$K^- p \rightarrow \Lambda\eta$ at the upper vertex is rather smaller than the $K^- N \rightarrow \Lambda\pi$ amplitude, say, by 4 to 10 times. Together, the η-π interference term is at most $\approx 10\%$ of the OPE cross section, Eqs. (2) and (4a).[13]

Another interfering process might be the nucleon-exchange graph, Fig. 1(d). (The corresponding graph for $pp \rightarrow d\pi^+$ has been discussed by Heinz[4] and by Mathews and Deo.[3]) However, because the deuteron vertex here involves a large relative momentum $[(p\text{-}n)^2 \approx 1.2\ (\text{Gev}/c)^2]$, this contribution is expected to be quite small. Moreover, because of its peripheral nature, its contribution to the energy behavior of the cross section would be smooth, with the Λ tending to go forward. Thus this graph, if it contributes at all, would only affect the OPE predictions.

We thank Dr. H. A. Thiessen and Dr. J. E. Young for discussions and Professor H. L. Anderson for permission to refer to his preliminary experimental results.

[13] These considerations also apply to η exchange in $pp \rightarrow d\pi^+$.

PHYSICAL REVIEW VOLUME 171, NUMBER 5 25 JULY 1968

Approximate Bounds on Coupling Constants from Analyticity, Crossing, and Unitarity

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(Received 26 February 1968)

If the exchange forces in a unitary scattering amplitude are too large, there is a breakdown of analyticity due to the appearance of ghosts. As an illustration of this, the condition that the singlet amplitudes of nucleon-nucleon scattering have no ghosts is formulated as a simple eigenvalue problem for the bound on the pion-nucleon coupling constant, in the approximation of neglecting all but one-pion-exchange terms. It is found that the pion-nucleon coupling constant g could not be more than $2\frac{1}{2}$ times its actual value without producing a ghost. It is suggested that the most stringent bounds should come from a consideration of the Pomeranchuk pole. It is further speculated that the electromagnetic and weak interactions may eventually be bounded in a similar way.

1. INTRODUCTION

In relativistic scattering theory, it is supposed that the interaction between two particles is caused by the exchange of particles in the crossed channels, and that the magnitude of a phase shift is thus determined, in a nonlinear manner, by the magnitudes of the coupling constants of the exchanged particles. At first sight, this requirement of crossing symmetry would not seem to restrict the magnitudes of these coupling constants, which could, apparently, be arbitrarily large. However, this is not so, since the scattering amplitudes must be unitary, and analytic, except for one-particle poles and multiparticle branch points. As is well known, these requirements are not consistent with arbitrarily large coupling constants: A large coupling leads to the appearance of ghosts. In this paper, the term "ghost" will be used to signify the appearance of any pole of the scattering amplitude that is disallowed, on physical grounds. A pole may be disallowed because, although it corresponds to a real mass, its residue has the wrong sign to correspond to a real coupling. Alternatively, a pole may have to be rejected because it does not correspond to a real mass; for example, if it lies on the negative real axis in the square of the total energy. In this case, the sign of the residue is immaterial; a pole anywhere on the physical sheet, except on the positive real axis below the normal threshold, constitutes a breakdown of analyticity. It is the exclusion of this type of ghost that is the concern of the present paper, and the object is to examine the consequent limitations on the magnitudes of coupling constants. As a simple example, restrictions on the size of the pion-nucleon coupling constant will be derived from a consideration of nucleon-nucleon scattering.

