Nonlocal Currents as Coordinates for Particles

J. Šoln
Department of Physics, University of Wisconsin—Milwaukee, Milwaukee, Wisconsin 53201
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We investigate the possibility of formulating the dynamical theories in terms of nonlocal "currents." As an example, we formulate the nonrelativistic system of n identical particles (with and without spin). From these models we learn that the extension to relativistic systems is possible. This is illustrated by treating the Thirring model in terms of nonlocal currents.

1. INTRODUCTION

Very recently, there have been a number of articles[1–3] in which dynamical theories have been formulated in terms of local currents, charge densities, and spin-density operators.

In Ref. 1, Dashen and Sharp show how a nonrelativistic system of many identical particles, for example, without spin, can be described in terms of a charge-density operator

$$\rho(\mathbf{x},t)=\psi^*(\mathbf{x},t)\psi(\mathbf{x},t) \tag{1}$$

and current operator

$$\mathbf{J}(\mathbf{x},t)=(1/2i)[\psi^*(\mathbf{x},t)\nabla\psi(\mathbf{x},t)-\nabla\psi^*(\mathbf{x},t)\psi(\mathbf{x},t)]. \tag{2}$$

In case the system of particles possesses spin, we must add to (1) and (2) a spin-density operator

$$\mathbf{\Sigma}(\mathbf{x},t)=\tfrac{1}{2}\psi^*(\mathbf{x},t)\boldsymbol{\sigma}\psi(\mathbf{x},t). \tag{3}$$

They have shown that the operators (1) and (2) form a closed equal-time algebra under commutation, and that $\rho(\mathbf{x},t)$ and $\mathbf{J}(\mathbf{x},t)$ are a satisfactory set of coordinates for a nonrelativistic system of spinless particles, in the sense that every operator O which commutes with $\rho(\mathbf{x},t)$ and $\mathbf{J}(\mathbf{x},t)$ is a function of the total charge

$$Q=\int\rho(\mathbf{x},t)d^3x.$$

This means that $\rho(\mathbf{x},t)$ and $\mathbf{J}(\mathbf{x},t)$ form an irreducible set of operators, at a given time, when acting on a space of states all having the same total charge, since in this case the charge operator Q is a multiple of identity when acting on this space of states.[4]

If the system consists of n identical spin-$\frac{1}{2}$ particles, the theory must then be formulated in terms of operators $\rho(\mathbf{x},t)$, $\mathbf{J}(\mathbf{x},t)$, and $\mathbf{\Sigma}(\mathbf{x},t)$, where again one can show that these operators form an irreducible set of operators at a given time when acting on a space of states all having the same total charge.[1]

Since $\rho(\mathbf{x},t)$ and $\mathbf{J}(\mathbf{x},t)$ for a system of n identical spinless particles form an irreducible set of operators, every operator is a function of $\rho(\mathbf{x},t)$ and $\mathbf{J}(\mathbf{x},t)$. For example, the total linear momentum $\mathbf{P}$ and the total angular momentum $\mathbf{L}$ are trivially given as

$$\mathbf{P}=\int\mathbf{J}(\mathbf{x},t)d^3x,\qquad \mathbf{L}=\int\mathbf{x}\times\mathbf{J}(\mathbf{x},t)d^3x. \tag{4}$$

As far as the total Hamiltonian is concerned, the expression for it in Ref. 1 is given as

$$H=\tfrac{1}{8}\int[\nabla\rho(\mathbf{x},t)-2i\mathbf{J}(\mathbf{x},t)]\frac{1}{\rho(\mathbf{x},t)}[\nabla\rho(\mathbf{x},t)+2i\mathbf{J}(\mathbf{x},t)]d^3x+\int\rho(\mathbf{x},t)V(|\mathbf{x}-\mathbf{y}|)\rho(\mathbf{y},t)d^3xd^3y, \tag{5}$$

where the particles interact through a central potential, and the system of units $\hbar=m=1$ is assumed.

As far as the interaction part of the Hamiltonian is concerned, one does not have any particular problem; it is a perfectly sound operator expressed in terms of ρ operators. However, the free part of the Hamiltonian is a singular expression in ρ, which certainly is one of the unpleasant features of the theory. Moreover, assuming that expressions such as $[\rho^{-1}(\mathbf{x},t),\rho(\mathbf{y},t)]$ are equal to zero, one derives with the help of charge conservation ($\dot\rho+\nabla\mathbf{J}=0$), that the Hamiltonian H must have the form

$$H=\tfrac{1}{2}\int d^3x J_i(\mathbf{x},t)\frac{1}{\rho(\mathbf{x},t)}J_i(\mathbf{x},t)+H', \tag{6}$$

where H' commutes with $\rho(\mathbf{x},t)$ and is therefore a function only of $\rho(\mathbf{x},t)$. [If the particle possesses spin, it is a function also of $\mathbf{\Sigma}(\mathbf{x},t)$.]

Although ρ and $\mathbf{J}$ form a complete set, it is not obvious that the domain of a well-defined operator made up of ψ's is the same as that when it is made up of ρ and $\mathbf{J}$.

More recently, Callan, Dashen, and Sharp,[3] solving the two-dimensional relativistic Thirring model,[5] have included the energy-momentum tensor $\theta_{\mu\nu}$ in the set of coordinates $j_\mu(x)$. In this way one avoids singular ex-

[1] R. F. Dashen and D. H. Sharp, Phys. Rev. **165**, 1857 (1968).
[2] D. H. Sharp, Phys. Rev. **165**, 1867 (1968).
[3] C. G. Callan, R. F. Dashen, and D. H. Sharp, Phys. Rev. **165**, 1883 (1968).
[4] The probelm of irreducibility of the equal-time current algebra, besides being treated in Ref. 1, has been also treated quite extensively by Sharp in Ref. 2 for the case of a model of self-interacting charged scalar mesons.
[5] W. E. Thirring, Ann. Phys. (N. Y.) **3**, 91 (1958); Nuovo Cimento **9**, 1007 (1958).

pressions like Eqs. (5) and (6). The price that one pays in this way is a larger number of coordinates. The electric current j_μ and the energy-momentum tensor $\theta_{\mu\nu}$ form a closed equal-time algebra under commutation. As a matter of fact, the components of $\theta_{\mu\nu}$ and j_μ form a closed equal-time algebra under commutation separately, and the authors of Ref. 3 were able to write down $\theta_{\mu\nu}$ as an explicit function of the current j_μ in such a way that the commutation rules for $\theta_{\mu\nu}$ and j_μ remain the same. Although that form of $\theta_{\mu\nu}$ does not show singular features as does H in the form in Eqs. (5) or (6), it is difficult to know in advance how to construct it, i.e., this form does not show a resemblance to $\theta_{\mu\nu}$ when expressed in terms of the fermion field ψ.

