As in the derivation of our sum rules discussed in the text, the use of $SU(3)$ approximation for the charge operator V_K and the existence of a vacuum state in the states under consideration restrict the intermediate states (in the one-particle approximation, which is the same as in the spectral-function case) to the ρ and κ states. The right-hand side of (A1) vanishes, so we have

$$\lim_{|\mathbf{q}|\to\infty} [E_\rho(\mathbf{q})-E_{K^*}(\mathbf{q})] \times\langle 0|V_0^{\pi^-}(x)|\rho^+\rangle\langle\rho^+|V_{\bar{K}^0}|K^{*+}(\mathbf{q})\rangle = \lim_{|\mathbf{q}|\to\infty} m_\kappa\langle 0|V_{\bar{K}^0}|\kappa^0\rangle\langle\kappa^0|V_0^{\pi^-}(x)|K^{*+}(\mathbf{q})\rangle .$$

Apart from the factor $[E_\rho(\mathbf{q})-E_{K^*}(\mathbf{q})]$, which vanishes in the limit $|\mathbf{q}|\to\infty$, the terms on the left-hand side of the above equation are finite. Therefore, we have

$$\lim_{|\mathbf{q}|\to\infty} \langle 0|V_{\bar{K}^0}|\kappa^0\rangle\langle\kappa^0|V_0^{\pi^-}(x)|K^*(\mathbf{q})\rangle=0 .$$

APPENDIX B

In a way similar to that of Appendix A, we can show that the pion electromagnetic form factor satisfies an unsubtracted dispersion relation. We take the commutator $[V_0^{\pi^0}(x),\dot{V}_{\pi^+}]=\partial_\mu A_\mu^{\pi^+}(x)$. As in (A1), we obtain

$$\lim_{|\mathbf{q}|\to\infty} \{\langle 0|V_0^{\pi^0}(x)|\rho^0\rangle\langle\rho^0|\dot{A}_{\pi^+}|\pi^-(\mathbf{q})\rangle - \langle 0|\dot{A}_{\pi^+}|\pi^-\rangle\langle\pi^-|V_0^{\pi^0}(x)|\pi^-(\mathbf{q})\rangle\} = \lim_{|\mathbf{q}|\to\infty} \langle 0|\partial_\mu A_\mu^{\pi^+}|\pi^-(\mathbf{q})\rangle . \quad \text{(B1)}$$

From the PCAC condition, the right-hand side of (B1) is zero. Then using the same argument as in Appendix A, we obtain

$$\lim_{|\mathbf{q}|\to\infty} \langle\pi^-(\mathbf{p})|V_0^{\pi^0}(x)|\pi(\mathbf{q})\rangle=0 ,$$

where $|\mathbf{p}|=0$.

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Subsidiary Condition in Quantum Electrodynamics

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The subsidiary condition $\partial A_\mu^{(+)}/\partial x_\mu|n\rangle=0$, usually known as the "Gupta-Bleuler" condition, is shown to be inadequate as a criterion for defining physical states in quantum electrodynamics in the Lorentz gauge. The condition is shown not to be covariant and to fail to define state vectors that remain in the physical subspace. An alternative subsidiary condition, which is satisfactory, is discussed and is shown to require an extensively different formulation of the collision problem in quantum electrodynamics. Some possible physical consequences of the inadequacy of $\partial A_\mu^{(+)}/\partial x_\mu|n\rangle=0$ are proposed; these include effects in the decays of short-lived particles, and the fact that in some types of strong interactions, acting simultaneously with electromagnetic ones, S-matrix elements may occur which predict transitions from the physical space into the part of space in which the subsidiary condition is violated. The solution to the collision problem for stable charged particles that have only electromagnetic interactions is shown to be identical to that obtainable from the present theory.

