highest energy, the "best-fit" parameters Reb_1 and Imb_0 are in agreement with the theoretical values. $|\mathrm{Re}b_0|$ is larger than predicted, as was found in other nuclei. Imb_1 was smaller than predicted; this can probably be accounted for by including correlations, which suppress inelastic effects in helium.[16]

However, fitting the 153-MeV data requires b's very different from the theoretical parameters, which suggests that the model is unreliable at these energies. Additional experimental data on various nuclei in the region above 100 MeV would help to clarify this point.

Assuming an effective Gaussian charge density for the π-α system, with $r_c^2=r_\alpha^2+r_\pi^2$, we have calculated the sensitivity to r_π at various energies, interpolating and extrapolating the best-fit optical parameters. In view of the uncertain validity of the model at 153 MeV, calculations above 100 MeV must be regarded with suspicion and will not be presented.

Table I gives the best-fit parameters and the interpolated and extrapolated values. Figure 1 shows the results for $20\leq T_\pi\leq 100$ MeV for $r_\pi=0$ and 2 F. It appears that the middle of this energy region offers the greatest sensitivity to r_π. This sensitivity is somewhat smaller than was suggested by the earlier estimates[3] based on high-energy approximations. An accuracy of about 0.01 mb/sr will give r_π to a few tenths of a fermi. Figure 2 shows that the energy dependence of $\pi^\pm$-α is not rapid and that a beam spread of a few MeV should not cause problems in the analysis.

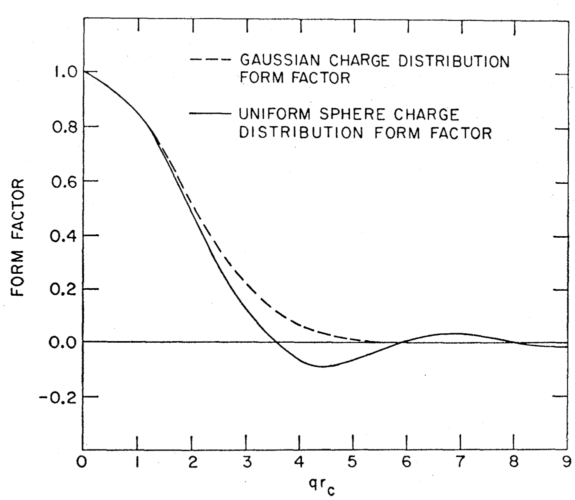

FIG. 3. Form factors F_G and F_S for Gaussian and uniform-sphere charge densities, respectively. See Eqs. (3) and (4).

[15] See Ref. 1, Eqs. (25) and (26) and Figs. 1–3.
[16] M. Ericson and T. Ericson, Ann. Phys. (N.Y.) **36**, 323 (1966).

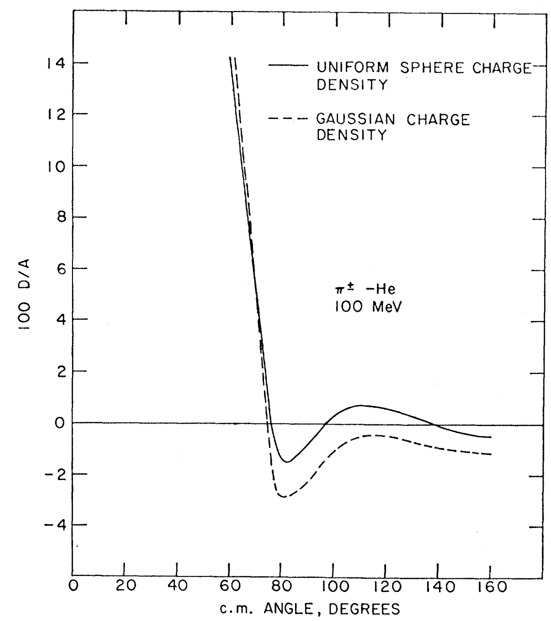

FIG. 4. D/A for $\pi^\pm$-α scattering with Gaussian and uniform-sphere charge densities at 100 MeV.

SHAPE PARAMETER

We examined the possibility of determining further form-factor details, or at least a "shape" parameter in addition to r_π. The form factor for the π-α Coulomb interaction is the product of π and α form factors. Thus if we assume that each of these is a Gaussian, the product is a Gaussian with $r_c^2=r_\alpha^2+r_\pi^2$, and the corresponding effective charge density is also a Gaussian. We compared the results of Gaussian calculations with those obtained with a uniformly charged sphere of the same rms radius. This latter density is not physically interesting, but was chosen as an illustration because of the simplicity of calculation.