If the crossed channels are the same as the direct channel, as in pion-pion scattering, then the coupling constants of the exchanged particles are not only limited by the requirement of no ghosts, but are, in principle, determined by the condition of consistency with the direct channel (the bootstrap principle). Indeed, the considerations of the above paragraph add nothing to the calculation of an amplitude that satisfies analyticity, crossing, and unitarity. Such a hypothetical calculation, whether or not it determined the couplings uniquely, would certainly limit them in magnitude. Martin[1] has shown, using analyticity, crossing, and only the unitarity inequality, that the pion-pion amplitude satisfies a

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[1] A. Martin, in *Proceedings of the Seminar on High-Energy Physics, Trieste, 1965* (International Atomic Energy Agency, Vienna, 1965), p. 155.

certain bound. If one thinks in terms of a $\lambda\varphi^4$ Lagrangian field theory, then the coupling constant λ is related to the scattering amplitude at the total symmetry point, and Martin's result is tantamount to a restriction on the size of λ. While Martin's numerical values are rigorous upper bounds, they are disappointingly rather large. What is offered in this paper are estimates of upper bounds that are only approximate. That is, although upper bounds certainly exist, the numbers given in this paper are not definitely greater than these bounds, they are only estimates of them. The reward for this semi-phenomenological approach is that the bounds appear not to be very large: The smallest estimate is $2\frac{1}{2}$ times the actual pion-nucleon coupling constant. (A much more stringent bound would obtain if one were prepared to exclude ghosts also in the odd-signature, $J=0$, $I=0$, singlet amplitude, which is, however, unphysical; see below). Another advantage of the present method is its extreme simplicity, for the problem reduces to a very straightforward eigenvalue calculation, which is considerably easier than a bootstrap.

The calculation will be performed in the N/D representation of a partial-wave amplitude. Crossing symmetry is approximated by inserting, as the discontinuity of the partial-wave amplitude across its left-hand cut, the imaginary part of the partial-wave projection of the one-particle-exchange terms in the crossed channels. In the nucleon-nucleon example, a first approximation consists in retaining only the one-pion-exchange term, so that the force depends only on the square of the pion-nucleon coupling constant (so that obviously all the bounds discussed below refer to the absolute value of the pion-nucleon coupling constant). Elastic unitarity will be assumed; it will be shown in the next section that an inelasticity factor would produce an even more stringent bound on the allowed range of coupling. Thus, the bounds obtained are rigorous, within the one-pion approximation for the force. The techniques here employed could easily be used with more complicated exchange diagrams, in particular the exchange of other bosons (ρ, ω, φ, etc.), but then one would only obtain a bound for a quadratic function of all the coupling constants of these bosons to the nucleon.

The calculation has been done for the singlet amplitudes of nucleon-nucleon scattering. This example is particularly instructive, since one pion exchange is attractive in the $I=1$ state, repulsive for $I=0$, where I is the isospin. The reader should perhaps be cautioned that the terms "attractive" and "repulsive" have rather ill-defined meanings in the context of an N/D calculation, in contrast to their meanings in potential theory. It is not simply a matter of the sign of the left-hand cut discontinuity, since this is not definite for higher partial waves. Nevertheless, in the absence of bound states, the $I=1$ phase shifts are predominantly positive, the $I=0$ phase shifts negative.

In the $I=1$, $J=0$, singlet state, generally designated ${}^1S_0$, a bound-state pole is produced if the pion-nucleon coupling constant is large enough. The condition that the bound state be just produced at the nucleon-nucleon threshold, with zero binding energy, can be expressed as a homogeneous system of N/D equations, in which the D function is subtracted at threshold. The corresponding value of the pion-nucleon coupling constant is then the smallest eigenvalue of the Fredholm kernel of this equation. This is explained in detail in the next section. As the pion-nucleon coupling is increased beyond this value, the state becomes bound more deeply, until, at a critical value of the coupling, the binding energy equals the rest mass of two nucleons; and, for larger values of the coupling, the "bound state" has an imaginary mass. This is an unacceptable situation; a pole in a partial-wave amplitude at negative energy-squared cannot be reinterpreted as a particle in a crossed channel, because such a particle's exchange contribution, when projected in partial waves in the direct channel, gives a cut, not a pole. In short, there is no physical interpretation for an "overbound state," and it must be rejected. The limiting condition for the occurrence of a zero-mass bound state can again be expressed as a homogeneous N/D equation, this time with subtraction at zero energy, as is shown in Sec. 2. Again, the problem is to locate the smallest eigenvalue of the kernel. For even larger values of the coupling, the pole moves to the left, becoming a ghost, i.e., a "state" with the square of the mass negative.