In this paper, we introduce nonlocal density operators and, later on, nonlocal currents, which for simplicity we shall call by the common name nonlocal "currents," as a set of dynamical coordinates.

In Sec. 2, we shall study a nonrelativistic system of many identical particles, first without spin and then with spin. First, we define a set of nonlocal "currents" as dynamical variables which form an irreducible set of operators, at a given time, when acting on a space of states all having the same total charge (or number of particles). The total Hamiltonian will not be a singular expression in terms of these "currents."

The case of the system of n identical spin-$\frac{1}{2}$ particles will be especially important, since the relativistic two-dimensional field theory with spin $\frac{1}{2}$, which we shall treat in Sec. 3, will have a very strong resemblance to the nonrelativistic system for the n identical spin-$\frac{1}{2}$ particles. In the Appendix, we shall present some mathematical techniques needed for solving the Thirring model.

2. NONRELATIVISTIC THEORY

A. System of n Identical Spinless Particles

We want to show that one can give a satisfactory description for the system of n identical spinless particles (either bosons or fermions) in terms of the nonlocal "currents" $\rho(\mathbf{x},\mathbf{x}',t)$,[6] for which we postulate the following equal-time algebra:

$$[\rho(\mathbf{x},\mathbf{x}',t),\rho(\mathbf{y},\mathbf{y}',t)]=\delta(\mathbf{x}'-\mathbf{y})\rho(\mathbf{x},\mathbf{y}',t)-\delta(\mathbf{x}-\mathbf{y}')\rho(\mathbf{y},\mathbf{x}',t). \quad (7)$$

From Eq. (7) we see that we can impose the condition $\rho^*(\mathbf{x},\mathbf{x}',t)=\rho(\mathbf{x}',\mathbf{x},t)$.

We shall demonstrate that $\rho(\mathbf{x},\mathbf{x}',t)$ is an adequate dynamical variable by showing that any operator that commutes with $\rho(\mathbf{x},\mathbf{x}',t)$ for all $\mathbf{x}$ and $\mathbf{x}'$ at a given time is a multiple of an operator, which we shall define as total charge Q. Since the total charge is a conserved quantity, the statement is then equivalent to saying that any operator commuting with $\rho(\mathbf{x},\mathbf{x}',t)$ is a multiple of the identity. To show this, we employ a technique similar to that in Ref. 1. Any operator O is supposed to be a function of ρ. If O is to commute with $\rho(\mathbf{x},\mathbf{x}',t)$ for all $\mathbf{x}$ and $\mathbf{x}'$, it must be invariant under the following similarity transformation:

$$U(\lambda,t)=\exp\left[i\int\lambda(\mathbf{x},\mathbf{x}')\rho(\mathbf{x},\mathbf{x}',t)d^3xd^3x'\right], \quad (8)$$

where $\lambda(\mathbf{x},\mathbf{x}')$ is an arbitrary function of $\mathbf{x}$ and $\mathbf{x}'$.

In what follows, it is very useful to introduce an infinite-dimensional Hilbert space whose vectors we write in the form of kets $|\mathbf{x}\rangle$. The basis vectors are normalized with the δ function:

$$\langle\mathbf{x}'|\mathbf{x}\rangle=\delta(\mathbf{x}-\mathbf{x}'). \quad (9)$$

The arbitrary function $\lambda(\mathbf{x},\mathbf{x}')$ that appears in (8) is itself the matrix element of the abstract operator λ:

$$\lambda(\mathbf{x},\mathbf{x}')=\langle\mathbf{x}|\lambda|\mathbf{x}'\rangle. \quad (10)$$

With this notation, we see that under the transformation $U(\lambda,t)$, the nonlocal "current" $\rho(\mathbf{y},\mathbf{y}',t)$ satisfies

$$U(\lambda,t)\rho(\mathbf{y},\mathbf{y}',t)U^{-1}(\lambda,t)=\int d^3xd^3x'\rho(\mathbf{x},\mathbf{x}',t)\times\langle\mathbf{x}|e^{i\lambda}|\mathbf{y}\rangle\langle\mathbf{y}'|e^{-i\lambda}|\mathbf{x}'\rangle. \quad (11)$$

We see that whatever we do with ρ (or whatever combination we take of the ρ's), because of the arbitrariness of $\lambda(\mathbf{x},\mathbf{x}')$, the invariance will never be achieved unless in (11) we take $\mathbf{y}'=\mathbf{y}$ and the integration $\int d^3y$ is performed, since $\int d^3y|\mathbf{y}\rangle\langle\mathbf{y}|=1$. In detail,

$$U(\lambda,t)\int d^3y\,\rho(\mathbf{y},\mathbf{y},t)U^{-1}(\lambda,t)=\int d^3xd^3x'\rho(\mathbf{x},\mathbf{x}',t)\langle\mathbf{x}|\mathbf{x}'\rangle=\int d^3x\,\rho(\mathbf{x},\mathbf{x},t). \quad (12)$$

In other words, the operator O depends only on the quantity

$$Q=\int d^3x\,\rho(\mathbf{x},\mathbf{x},t), \quad (13)$$

which we shall define as the total charge of the system. For that reason the elements of the equal-time algebra [Eq. (7)] are irreducible at a given time when acting on states all having the same total charge Q. This means that every operator of physical interest is a function of $\rho(\mathbf{x},\mathbf{x}',t)$.