The corresponding form factors are

$$F_G(q^2)=\exp(-\tfrac{1}{6}qr_c^2)=1-\tfrac{1}{6}q^2r_c^2+\tfrac{1}{2}(\tfrac{1}{6}q^2r_c^2)^2+\cdots, \quad (3)$$

$$\begin{aligned}F_S(q^2)&=3(\sin qb-qb\cos qb)/(qb)^3\\&=1-\tfrac{1}{6}q^2r_c^2+(5/14)(\tfrac{1}{6}q^2r_c^2)^2+\cdots, \quad (4)\\ b&\equiv(5/3)^{1/2}r_c.\end{aligned}$$

They are plotted in Fig. 3 as functions of qr_c. For $qr_c<3$ they differ by less than 0.1. For large arguments they are quite different, but unfortunately are also rather small. Thus the computed differential cross sections are quite similar.

If we assume $r_\pi=1$ F or $r_c=1.9$ F, we find D/A changes by at most 0.015 when we go from a Gaussian to a uniform-sphere density for all energies up to 150 MeV. Figure 4 shows the results for 100 MeV, which are typical.

Thus very good statistics and a high level of confidence in the computation of the distortion amplitude

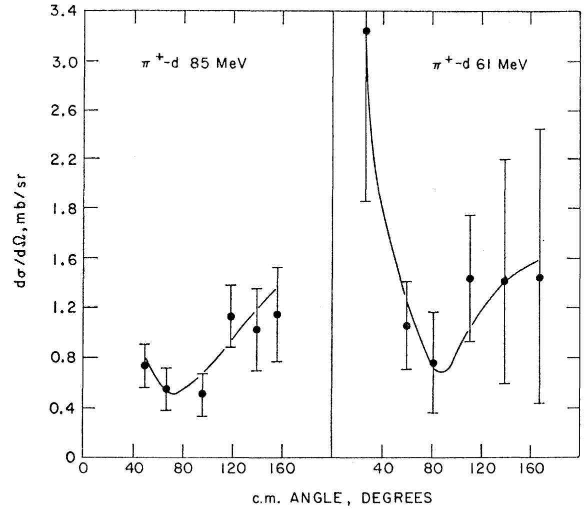

FIG. 5. $\pi^{\pm}$-d scattering at 61 and 85 MeV. Data are from Refs. 17 and 18, respectively. See Table I for the optical parameters and densities used.

would be required to distinguish F_G from F_S. More realistic calculations based on a Gaussian α form factor and some other form factor for the pion would probably lead to even smaller deviations from the Gaussian calculations. The situation would improve only slightly if r_π turns out to be somewhat larger than 1 F.

OTHER TARGETS

We have calculated the sensitivity of $\pi^{\pm}$-d and $\pi^{\pm}$-C scattering to r_π to learn whether the α particle is the best choice for measuring r_π.

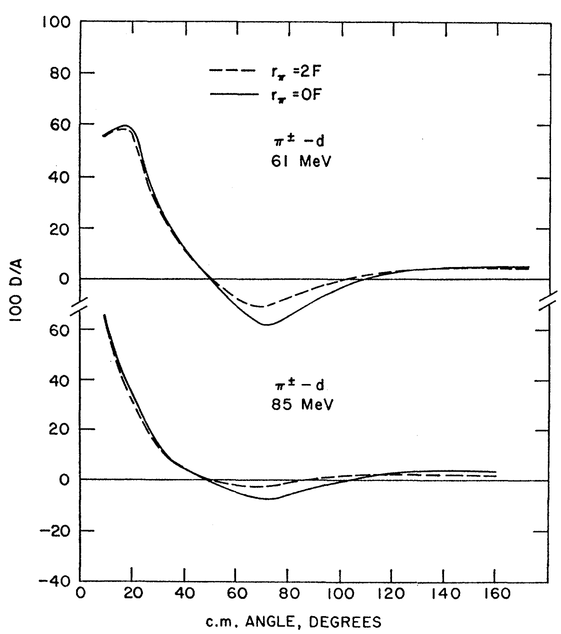

FIG. 6. D/A for $\pi^{\pm}$-d scattering at 61 and 85 MeV. See Table I for optical parameters and densities.

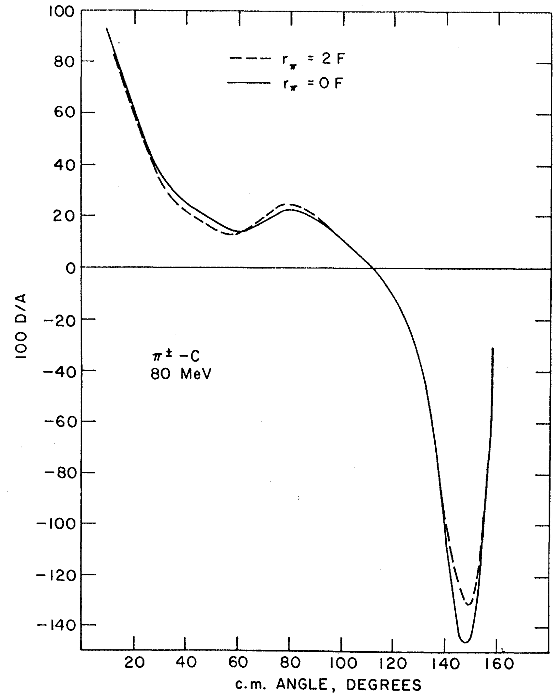

FIG. 7. D/A for $\pi^{\pm}$-C scattering at 80 MeV. See Table I for optical parameters and density.