Although this procedure does yield an upper bound to the pion-nucleon coupling constant, one should not expect it to be an interesting one. For one thing, a more stringent bound would be implied by the requirement of no bound state at all (as is known to be the case experimentally), even though this would be a phenomenological rather than a fundamental bound. Moreover, since the occurrence of an overbound ghost requires the build-up of a very large binding energy (two nucleon masses), one should expect the corresponding bound on the coupling constant to be exceedingly large. This is indeed the case (see below).[2]

A more interesting situation prevails in the $I=0$ states, in which the one-pion-exchange force is "repulsive." Here, it is expected that, for a critical value of the coupling, a pole will be produced at $s=-\infty$, where s is the energy squared. As is shown in the next section, this critical value corresponds to the smallest eigenvalue of a homogeneous, unsubtracted N/D kernel,

[2] If one were to take the calculation of the condition for an overbound state seriously, one should of course consider the coupled-triplet amplitudes, in which the attraction is most effective, since this is the only case in which there actually is a bound state, the deuteron. However, the calculation would be appreciably more complicated, and it would again produce a poor bound on the pion-nucleon coupling constant, namely, the value required to raise the magnitude of the deuteron's binding energy to 2 GeV. In fact, the well-known virtual state occurs in the ${}^1S_0$ state that is considered in this paper.

and in fact it yields a fairly stringent bound on the pion-nucleon coupling constant. However, it might be objected that, in practice, one would not treat seriously a ghost at $s=-\infty$, or even at some large, finite value of $-s$, especially in an approximate calculation. Indeed, it is regular practice[3] in numerical calculations to insert spurious poles far from the physical region, in order to ensure correct threshold behavior. This is indeed true; but it must be emphasized again that the present object is only to obtain approximate bounds on the pion-nucleon coupling. Moreover, for larger values of the coupling [and actually not much larger values (see below)], the ghost will move to the right, eventually reaching $s=0$ from the left. This somewhat larger value of the coupling, corresponding to an N/D system subtracted at $s=0$, can also be calculated, although of course the scattering amplitude would long since have been unacceptable because of the ghost. For even larger values of the coupling, one would expect the pole to move to the right of $s=0$, but with a residue of the wrong sign for a bound state.

One difficulty in the $I=0$ state is that, because of Fermi statistics, there is no singlet S wave. It may still be of some interest to consider the "unnatural" J-parity S wave, in which the t and u channels combine constructively instead of destructively; and the results for this wave are collated in the next section. The corresponding bound on the coupling constant g is remarkably stringent, namely, 1.4 times the experimental value. However, one cannot strictly require the absence of ghosts, since this is not a physical partial-wave amplitude, but only the analytic continuation in J of the odd partial waves. Accordingly, for a physical case one may consider the wave $I=0$, $J=1$, a state that is usually designated ${}^1P_1$. A problem in dealing with a P wave, or higher waves, is that the threshold condition must be observed. It is true that this condition can always be satisfied by considering a solution of the N/D equations of a sufficiently high Castillejo-Dalitz-Dyson (CDD) class.[4] On the other hand, if it is supposed that there are no CDD poles, then the left-hand-cut discontinuity must satisfy suitable moment conditions. It is known that the one-particle-exchange terms do not satisfy these conditions, but the point of view adopted here is that the one-pion-exchange term is an acceptable approximation to the exact discontinuity, which itself would yield the correct threshold behavior. Thus, it is argued that no special provision need be made to observe the threshold behavior, since one is interested (in particular in the repulsive P wave) in the occurrence of a ghost and not in the details of the low-energy phase shift. Once again, the reader is reminded that there certainly is an (unknown) rigorous upper bound on the pion-nucleon coupling; the results of this paper are merely approximate estimates of this bound.