The charge-density operator can be defined, from (13), as

$$\rho(\mathbf{x},t)=\rho(\mathbf{x},\mathbf{x},t). \quad (14)$$

In defining the free part of the Hamiltonian H_0, we must be careful, since we want $\dot{Q}=0$. We can see that

[6] From (14) we shall see that the most appropriate name for this quantity is probably *density-matrix operator*; see S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson, and Co., Evanston, Ill., 1961).

H_0, defined through the limiting process[7]

$$H_0=\lim_{f(\mathbf{x},\mathbf{x}')\to\delta(\mathbf{x}-\mathbf{x}')}\tfrac{1}{2}\int f(\mathbf{x},\mathbf{x}')d^3xd^3x'\sum_i\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i'}\times\rho(\mathbf{x},\mathbf{x}',t), \quad (15)$$

will commute with Q, and therefore Q is the constant of motion.

To define the current operator $\mathbf{J}(\mathbf{x},t)$, we use current conservation:

$$\dot{\rho}(\mathbf{y},\mathbf{y},t)=-\frac{\partial}{\partial y_i}J_i(\mathbf{y},t). \quad (16)$$

Then, with the help of the Heisenberg equation of motion and (7), we have

$$\begin{aligned}\frac{\partial}{\partial y_i}J_i(\mathbf{y},t)&=i[H_0,\rho(\mathbf{y},\mathbf{y},t)]\\&=\lim_{f(\mathbf{x},\mathbf{y})\to\delta(\mathbf{x}-\mathbf{y})}\tfrac{1}{2}i\int f(\mathbf{x},\mathbf{y})d^3x\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_i}\rho(\mathbf{x},\mathbf{y},t)\\&-\lim_{f(\mathbf{y},\mathbf{x}')\to\delta(\mathbf{y}-\mathbf{x}')}\tfrac{1}{2}i\int f(\mathbf{y},\mathbf{x}')d^3x'\frac{\partial}{\partial x_i'}\frac{\partial}{\partial y_i}\rho(\mathbf{y},\mathbf{x}',t). \quad (17)\end{aligned}$$

Consequently, from Eq. (17), $J_i(\mathbf{y},t)$ can be defined as

$$J_i(\mathbf{y},t)=\lim_{\mathbf{x}\to\mathbf{y}}\frac{1}{2i}\left(\frac{\partial}{\partial x_i}\rho(\mathbf{y},\mathbf{x},t)-\frac{\partial}{\partial x_i}\rho(\mathbf{x},\mathbf{y},t)\right). \quad (18)$$

Finally, let us see what differential equation is satisfied by the elements of the algebra of Eq. (7) in the absence of interaction. From the Heisenberg equation of motion and Eq. (7) we have

$$i\frac{\partial}{\partial t}\rho(\mathbf{x},\mathbf{x}',t)=[\rho(\mathbf{x},\mathbf{x}',t),H_0]=-\tfrac{1}{2}\nabla^2(x')\rho(\mathbf{x},\mathbf{x}',t)+\tfrac{1}{2}\nabla^2(x)\rho(\mathbf{x},\mathbf{x}',t). \quad (19)$$

Equation (19) is invariant under three-dimensional rotation, provided that ρ transforms as a scalar.

The form (15) for H_0 does not show the unpleasant features of (5), i.e., being singular in ρ. The commutator of H_0 with $\rho(\mathbf{y},\mathbf{y}',t)$ (element of the algebra) is well defined in the limit $f\to\delta$. One could try to construct different expressions for H_0 and $\mathbf{J}$, but keeping in mind, of course, that $\dot{Q}=0$ and the conservation law (16).

The equal-time algebra of $\rho(\mathbf{x},\mathbf{x}',t)$ in Eq. (7) can be realized, however, with the field operators $\psi(\mathbf{x},t)$ and $\psi^*(\mathbf{x},t)$, which satisfy the usual canonical commutation or anticommutation relations, if one defines ρ as follows:

$$\rho(\mathbf{x},\mathbf{x}',t)=\psi^*(\mathbf{x},t)\psi(\mathbf{x}',t). \quad (20)$$

[7] Note that in order to maintain the Hermiticity of H_0 [Eq. (15)] through the limiting process, the function $f(\mathbf{x},\mathbf{x}')$ must satisfy $f^*(\mathbf{x},\mathbf{x}')=f(\mathbf{x}',\mathbf{x})$, since $\rho^*(\mathbf{x},\mathbf{x}',t)=\rho(\mathbf{x}',\mathbf{x},t)$.

The expressions for $\rho(\mathbf{x},t)$ [Eq. (14)], $J_i(\mathbf{x},t)$ [Eq. (18)], and H_0 [Eq. (15)] can be linked to the expressions (1), (2), and

$$H_0=\tfrac{1}{2}\int\nabla\psi^*(\mathbf{x},t)\nabla\psi(\mathbf{x},t)d^3x.$$

This, together with (19), confirms that the theory presented above in terms of $\rho(\mathbf{x},\mathbf{x}',t)$ describes the nonrelativistic system of n identical spinless particles. However, we want to point out that at no place did we need the canonical commutation or anticommutation relations for the field operators.

This link between the operators, when expressed in terms of ψ and ρ, respectively, can be of great help when dealing with concrete problems, i.e., when we want to postulate the theory for a given problem *a priori*. For example, from (5) we see that the interaction of particles through a central potential, taking into account (14), will be described with H_{int}, when expressed in terms of the ρ's ,as follows:

$$H_{\text{int}}=\int d^3xd^3y\,\rho(\mathbf{x},\mathbf{x},t)V(|\mathbf{x}-\mathbf{y}|)\rho(\mathbf{y},\mathbf{y},t). \quad (21)$$

Note, however, that if H_0 is not taken in the form (15), H_{int} must then be determined by some other considerations.