The deuteron has a larger charge radius than the α, and half the charge. Thus by using optical parameters obtained by fitting the π^+-d data at 61[17] and 85[18] MeV, we find that the D/A sensitivity to r_π is about half that of $\pi^{\pm}$-α scattering (see Figs. 5 and 6).

The large carbon radius more than offsets its larger charge. The sensitivity of D/A for $\pi^{\pm}$-C to r_π is a third of that of $\pi^{\pm}$-α, except near the second minimum at 150°, where it is apparently comparable. However, the very small ($\sim$0.04 mb/sr) cross sections at this minimum depend critically on the particular nuclear density used,[1] and have not been measured. See Fig. 7.

Thus it appears that the α particle is the most sensitive isoscalar target available. An advantage of the deuteron is the fact that good nuclear wave functions are available, making possible calculations based on more rigorous foundations than the present ones. In any case, good π-d scattering data would be of considerable value in testing our basic understanding of pion-nucleon and pion-nucleus interactions. For example, information concerning the pion-nucleon off-shell amplitude could be obtained.[19]

Use of $T=\frac{1}{2}$ targets does not appear feasible. The $\pi^{\pm}$ scattering here involves both $T=\frac{1}{2}$ and $\frac{3}{2}$ amplitudes, so that D/A for $\pi^{\pm}$-p is no longer simply related

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[19] S. D. Drell and L. Verlet, Phys. Rev. **99**, 849 (1955).

to r_π. Scattering $\pi^\pm$ from both ^{3}He and ^{3}H would be similar in principle to the $\pi^\pm$-α experiment, but much more complex to analyze, since ^{3}He and ^{3}H have different charge distributions. Such experiments might be of more interest in the context of probing nuclear structure, e.g., in determining the amount of S' or mixed-symmetry state.[20,21]

[20] K. Ananthanarayanan, Phys. Letters **19**, 43 (1965).
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A Sum Rule Based on Unitarity*

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Unitarity, analyticity, Regge asymptotic behavior, and a resonance approximation are combined to derive a new sum rule. The sum rule is very convergent; the contribution of high-mass resonances is suppressed by a decreasing weight function. The spin-flip and non-spin-flip residues of the ρ meson in the $\pi\pi \to N\bar{N}$ amplitude are evaluated at the mass of the ρ, and in conjunction with the first-moment finite-energy sum rule, a calculation of the ρ-meson mass is performed. The results are in good agreement with experiment. A calculation of the ρ and f_0 resonance parameters in the $\pi\pi \to \pi\pi$ amplitude is also discussed.

I. INTRODUCTION

THE recent calculations of strong-interaction parameters from the finite-energy sum rules have been quite successful. The results have been in agreement with experiment to within the limits of error imposed by the model. Moreover, they have provided the bootstrap problem with a new approach which has already enjoyed some successes.[1,2]

The finite-energy sum rules relate all the moments of the discontinuity of the amplitude over a finite region in energy to the Regge parameters. However, in practice only the first few positive-moment sum rules have been used. The higher-moment sum rules emphasize higher-energy behavior so that in the context of most models they become redundant. As the negative-moment sum rules each contain the value of the amplitude or one of its derivatives at some point, they supply no additional constraints without prior knowledge of these unknown constants. It would be useful to have sum rules in which the weight function decreases, since even the low-positive-moment sum rules already put an uncomfortable emphasis on the higher-energy behavior of the discontinuity of the amplitude.

In Sec. II of this paper we derive a sum rule with a decreasing weight function by using two-body unitarity in the complex-J plane in addition to analyticity and Regge behavior. The weight function that multiplies the imaginary part of the amplitude is $Q_J(z)$. The derivation involves a small-width resonance approximation (*not* the usual narrow-width approximation), which we discuss in detail. By a small-width approximation we mean the width of the resonances we consider are small enough that the Breit-Wigner formula is reasonably accurate, but we do not take the limit $\mathrm{Im}\alpha \to 0$ in the discussion of this paper.

Finite-energy sum rules in general contain a parameter N, the upper limit of the integral of the imaginary part of the amplitude multiplied by some weight function. In order for these sum rules to be useful in bootstrap-type calculations, N must correspond to the "intermediate energies," so that the integrand may be parametrized by a sum of resonances. We find that for the sum rule presented here, the value of N depends on the magnitude of $\mathrm{Im}\alpha$. If N is to correspond to intermediate energies, $\mathrm{Im}\alpha$ can not be very small. Thus, for the

* This work was supported in part by the U. S. Atomic Energy Commission.
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