TABLE I. Critical values of the square of the πN coupling constant, $g^2/(4\pi)$ (experimental value about 14).

		Position of pole in s plane			
I	J	$-\infty$	0	$4m^2$	Remarks
0	0	26	33	⋯	Unphysical, repulsive
0	1	92	180	10^4	Physical, mainly repulsive
1	0	⋯	10^4	110	Physical, attractive
1	1	2.10^3	2.10^3	210	Unphysical, mainly attractive

It is found that the value[5] of $g^2/(4\pi)$ that would be needed to produce a pole at $s=0$ in the physical ${}^1S_0$ wave is of the order of 10^4, compared to the experimental value of 14. On the other hand, if $g^2/(4\pi)=92$, a pole just appears at $s=-\infty$ in the physical ${}^1P_1$ wave, while a ghost appears in the unphysical $I=0$ S wave for $g^2/(4\pi)=26$. These numbers, and those for other cases, are listed in Table I. Thus, it can be said that the physical value of the pion-nucleon coupling constant is not such as to produce ghosts in the singlet amplitudes, but that in fact this coupling could not be very much larger than it is, without leading to a breakdown of analyticity, unitarity, and crossing.

It is not surprising that the actual pion-nucleon coupling constant falls short of these upper bounds. In the first place, they are only upper bounds, and there is no necessity that they be attained. Moreover, these results have been obtained by using elastic unitarity, and, as is indicated below, the inclusion of inelastic effects would make the bounds more stringent.

It should also be pointed out that there are many other partial waves, and many other reactions, and in none of these must there be a ghost. It is interesting to speculate upon which scattering amplitude would lead to the most stringent bound for the magnitude of strong interactions. Following Chew and Frautschi,[6] one would expect this to be the $J=1$, $I=0$, $B=0$, even-signature amplitude that contains the Pomeranchuk pole as a "bound state." Although this is an unphysical amplitude, the Froissart bound,[7] which is a consequence of analyticity, crossing, and unitarity, is equivalent to the requirement that the "Pomeranchuk state" have positive, or zero, mass squared. Moreover, if Chew's principle of maximal strength[6] is correct, and the Pomeranchuk actually has zero mass, as appears to be the case experimentally, then one might expect that the smallest eigenvalue of the relevant homogeneous N/D equation gives, for the exchange force, not an *in*equality but an equality. This equality, however, would not yield directly the value of one coupling constant, but rather the value of a combination of coupling constants,

[3] J. S. Ball, A. Scotti, and D. Y. Wong, Phys. Rev. **142**, 1000 (1966).

[4] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956); for a discussion of CDD classes, see also D. Atkinson and D. Morgan, Nuovo Cimento **41**, 559 (1966).

[5] Note that the values given here (and in Table I) are those of $g^2/(4\pi)$, whereas in the previous discussion the numbers referred to the pion-nucleon coupling constant g itself.

[6] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 384 (1961).

[7] M. Froissart, Phys. Rev. **123**, 1053 (1961).

with the $\pi\pi$ coupling constant presumably playing a prominent role.

In the same vein of speculation, it may be wondered whether similar considerations might not be applicable to the electromagnetic and perhaps even the weak interactions. The reason why the fine-structure constant has its actual value is a long-standing puzzle. In the renormalization of the electron charge, one has apparently given up the possibility of calculating the electromagnetic coupling, as the price of having removed, or at least suppressed, the ultraviolet divergences. It is fascinating to speculate, therefore, that, if the ultraviolet catastrophe could be averted in some more reasonable manner, perhaps one could obtain a bound on the electromagnetic coupling, and that, once the ultraviolet divergences were treated properly, this bound might have the order-of-magnitude characteristic of the electromagnetic rather than the strong interaction. However, at the moment one does not possess an unequivocal way of avoiding the ultraviolet problem.[8] In the same way, it may be wondered if the weak interaction, also, might be limited in strength by similar considerations. Since the weak interactions are not even renormalizable, one may wonder if perhaps this fact indicates that, in a future theory, the corresponding kernels will be much larger, with correspondingly smaller lowest eigenvalues, of, perhaps, the order of magnitude of the weak coupling constant. Of course, in these cases the bounds should come from the exclusion of ghosts due to "repulsive" interactions.