I. INTRODUCTION

THE correct formulation of quantum electrodynamics (QED) in the Lorentz gauge requires the imposition of a subsidiary condition [namely, $\chi^{(+)}(x)|n\rangle=0$, where $\chi^{(+)}(x)=\partial A_\mu^{(+)}(x)/\partial x_\mu$] on the "physical" state vectors and involves the use of a nondegenerate indefinite metric space instead of the usual Hilbert space in which quantum theories are ordinarily framed. The reasons for this have to do with the incompatibility of the subsidiary condition $\partial A_\mu(x)/\partial x_\mu=0$, as an operator identity, with the canonical quantization procedure and the commutation rules for the four-dimensional vector potential. This situation has been understood for a very long time and is discussed in detail in most standard texts.[1] For a set of noninteracting photons the resulting theory is clear and has the follow-

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ing features: The subsidiary condition in the noninteracting Heisenberg (or "interaction") picture

$$\chi^{(+)}(x)|n\rangle=0 \quad (1)$$

effects an unambiguous separation into physical and unphysical states. The operator $\chi^{(+)}(x)$ is a four-dimensional scalar operator and Eq. (1) is trivially covariant. If the subsidiary condition is true at a space-time point x, it is also true at a space-time point x' if $(x'-x)$ is timelike; i.e., in that case Eq. (1) implies

$$\exp[-iP_\mu(x'-x)_\mu]\chi^{(+)}(x)\exp[+iP_\mu(x'-x)_\mu]|n\rangle=0.$$

QED, for this set of noninteracting photons, takes place as it were, wholly within the physical subspace, and the imposition of Eq. (1) manifestly leads to no dilemmas or paradoxes.

The situation for the case of interacting photons and charged particles is generally assumed to differ only trivially from the state of affairs that prevails for noninteracting photons. On the basis of various arguments,[2] it is believed that state vectors that are wholly within the physical subspace at any one time remain in it forever after; the permanent inclusion of state vectors within the physical subspace is invoked to guarantee the continued validity of Maxwell's equations for the expectation values of the electric and magnetic fields, and also the nonappearance of "negative probability" states.[1]

In this paper we first point out that the subsidiary condition[3] $\chi^{(+)}(x)|n\rangle=0$ is inadequate in the presence of interactions because, first of all, it is not a covariant condition (this is proven in Appendix A); in addition, the normal time evolution of a state vector, in the presence of charged-particle–photon interactions, suffices to invalidate this condition at times later than x_0 so that the state vector "leaks" from the physical to the unphysical state. We describe an alternative subsidiary condition which is both covariant and permanently persistent but which has a complicated set of eigenstates (which, for example, do not include the vacuum or photon-free states such as n-electron states). We shall prove the validity of the usual computational practices in QED for a restricted set of circumstances, and also discuss some possible physical consequences of the difference between our formulation and the usual one.

[2] K. Bleuler, Helv. Phys. Acta **23**, 567 (1950); F. Coester and J. M. Jauch, Phys. Rev. **78**, 149 (1950); Ning Hu, *ibid.* **76**, 391 (1949); **77**, 150 (1950); J. M. Jauch and F. Rohrlich (Ref. 1); H. Umezawa, *Quantum Field Theory* (North-Holland Publishing Company, Amsterdam, 1956).

[3] The following notation will be used to designate operators: The operator P or $P(\mathbf{x})$ will designate a Schrödinger operator, $P(x)$ or $P(t)$ will designate the operator in the interaction picture, and $\bar{P}(x)$ or $\bar{P}(t)$ will designate the operator in the Heisenberg picture. $\mathbf{x}$ refers to the three-dimensional position, and x to the four-vector $\mathbf{x}$, it. Boldfaced operators $\mathbf{A}$ designate three-vectors in whatever picture their argument (or barring) signifies.