B. System of n Identical Spin-$\frac{1}{2}$ Particles

The system can consist of either fermions or bosons. We claim that the system can be described in terms of the nonlocal "currents" $\rho_\mu(\mathbf{x},\mathbf{x}',t)$ as dynamical variables, for which we postulate the following equal-time algebra:

$$[\rho_\mu(\mathbf{x},\mathbf{x}',t),\rho_\nu(\mathbf{y},\mathbf{y}',t)]=F_{\mu\nu\kappa}\delta(\mathbf{x}'-\mathbf{y})\rho_\kappa(\mathbf{x},\mathbf{y}',t)-F_{\nu\mu\kappa}\delta(\mathbf{x}-\mathbf{y}')\rho_\kappa(\mathbf{y},\mathbf{x}',t). \quad (22)$$

The constants $F_{\mu\nu\kappa}$ are defined by the relation

$$\sigma_\mu\sigma_\nu=F_{\mu\nu\kappa}\sigma_\kappa, \quad (23)$$

where $\sigma_0=1$, and σ_1, σ_2, and σ_3 are Pauli spin matrices. From (23) it is easy to calculate the $F_{\mu\nu\kappa}$, and we list them explicitly:

$$F_{ijk}=i\epsilon_{ijk}, \quad (24a)$$

where i, j, and k run from 1 to 3 cyclically;

$$F_{ii0}=F_{0ii}=F_{i0i}=1,\quad F_{000}=1, \quad (24b)$$

where $i=1$, 2, or 3. The remaining F's are zero.

If the coordinates $\rho_\mu(\mathbf{x},\mathbf{x}',t)$ are to describe a system of spin-$\frac{1}{2}$ particles, we must be able to construct a spin operator. It is easy to see that with the definition

$$S_i=\tfrac{1}{2}\int\rho_i(\mathbf{x},\mathbf{x},t)d^3x, \quad (25)$$

the S_i's satisfy the following algebra:

$$[S_i,S_j]=i\epsilon_{ijk}S_k\,, \tag{26}$$

which is a familiar $SU(2)$ algebra formed by spin operators.

However, we want to show, in general, that the ρ_μ's are an adequate set of coordinates. For that reason, as before in the spinless case, we show that any operator O that commutes with the ρ_μ's is a multiple of a certain operator, which we shall call the total charge Q.

If O is to commute with $\rho_\mu(\mathbf{x},\mathbf{x}',t)$ for all μ's and all $\mathbf{x}$ and $\mathbf{x}'$, it must be invariant under the similarity transformation

$$U(\lambda,t)=\exp\left[i\int\lambda_\mu(\mathbf{x},\mathbf{x}')\rho_\mu(\mathbf{x},\mathbf{x}',t)d^3xd^3x'\right] \tag{27}$$

(summation over μ implied), where $\lambda_\mu(\mathbf{x},\mathbf{x}')$ is an arbitrary function of $\mathbf{x}$ and $\mathbf{x}'$ for every μ. Since O is supposed to be expressed through the ρ's, we examine how $\rho_\nu(\mathbf{y},\mathbf{y}',t)$ behaves under the transformation $U(\lambda,t)$. For simplicity, we consider infinitesimal λ's:

$$U(\lambda,t)\rho_\nu(\mathbf{y},\mathbf{y}',t)U^{-1}(\lambda,t)=\rho_\nu(\mathbf{y},\mathbf{y}',t)+i\int d^3xd^3x'\rho_\kappa(\mathbf{x},\mathbf{x}',t)$$
$$\times[\delta(\mathbf{y}'-\mathbf{x}')F_{\mu\nu\kappa}\lambda_\mu(\mathbf{x},\mathbf{y})-\delta(\mathbf{x}-\mathbf{y})F_{\nu\mu\kappa}\lambda_\mu(\mathbf{y}',\mathbf{x}')]. \tag{28}$$

As in the spinless case, we see that whatever combination of ρ's we take, because of the arbitrariness of $\lambda_\mu(\mathbf{x},\mathbf{x}')$ the invariance will never be achieved unless in (28) we take $\mathbf{y}'=\mathbf{y}$ and the integration $\int d^3y$ is performed, and, furthermore, only if the constants $F_{\mu\nu\kappa}$ are symmetric in μ and ν. It is easy to see that then ν can only be zero.

It follows, then, that the operator O is a multiple of the operator

$$Q=\int d^3y\,\rho_0(\mathbf{y},\mathbf{y},t)\,, \tag{29}$$

which we define to be the total charge of the system. Therefore $\rho_\mu(\mathbf{x},\mathbf{x}')$ for $\mu=0$, 1, 2, 3 form an irreducible set of operators at a given time, when acting on a space of states all having the same total charge. From (25) and (29) we see that we can define spin density

$$\Sigma_i(\mathbf{x},t)=\tfrac{1}{2}\rho_i(\mathbf{x},\mathbf{x},t)$$

and charge density

$$\rho(\mathbf{x},t)=\rho_0(\mathbf{x},\mathbf{x},t)\,,$$

and that, because of (22), (24a), and (24b), we have

$$[Q,S_i]=0\,,$$

as it should be.

We could write down the rest of the operators of physical interest in terms of ρ_μ's. This is hardly necessary, since how this can be done is illustrated above and in the spinless case. Instead, we shall now turn to the Thirring model and try to solve it in terms of nonlocal "currents."

Finally, let us note that the equal-time algebra (22) can be realized with the two-component field operators $\psi_i(\mathbf{x},t)$ and $\psi_i^*(\mathbf{x},t)$ satisfying the usual anticommutation or commutation relations. The variable $\rho_\mu(\mathbf{x},\mathbf{x}',t)$ is given as

$$\rho_\mu(\mathbf{x},\mathbf{x}',t)=\psi^*(\mathbf{x},t)\sigma_\mu\psi(\mathbf{x}',t)\,.$$

3. THIRRING MODEL

A. Solution in Terms of Nonlocal "Currents"

As in well known, the Thirring model[5] is a relativistic model of self-interacting fermions in two-dimensional space-time. It is certainly a good idea to try to solve it in terms of nonlocal variables, since, because of self-interaction, we need only one type of them, those for a Fermi system. In our formulation of the model it will not be immediately apparent that we are dealing with the Thirring model. However, in Sec. 3B, we shall show the connection with the usual formulation in terms of fields.