Finally, before proceeding with the mathematical details, one should point out that a number of papers have already appeared[9] that also address themselves to the problem of establishing upper bounds on the magnitudes of coupling constants. However, these papers are concerned with limitations on coupling constants due to analyticity and unitarity alone, with no crossing symmetry, and special assumptions have to be made, either concerning an "interaction range," or about the number of zeros on the left-hand-cut discontinuity. Thus, these methods, and their starting assumptions, are quite different from those of the present paper; as for the results, those that are sufficiently explicit to allow a comparison are less stringent than those of the present work.

2. CONDITIONS FOR THE ONSET OF GHOSTS IN THE NUCLEON-NUCLEON SYSTEM

Although the method is quite general, for definiteness the mathematical method will be illustrated by a consideration of nucleon-nucleon scattering. This can occur in states of isospin zero and one. For each isospin, there are five partial-wave amplitudes: the singlet, the uncoupled-triplet, and the three coupled-triplet amplitudes.[3,10] In this paper, as has been explained already, only the singlet amplitudes will be considered. The Pauli principle implies the vanishing of the even partial waves for $I=0$, and of the odd ones for $I=1$. Accordingly, attention is focused on the $I=1$, $J=0$, and the $I=0$, $J=1$ states. However, results are also given for the unphysical odd-signature $I=0$, $J=0$, and even-signature $I=1$, $J=1$, partial-wave amplitudes.

Let $f(s)$ be a typical partial-wave amplitude, normalized so that the unitarity relations reads

$$\mathrm{Im} f(s)=[(s-4m^2)/s]^{1/2}|f(s)|^2, \qquad (2.1)$$

where s is the square of the total energy in the center-of-mass system and m is the nucleon mass. This amplitude is susceptible to an N/D decomposition in the usual way:

$$f(s)=N(s)/D(s), \qquad (2.2)$$

where

$$N(s)=\frac{1}{\pi}\int_{-\infty}^{4m^2-\mu^2}\frac{ds'}{s'-s}\alpha(s')D(s') \qquad (2.3)$$

and

$$D(s)=1-\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'-s}\left(\frac{s'-4m^2}{s'}\right)^{1/2}N(s'). \qquad (2.4)$$

Here, μ is the mass of the pion, the least massive particle that can be exchanged in the crossed channels, and $\alpha(s)$ is the discontinuity of $f(s)$ on its left-hand cut. The inelasticity factor has been omitted from Eq. (2.1) or (2.4); its possible effects will be considered at the end of this section.

In the nucleon-nucleon problem, a good first approximation to the left-hand-cut discontinuity is provided by the one-pion-exchange terms in the t and u channels. For the S wave this contribution is[3]

$$\alpha(s)=\tfrac{1}{8}g^2\frac{\mu^2}{s-4m^2}U_I, \quad s\leqslant 4m^2-\mu^2, \qquad (2.5)$$

where U_I is an isospin crossing-matrix element, equal to $\frac{1}{2}$ for the (physical) $I=1$ state, and $-\frac{3}{2}$ for the (unphysical) $I=0$ state, and where g is the renormalized pion-nucleon coupling constant. In order to specify the normalization, the Lagrangian density is

$$\mathcal{L}=g\bar{\psi}\gamma_5\tau_k\psi\varphi_k. \qquad (2.6)$$

Here, ψ is the nucleon field and φ_k the pion field, k being the isospin label. Experimentally,

$$g^2/(4\pi)\approx 14. \qquad (2.7)$$

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[8] See however, in this connection, a forthcoming paper by D. Atkinson and F. Calogero.

[9] M. A. Ruderman and S. Gasiorowicz, Nuovo Cimento **8**, 861 (1958). V. N. Gribov, Ya. B. Zel'dovich, and A. M. Perelomov, Zh. Eksperim. i Teor. Fiz. **40**, 1190 (1961) [English transl.: Soviet Phys.—JETP **13**, 836 (1961)]; A. A. Ansel'm, V. N. Gribov, G. S. Danilov, I. T. Dyatlov, and V. M. Shekhter, Zh. Eksperim. i Teor. Fiz. **41**, 619 (1961) [English transl: Soviet Phys.—JETP **14**, 444 (1962)]; M. A. Ruderman, Phys. Rev. **127**, 312 (1962); Y. S. Jin and A. Martin, *ibid.* **135**, B1369 (1964); H. D. Dahmen and A. S. Reiner, *ibid.* **143**, 1295 (1966).

[10] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

For the P waves, the one-pion-exchange term is[3]

$$\alpha(s)=\tfrac{1}{8}g^2\frac{\mu^2}{s-4m^2}\left(1+\frac{2\mu^2}{s-4m^2}\right)U_I. \qquad (2.8)$$

This time the physical state corresponds to $I=0$, the unphysical to $I=1$.

The discontinuity for the physical $^1S_0$ wave is negative for all $s\leqslant 4m^2-\mu^2$ [Eq. (2.5), with $I=1$], and this corresponds to an attractive interaction. On the other hand, for the $^1P_1$ wave [Eq. (2.8), with $I=0$], $\alpha(s)$ is negative for $4m^2-2\mu^2<s<4m^2-\mu^2$, and positive for $s<4m^2-2\mu^2$. This apparently corresponds, in the imprecise normal usage, to a predominantly repulsive interaction, with a long-range attraction. This sign change is typical of a P wave, and for most purposes the "attraction" is without great effect. This is borne out by the experimental situation: The $^1P_1$ phase shift is small and negative, compared with a large, positive $^1S_0$ phase shift, at low energies.

The purpose is to show that the form of the N/D equations restricts the possible size of the πN coupling constant. Suppose that g were such as to produce a zero of the D function, for a given partial-wave amplitude, at a point $s=c\leqslant 4m^2$. Then Eq. (2.4) would imply

$$0=1-\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'-c}\left[\frac{s'-4m^2}{s'}\right]^{1/2}N(s'). \qquad (2.9)$$

Upon subtraction of Eq. (2.9) from Eq. (2.4), one finds

$$D(s)=-\frac{s-c}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'-s}\left[\frac{s'-4m^2}{s'}\right]^{1/2}\frac{N(s')}{s'-c}. \qquad (2.10)$$

Now, this can be substituted into Eq. (2.3) to give the following homogeneous equation for $N(s)$:

$$N(s)=\frac{1}{\pi^2}\int_{4m^2}^{\infty}ds'N(s')\left[\frac{s'-4m^2}{s'}\right]^{1/2}\frac{1}{s'-c}\times\int_{-\infty}^{4m^2-\mu^2}ds''\frac{\alpha(s'')(s''-c)}{(s''-s)(s''-s')}. \qquad (2.11)$$

With either of the expressions (2.5) and (2.8), this is a Fredholm equation.