First, let us define the nonlocal "currents" as dynamical variables. The variables denoted as $\tilde{\rho}_\mu(x,x',t)$ satisfy the following equal-time algebra:

$$[\tilde{\rho}_\mu(x,x',t),\tilde{\rho}_\nu(y,y',t)]=F_{\mu\nu\kappa}\delta(x'-y)\tilde{\rho}_\kappa(x,y',t)$$
$$-F_{\nu\mu\kappa}\delta(x-y')\tilde{\rho}_\kappa(y,x',t)\,, \tag{30}$$

where $F_{\mu\nu\kappa}$ are defined by the relation (23) and are explicitly given by (24a) and (24b). The set of nonlocal coordinates in (30) is the same as the set for a nonrelativistic system of n identical spin-$\frac{1}{2}$ particles [Eq. (22)], except that it depends nonlocally on one-dimensional space. It is quite clear that the set of coordinates $\tilde{\rho}_\mu$, which are the elements of the equal-time algebra (30), are irreducible when acting on states all having the same total charge. Obviously the total charge is defined as

$$Q_0=\int\tilde{\rho}_0(x,x,t)dx\,. \tag{31}$$

Of course, Q_0 is a constant of the motion. We define the free part of the Hamiltonian for the Thirring model (with mass equal to zero) as follows:

$$H_0=\lim_{f(\mathbf{x},\mathbf{x}')\to\delta(\mathbf{x}-\mathbf{x}')}\frac{1}{2i}\int dxdx'f(x,x')$$
$$\times\left(\frac{\partial}{\partial x'}\tilde{\rho}_3(x,x',t)-\frac{\partial}{\partial x}\tilde{\rho}_3(x,x',t)\right). \tag{32}$$

It is not difficult to see that $[Q_0,H_0]=0$. However, we know that the Thirring model has another constant of the motion, which corresponds—in the field-theoretic formulation—to the conservation of the axial-vector current. It is not difficult to see, however, that $[Q_3,H_0]=0$, where

$$Q_3=\int \tilde{\rho}_3(x,x,t)dx. \qquad (33)$$

Therefore, we have found two constants of the motion, Q_0 and Q_3. It is convenient to define

$$Q_{1,2}=\tfrac{1}{2}(Q_0\pm Q_3), \qquad (34)$$

where Q_1 and Q_2 are again constants of the motion. For that reason, we shall define a new set of coordinates:

$$\rho_{1,2}(x,x',t)=\tfrac{1}{2}[\tilde{\rho}_0(x,x',t)\pm\tilde{\rho}_3(x,x',t)]. \qquad (35)$$

With a little work, using (24a), (24b), and (30), it is not difficult to show that $\rho_{1,2}$ form the following algebra under commutation:

$$\begin{aligned}[][\rho_1(x,x',t),\rho_1(y,y',t)]&=\delta(x'-y)\rho_1(x,y',t)-\delta(x-y')\rho_1(y,x',t),\\ [\rho_1(x,x',t),\rho_2(y,y',t)]&=0,\\ [\rho_2(x,x',t),\rho_2(y,y',t)]&=\delta(x'-y)\rho_2(x,y',t)-\delta(x-y')\rho_2(y,x',t),\end{aligned} \qquad (36)$$

with

$$\rho_{1,2}^*(x,x',t)=\rho_{1,2}(x',x,t).$$

In terms of ρ_1 and ρ_2, the quantities Q_1 and Q_2 are given as follows:

$$Q_{1,2}=\int dx\,\rho_{1,2}(x,x,t). \qquad (34')$$

It is easy to show now that the set of nonlocal "currents" ρ_1 and ρ_2 is a satisfactory set of coordinates, since any operator O that commutes with $\rho_1(x,x',t)$ and $\rho_2(y,y',t)$ for all x, x' and y, y' at a given time is a function of the charges Q_1 and Q_2. This can be shown, as in the case of the theory of a nonrelativistic system of n identical spinless particles, by demanding that the operator O be invariant under the similarity transformations

$$U(\lambda_1,t)=\exp\left[i\int \lambda_1(x,x')\rho_1(x,x',t)dxdx'\right], \qquad (37a)$$

$$U(\lambda_2,t)=\exp\left[i\int \lambda_2(x,x')\rho_2(x,x',t)dxdx'\right], \qquad (37b)$$

where λ_1 and λ_2 are two arbitrary functions of x and x'.

The reason that the number of variables has decreased from four $\tilde{\rho}_\mu$'s to two (ρ_1 and ρ_2) is that the number of constants of the motion has increased from one to two. It is clear that this phenomenon is not restricted to two-dimensional theories.

In our further discussion, we shall meet the quantities

$$[\rho_{1,2}(x,x,t),\rho_{1,2}(y,y,t)]=iC_{1,2}(x,y,t). \qquad (38)$$

One can easily show that $C_{1,2}$ are 0 by employing the similarity transformations $U(\lambda_1,t)$ and $U(\lambda_2,t)$. For example,

$$\begin{aligned}&U(\lambda_1,t)[\rho_1(x,x,t),\rho_1(y,y,t)]U^{-1}(\lambda_1,t)\\ &=\delta(x-y)\Bigg[\int dx_1dx_2'\langle x_1|e^{i\lambda_1}|x\rangle\langle y|e^{-i\lambda_1}|x_2'\rangle\rho_1(x_1,x_2',t)\\ &\quad-\int dx_1'dx_2\langle x_2|e^{i\lambda_1}|y\rangle\langle x|e^{-i\lambda_1}|x_1'\rangle\\ &\quad\times\rho_1(x_2,x_1',t)\Bigg]=0,\end{aligned} \qquad (38')$$

where we have used the fact that $\rho(y,y',t)$ transforms in the same way as $\rho(\mathbf{y},\mathbf{y}',t)$ from Sec. 2 A [see Eq. (11)]. From (38') we conclude that C_1 must be zero. Similiarly, we conclude that C_2 must be zero. Defining the interaction part of the Hamiltonian as follows:

$$H_{\text{int}}=2g\int dx\,\rho_1(x,x,t)\rho_2(x,x,t),$$

and taking into account (35) and (32), the total Hamiltonian then can be written as

$$\begin{aligned}H=&\lim_{f(\mathbf{x},\mathbf{x}')\to\delta(\mathbf{x}-\mathbf{x}')}\frac{1}{2i}\int dxdx'f(x,x')\left\{\frac{\partial}{\partial x'}[\rho_1(x,x',t)\right.\\ &\left.-\rho_2(x,x',t)]+\frac{\partial}{\partial x}[\rho_2(x,x',t)-\rho_1(x,x',t)]\right\}\\ &+2g\int dx\,\rho_1(x,x,t)\rho_2(x,x,t).\end{aligned} \qquad (39)$$

Having the total Hamiltonian, we can write down the equations of motion:

$$\begin{aligned}&-i\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial x'}\right)\rho_1(x,x',t)\\ &\qquad=2g[\rho_2(x,x,t)-\rho_2(x',x',t)]\rho_1(x,x',t),\\ &-i\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)\rho_2(x,x',t)\\ &\qquad=2g[\rho_1(x,x,t)-\rho_1(x',x',t)]\rho_2(x,x',t).\end{aligned} \qquad (40)$$

We would like to show now that the dynamical coordinates ρ_1 and ρ_2, that depend locally on space variables, are the same as the interaction-free dynamical coordinates for $x'=x$. To show this, we introduce a new set of variables: t, $(x+x')/2$, and $(x-x')/2$; then the equations of motion (40) become

$$-i\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial(x+x')/2}\right)\rho_1(x,x',t)$$
$$=2g[\rho_2(x,x,t)-\rho_2(x',x',t)]\rho_1(x,x',t)\,, \tag{41}$$
$$-i\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial(x+x')/2}\right)\rho_2(x,x',t)$$
$$=2g[\rho_1(x,x,t)-\rho_1(x',x',t)]\rho_2(x,x',t)\,.$$

Setting $x'=x$ in (41), we have

$$-i\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\rho_1(x,x,t)=0\,,$$
$$-i\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\rho_2(x,x,t)=0\,. \tag{42}$$

The interaction-free dynamical coordinates obviously satisfy

$$-i\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial x'}\right)\rho_1{}^0(x,x',t)=0\,,$$
$$-i\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)\rho_2{}^0(x,x',t)=0\,. \tag{43}$$

Therefore, we can set

$$\rho_1(x,x,t)=\rho_1{}^0(x,x,t)\,,$$
$$\rho_2(x,x,t)=\rho_2{}^0(x,x,t)\,. \tag{43$'$}$$

Moreover, from (42) and (43) it follows that

$$\rho_1{}^0(x,x,t)=\rho_1{}^0(t-x)\,,$$
$$\rho_2{}^0(x,x,t)=\rho_2{}^0(x+t)\,. \tag{44}$$

Equation (40) can now be rewritten, taking into account the second relation in (36) and (43), as follows:

$$-i\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial u'}\right)\rho_1(u,v;\,u',v')$$
$$=g[\rho_2{}^0(u)\rho_1(u,v;\,u',v')-\rho_1(u,v;\,u',v')\rho_2{}^0(u')]\,, \tag{45}$$
$$-i\left(\frac{\partial}{\partial v}+\frac{\partial}{\partial v'}\right)\rho_2(u,v;\,u',v')$$
$$=g[\rho_1{}^0(v)\rho_2(u,v;\,u',v')-\rho_2(u,v;\,u',v')\rho_1{}^0(v')]\,,$$

where of the four variables $u=x+t$, $v=t-x$, $u'=x'+t$, and $v'=t-x'$, only three are independent, since

$$u+v=u'+v'.$$

The general solution of (45) is (see Appendix)

$$\rho_1(u,v;\,u',v')=\exp\left[ig\int_{-\infty}^{u}\rho_2{}^0(u_1)du_1\right]$$
$$\times\exp\left[-ig\int_{-\infty}^{u'}\rho_2{}^0(u_1)du_1\right]\rho_1{}^0(v,v')\,, \tag{46a}$$

$$\rho_2(u,v;\,u',v')=\exp\left[ig\int_{-\infty}^{v}\rho_1{}^0(v_1)dv_1\right]$$
$$\times\exp\left[-ig\int_{-\infty}^{v'}\rho_1{}^0(v_1)dv_1\right]\rho_2{}^0(u,u')\,, \tag{46b}$$

where $\rho_1{}^0(v,v')$ and $\rho_2{}^0(u,u')$, the interaction-free solutions of (45) ($t\to-\infty$; u, v, u', $v'\to-\infty$), are two arbitrary functions of v, v' and of u, u', respectively [this follows from (43)]. The solutions $\rho_{1,2}$ go asymptotically into $\rho_{1,2}{}^0$ as $t\to-\infty$.

Using the fact that C_1 and C_2 are zero [see (A5) and the discussion in the Appendix], one can easily show that the solutions (46a) and (46b) satisfy the equal-time algebra (36), provided that the interaction-free solutions

$$\rho_1{}^0(v,v')=\rho_1{}^0(t-x,\,t-x')=\rho_1{}^0(x,x',t)$$

and

$$\rho_2{}^0(u,u')=\rho_2{}^0(x+t,\,x'+t)=\rho_2{}^0(x,x',t)$$

satisfy it.