Consider first the case of an attraction, for example, the physical $^1S_0$ state. The condition whereby a zero of D just appears at the normal threshold is that (2.11) have a solution for $c=4m^2$. Suppose that one writes g^2, in Eq. (2.5), as a factor in (2.11), and considers the kernel to be the rest of the expression, without this factor. Then it is clear that there will be no nontrivial solution of Eq. (2.11) if g^2 is less than the smallest eigenvalue of the kernel. That is, for g^2 smaller than this critical smallest eigenvalue, say g_B^2, there can be no zero of $D(s)$ at $s=4m^2$. When g^2 is increased to the value g_B^2, one has $D(4m^2)=0$; and as g^2 is increased beyond g_B^2, a zero of $D(s)$ moves to the left. It is easy to see that the residue of the corresponding pole in $f(s)$, for the physical state $I=1$, $J=0$, is of the correct sign for a bound state. Hence the condition for the absence of a bound state in the S wave is

$$g^2<g_B^2, \qquad (2.12)$$

where g_B^2 is the smallest eigenvalue of the kernel

$$U_I\frac{\mu^2}{8\pi^2}[s'(s'-4m^2)]^{-1/2}\int_{-\infty}^{4m^2-\mu^2}\frac{ds''}{(s''-s)(s''-s')}. \qquad (2.13)$$

If g^2 is increased still further, the bound state becomes ever more deeply bound, until eventually it has zero mass. This occurs when g^2 is equal to g_0^2, the smallest eigenvalue of the kernel of (2.11), corresponding to $c=0$. For still greater values of g^2, the bound state has "imaginary mass," and, as pointed out in the Introduction, it is rigorously excluded by the assumption of analyticity. Thus the requirement that the state should not be "overbound" is

$$g^2<g_0^2, \qquad (2.14)$$

where g_0^2 is the smallest eigenvalue of the kernel

$$U_I\frac{\mu^2}{8\pi^2}\left[\frac{s'-4m^2}{s'}\right]^{1/2}\frac{1}{s'}\times\int_{-\infty}^{4m^2-\mu^2}ds''\frac{s''}{(s''-4m^2)(s''-s)(s''-s')}. \qquad (2.15)$$

For the wave $^1P_1$, the force is predominantly repulsive; and, for $c=4\,m^2$, the corresponding eigenvalues of Eq. (2.11) would be expected to be negative. [This is not a rigorous statement, because $\alpha(s)$ does not have a constant sign in this case.] Thus one does not expect a positive value of g^2 for which $D(4m^2)=0$. This corresponds to the obvious physical fact that a bound state cannot be produced by a repulsion. What happens as g^2 is increased is that a zero of $D(s)$ appears at $s=-\infty$, and then moves to the right. This zero corresponds to a ghost, and the condition for its exclusion is

$$g^2<g_G^2, \qquad (2.16)$$

where g_G^2 is the smallest eigenvalue of the kernel of Eq. (2.11), with $c=-\infty$. Notice that, for large negative c, the sign of the kernel changes. Thus g_G^2 is the smallest eigenvalue of the kernel

$$U_I\frac{\mu^2}{8\pi^2}\left[\frac{s'-4m^2}{s'}\right]^{1/2}\int_{-\infty}^{4m^2-\mu^2}\frac{ds''[1+2\mu^2/(s''-4m^2)]}{(s''-4m^2)(s''-s)(s''-s')}. \qquad (2.17)$$

For a larger value of the coupling, namely, the smallest eigenvalue of the kernel

$$U_I\frac{\mu^2}{8\pi^2}\left[\frac{s'-4m^2}{s'}\right]^{1/2}\frac{1}{s'}$$

$$\times\int_{-\infty}^{4m^2-\mu^2}\frac{ds''[1+2\mu^2/(s''-4m^2)]}{(s''-4m^2)(s''-s)(s''-s')}, \quad (2.18)$$

the ghost will have moved from infinity to the point $s=0$.