From the solutions (46) we conclude that the physical result of interest is very simple. Taking the limit $t\to\infty$ ($u,u',v,v'\to\infty$) in (46), we have

$$\rho_1{}^{\text{out}}(v,v')=\lim_{t\to\infty}\rho_1(u,v;\,u',v')=\rho_1{}^0(v,v')\,,$$
$$\rho_2{}^{\text{out}}(u,u')=\lim_{t\to\infty}\rho_2(u,v;\,u',v')=\rho_2{}^0(u,u')\,. \tag{47}$$

From (47) it follows that the S matrix is simply unity, because of the completeness of the operators ρ_1 and ρ_2. Therefore no process is possible in the Thirring model, as is already well known. Now, since all the theories are equivalent as long as they give the same S matrix, we can conclude from (47) that the solution of the Thirring model in terms of fields can be written in its simplest form as $\psi=\phi^0$ (ϕ^0 being an interaction-free solution at $t\to-\infty$).

Finally, let us note that by similar methods one could obtain the solutions for ρ_1 and ρ_2 for the case of the interaction of a zero-mass Dirac field with an external field,[8] which, however, does not have an S matrix equal to 1.

[8] J. Šoln, Nuovo Cimento 18, 914 (1960).

B. Connection with Thirring Model Formulated in Terms of Field Operators

The equation of motion for the field operator ψ is

$$-i\gamma^\mu\partial_\mu\psi(x,t)=g\bar{\psi}\gamma_\mu\psi\gamma^\mu\psi\,, \tag{48}$$

where

$$\{\gamma^\mu,\gamma^\nu\}=-2g^{\mu\nu}\,,\quad \bar{\psi}=\psi^*\gamma^2\,,\quad g^{11}=-g^{22}=1\,,$$

and

$$g^{12}=g^{21}=0\,.$$

According to Thirring,[5] it is convenient to take the following representation of the Dirac matrices:

$$\gamma^1=i\sigma_1\,,\quad \gamma^2=\sigma_2\,,\quad \gamma^5=\gamma^2\gamma^1=\sigma_3\,, \tag{49}$$

where the σ_r are the Pauli matrices. The field operators satisfy the following equal-time anticommutation relation:

$$\begin{aligned}\{\psi_\alpha^*(x,t),\psi_\beta(x',t)\}&=\delta_{\alpha\beta}\delta(x-x')\,,\\ \{\psi_\alpha(x',t)\psi_\beta(x,t)\}&=0\,,\end{aligned} \tag{50}$$

where α and β represent the spinor indices.

From (48) it is easy to see that the vector current is conserved: $\partial_\mu\bar{\psi}\gamma^\mu\psi=0$ (which is true regardless of whether m is equal to or different from zero). Consequently, the total charge $Q_0=\int dx\,\psi^*\psi$ is a constant of motion. It is readily seen that the equal-time algebra (30) satisfied by $\tilde{\rho}_\mu(x,x',t)$ can be realized if we define

$$\tilde{\rho}_\mu(x,x',t)=\psi^*(x,t)\sigma_\mu\psi(x',t) \tag{51}$$

($\sigma_0=1$, σ_r are Pauli matrices), taking into account the anticommutation relations (50).

The Hamiltonian that yields the equation of motion (48) is

$$H=\frac{1}{2i}\int dx\left[\bar{\psi}\gamma^1\frac{\partial}{\partial x}\psi-\left(\frac{\partial}{\partial x}\bar{\psi}\right)\gamma^1\psi\right]-\tfrac{1}{2}g\int dx\; j_\mu j^\mu\,. \tag{52}$$

The free part of H in (52) can be linked to (32) with the help of (49) and (51), since $\gamma^2\gamma^1=\sigma_3$.

However, the mass of the fermion is zero for the Thirring model. For this reason, the axial-vector current is also conserved: $\partial_\mu\bar{\psi}\gamma^\mu\gamma_5\psi=0$. Equivalently, we have another constant of motion

$$Q_3=\int\psi^*\sigma_3\psi dx\,,$$

since $\gamma_5=\sigma_3$.

Defining $Q_{1,2}=\frac{1}{2}(Q_0\pm Q_3)$, we have

$$Q_{1,2}=\int dx\,\psi_{1,2}{}^*\psi_{1,2}\,,$$

because of (49). Therefore a new complete set of coordinates ρ_1 and ρ_2 can be realized:

$$\begin{aligned}\rho_1(x,x',t)&=\psi_1^*(x,t)\psi_1(x',t)\,,\\ \rho_2(x,x',t)&=\psi_2^*(x,t)\psi_2(x',t)\,.\end{aligned} \tag{53}$$

One can easily see that the algebra (36) is satisfied.

With the help of (49), we can write

$$j^1=\psi_1^*\psi_1-\psi_2^*\psi_2\,,\quad j^2=\psi_1^*\psi_1+\psi_2^*\psi_2\,. \tag{54}$$

Therefore Eq. (52) can be rewritten as

$$\begin{aligned}H=\frac{1}{2i}\int dx\,&\left\{\left[\psi_1^*\frac{\partial}{\partial x}\psi_1-\psi_2^*\frac{\partial}{\partial x}\psi_2\right]\right.\\ &\left.+\left[\left(\frac{\partial}{\partial x}\psi_2^*\right)\psi_2-\left(\frac{\partial}{\partial x}\psi_1^*\right)\psi_1\right]\right\}\\ &+2g\int dx\,\psi_1^*\psi_1\psi_2^*\psi_2\,.\end{aligned} \tag{55}$$

The form (55) for H expressed in terms of ψ's can be linked to the form (39) in terms of ρ's.

The equations of motion for $\psi_{1,2}$ can be derived from (48) with the help of (54) or directly, using the Hamiltonian (55):

$$\begin{aligned}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\psi_1(x,t)+2ig\rho_2{}^0(x,x,t)\psi_1(x,t)&=0\,,\\ \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)\psi_2(x,t)+2ig\rho_1{}^0(x,x,t)\psi_2(x,t)&=0\,,\end{aligned} \tag{56}$$

where we have denoted

$$\begin{aligned}\rho_1{}^0(x,x,t)&=\psi_1^*(x,t)\psi_1(x,t)=\phi_1{}^{0*}(t-x)\phi_1{}^0(t-x)\,,\\ \rho_2{}^0(x,x,t)&=\psi_2^*(x,t)\psi_2(x,t)=\phi_2{}^{0*}(x+t)\phi_2{}^0(x+t)\end{aligned} \tag{57}$$

[see (43′) and (44), which are naturally valid here too]. $\phi_{1,2}{}^0$ are interaction-free solutions.