The method of locating these smallest eigenvalues is extremely easy. By symmetrizing the kernel, and transforming the variables according to $y=4m^2/s$ and $y'=4m^2/s'$, Eq. (2.11) becomes

$$\psi(y)=\frac{g^2}{4\pi}\int_0^1 dy'H(y,y')\psi(y'), \quad (2.19)$$

where

$$\psi(y)=s^{3/4}(s-4m^2)^{1/4}(s-c)^{-1/2}N(s). \quad (2.20)$$

The expressions for the symmetric kernel $H(y,y')$ in the various cases (S and P waves, $c=-\infty$, 0, and $4m^2$) are given in the Appendix, in case the reader should care to check the numerical results.

The integral equation (2.19) was converted into a matrix equation by replacing the integral by a sum, and all the eigenvalues of the matrix were found by a standard computer-library subroutine. The largest eigenvalue of the matrix, corresponding to the smallest eigenvalue of the integral equation, was found to be very stable against increase in the order of the matrix, and a 30×30 matrix in most cases gave errors of only a few percent. The programs were also run for 50×50 matrices; this required about a minute of IBM-7040 computer time.

The results are displayed in Table I. They justify the discussion of the Introduction. The following additional remarks appear appropriate:

(i) The difference between the values of the coupling required to produce ghosts at $-\infty$ and at 0 is rather small in the completely repulsive $I=0$, $J=0$ case. It is also relatively small in the $I=0$, $J=1$ case, even though here the long-range attraction weakens the repulsion more effectively when the ghost occurs at 0 than when it occurs at $-\infty$.

(ii) The difference between the values of the coupling required to produce a bound state at threshold and an overbound state at $s=0$ is very large in the completely attractive $I=1$, $J=0$ case. This reflects the fact that an overbound state at $s=0$ has the enormous binding energy $2m$.

APPENDIX: SYMMETRIZED KERNELS IN THE VARIOUS CASES

Let

$$\Psi(y,y';A;B)=(y-y')^{-1}\left[\frac{A-yB}{1-y}\ln\left(1+\lambda\frac{1-y}{y}\right)-\frac{A-y'B}{1-y'}\ln\left(1+\lambda\frac{1-y'}{y'}\right)\right], \quad (A1)$$

where

$$\lambda=4m^2/\mu^2. \quad (A2)$$

Then the kernel of Eq. (2.19) has the following forms:

S wave,

$$c=-\infty:\quad H(y,y')=(2\pi\lambda)^{-1}U_I[(1-y)(1-y')]^{1/4}\times\Psi(y,y';0;1), \quad (A3)$$

$$c=0:\quad H(y,y')=-(2\pi\lambda)^{-1}U_I[yy']^{1/2}\times[(1-y)(1-y')]^{1/4}\Psi(y,y';1;0), \quad (A4)$$

$$c=4m^2:\quad H(y,y')=-(2\pi\lambda)^{-1}U_I(yy')^{1/2}\times[(1-y)(1-y')]^{1/4}\Psi(y,y';1;1); \quad (A5)$$

P wave,

$$c=-\infty:\quad H(y,y')=(\pi\lambda)^{-1}U_I[(1-y)(1-y')]^{-3/4}\times\left[1+\lambda^{-1}yy'\Psi\left(y,y';1;\frac{\lambda(1-y)(1-y')}{2yy'}+\frac{1}{y}+\frac{1}{y'}-1\right)\right], \quad (A6)$$

$$c=0:\quad H(y,y')=(2\pi\lambda)^{-1}U_I[yy']^{1/2}[(1-y)(1-y')]^{-3/4}\times\{2-[(1-y)(1-y')-2\lambda^{-1}yy']\times\Psi(y,y';1;yy'-\tfrac{1}{2}\lambda(1-y)(1-y'))\}, \quad (A7)$$

$$c=4m^2:\quad H(y,y')=-(2\pi\lambda)^{-1}U_I[yy']^{1/2}\times[(1-y)(1-y')]^{-1/4}\Psi(y,y';1;1-2/\lambda), \quad (A8)$$

where U_I is the isospin factor:

$$U_0=-\tfrac{3}{2},\quad U_1=\tfrac{1}{2}. \quad (A9)$$