The solutions for $\psi_{1,2}$ can be obtained by the technique presented in the Appendix. Again, from the fact that C_1 and C_2 are equal to zero [see the Appendix and particularly (A5)], we obtain the following solutions:

$$\begin{aligned}\psi_1(u,v)&=\exp\left(-ig\int_{-\infty}^{u}\rho_2{}^0(u_1)du_1\right)\phi_1{}^0(v)\,,\\ \psi_2(u,v)&=\exp\left(-ig\int_{-\infty}^{v}\rho_1{}^0(v_1)dv_1\right)\phi_2{}^0(u)\,.\end{aligned} \tag{58}$$

The solutions (58) are those of Glaser[9] (with $g\to-g$).

It is possible to conclude from (58) that the S matrix is equal to unity (see Ref. 9). However, this result follows unambiguously from the solutions (46) for ρ_1 and ρ_2.

4. DISCUSSION

We have shown in the last three sections how both nonrelativistic and relativistic theories can be written in terms of nonlocal "currents," actually nonlocal charge

[9] V. Glaser, Nuovo Cimento 9, 940 (1958).

and spin densities. Although we have formulated and solved a two-dimensional relativistic Thirring model, it is quite clear that one can introduce the nonlocal "currents" as dynamical variables also for the four-dimensional relativistic Fermi system. In this case, under the assumption that the charge is conserved, we would have 16 variables altogether. There is one quite important thing (which turns out to be generally true) that we have learned from the Thirring model: The number of nonlocal dynamical variables necessary to describe the system decreases if the number of constants of the motion increases.

We have left open the problem of how to construct the representations of the "current" algebras (7), (22), or (36). We do not feel that this should pose a particular problem (Sharp showed how this problem can be treated for cases of local currents in Ref. 2). Also, it should be possible to extend the present formulation to one which would include the interaction of charged particles with a quantized electromagnetic field, without particular difficulty. How this can be done for a system of charged mesons interacting with photons, which is described in terms of local currents, was shown by Sharp in Ref. 2. The treatment of these problems is beyond the scope of this paper.

Although the local currents have counterparts in classical observables, which the nonlocal ones do not have, the formulation of the theory in terms of the nonlocal currents shows a certain advantage over the formulation in terms of local currents. If one tries to construct, for example, H_0 (the free part of the Hamiltonian) in terms of local currents, looking at H_0 expressed in terms of field operators, one runs into singular expressions like (6). As we saw, H_0 expressed in terms of nonlocal currents does not give this difficulty. It is true, however, that one can achieve nonsingular expressions for H_0 or $\theta_{\mu\nu}$ in terms of local currents simply by expressing them as polynomials in the currents and by demanding Lorentz covariance (see Refs. 3 and 10). However, in this case, we have model theories rather than theories which are supposed to deal with the problem *a priori*.

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[10] H. Sugawara, Phys. Rev. **170**, 1659 (1968).

APPENDIX

To illustrate the technique of solving the differential equations for operators, we take the first of the equations in (45):

$$-i\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial u'}\right)\rho_1(u,v;\,u',v')=g[\rho_2{}^0(u)\rho_1(u,v;\,u',v')-\rho_1(u,v;\,u',v')\rho_2{}^0(u')].\quad\text{(A1)}$$

Since $\rho_1{}^*(x,x',t)=\rho_1(x',x,t)$, we can set

$$\rho_1(u,v;\,u',v')=U_1(u)\rho_1{}^0(v,v')U_1{}^*(u'),\qquad\text{(A2)}$$

where $\rho_1{}^0(v,v')$ is the interaction-free solution of (A1) at $t\rightarrow-\infty$. Equation (A1) is going to be satisfied if we put

$$i\frac{\partial}{\partial u}U_1(u)=-g\rho_2{}^0(u)U_1(u).\qquad\text{(A3)}$$

Note that $U_1\rightarrow 1$ as $t\rightarrow-\infty$.

In what follows, we shall need the quantities

$$[\rho_1{}^0(v),\rho_1{}^0(v')]=iC_1(v,v'),\qquad\text{(A4a)}$$

$$[\rho_2{}^0(u),\rho_2{}^0(u')]=iC_2(u,u'),\qquad\text{(A4b)}$$

where $\rho_1{}^0(v)=\rho_1{}^0(t-x)=\rho_1{}^0(x,x,t)$ and $\rho_2{}^0(u)=\rho_2{}^0(x+t)=\rho_2{}^0(x,x,t)$ [see (43′) and (44)]. We claim that $C_1(v,v')$ and $C_2(u,u')$ are equal to zero regardless of whether the times are equal or not in v, v' and u, u', respectively. Let us show this, for example, for (A4a). From (38), (38′), and the discussion below (38) we have

$$[\rho_1{}^0(t-x),\,\rho_1{}^0(t-x')]=0.$$

Making the translation $x'\rightarrow x'+\Delta$ and defining $t'=t-\Delta$, we have

$$[\rho_1{}^0(t-x),\,\rho_1{}^0(t'-x')]=0.$$

Therefore

$$C_1(v,v')=0,\quad C_2(u,u')=0,\qquad\text{(A5)}$$

regardless of whether the times are equal or different in v, v' and u, u', respectively.

The solution of (A3) is then simply given as

$$U_1(u)=\exp\left(ig\int_{-\infty}^{u}\rho_2{}^0(u_1)du_1\right).\qquad\text{(A6)}$$

Equation (A6) together with (A2) gives the solution (45) for ρ_1. Similarly, one derives the solution for ρ_2.