II. INDEFINITE METRIC SPACE

The indefinite metric space is spanned by a set of state vectors and their adjoints $\langle m^\star|=\langle m|\eta$, where $\langle m|$ denotes the Hermitian conjugate and η is a Hermitian, unitary operator which obeys

$$[\eta,\mathbf{A}(\mathbf{x})]=0,\quad \{\eta,A_4(\mathbf{x})\}=0, \quad (2)$$

and which commutes with all other fields. All four components of A_μ, including $A_4=iA_0$, are Hermitian and $\langle n^\star|A_4|n\rangle$ is an imaginary quantity because of Eq. (2). Observable operators must be self-adjoint in this space, i.e., $P=P^\star$ for $P^\star=\eta^{-1}P^\dagger\eta$, and transforms are length-preserving if they are pseudo-unitary, i.e., if $T^{-1}=T^\star$. The following photon operators, in the Schrödinger picture, can be defined:

$$a^\dagger_{\mathbf{k},\epsilon(i)}=a^\dagger_{\mathbf{k},l}\epsilon_l(i),$$
$$a^\dagger_{\mathbf{k},L}=a^\dagger_{\mathbf{k},l}k_l/k$$

as creation operators and their Hermitian adjoints as annihilation operators. We see that

$$[a_{\mathbf{k},\epsilon(i)},a^\dagger_{\mathbf{k}',\epsilon'(i')}]=\delta_{\mathbf{k},\mathbf{k}'}\delta_{i,i'},\quad [a_{\mathbf{k},L},a^\dagger_{\mathbf{k}',L}]=\delta_{\mathbf{k},\mathbf{k}'},$$

and $[a_{\mathbf{k},\epsilon(i)},a^\dagger_{\mathbf{k}',L}]=0$. Moreover, since A_4 is Hermitian, we have $[a_{\mathbf{k},4},a^\dagger_{\mathbf{k}',4}]=\delta_{\mathbf{k},\mathbf{k}'}$.

It is, moreover, useful to define the following sets of operators:

$$a^\dagger_{\mathbf{k},R}=(2)^{-1/2}[a^\dagger_{\mathbf{k},L}+ia^\dagger_{\mathbf{k},4}],$$
$$a^\dagger_{\mathbf{k},Q}=(2)^{-1/2}[a^\dagger_{\mathbf{k},L}-ia^\dagger_{\mathbf{k},4}], \quad (3)$$

and their Hermitian adjoints. We find that

$$[a_{\mathbf{k},R},a^\dagger_{\mathbf{k}',R}]=\delta_{\mathbf{k},\mathbf{k}'},\quad [a_{\mathbf{k},Q},a^\dagger_{\mathbf{k}',Q}]=\delta_{\mathbf{k},\mathbf{k}'},$$

and

$$[a_{\mathbf{k},R},a^\dagger_{\mathbf{k}',Q}]=[a_{\mathbf{k},Q},a^\dagger_{\mathbf{k}',R}]=0.$$

The significance of these operators can be seen by writing $\chi^{(+)}(\mathbf{x})$ in the momentum representation. In the Schrödinger picture, the operator $\chi^{(+)}=\nabla\cdot\mathbf{A}^{(+)}-i\Pi_4{}^{(+)}$ (Π_μ is the canonically conjugate field to A_μ); we find that

$$\chi^{(+)}(\mathbf{x})=i\sum_{\mathbf{k}}k^{1/2}a_{\mathbf{k},Q}e^{i\mathbf{k}\cdot\mathbf{x}}.$$

In the Schrödinger picture, the subsidiary condition becomes

$$a_{\mathbf{k},Q}|n\rangle=0, \quad (4a)$$

i.e., there are no "Q"-type photons in physical ket-state vectors. Since $a^\star_{\mathbf{k},Q}=\eta^{-1}a^\dagger_{\mathbf{k},Q}\eta=a^\dagger_{\mathbf{k},R}$, the subsidiary condition also can be written

$$\langle n^\star|a^\dagger_{\mathbf{k},R}=0, \quad (4b)$$

and asserts that there are no R-type photons in physical bra-state vectors. It is indeed possible to relabel these operators as physical and unphysical by

$$a^\dagger_{\mathbf{k},R}=a^\star_{\mathbf{k},u},\quad a_{\mathbf{k},Q}=a_{\mathbf{k},u},$$
$$a^\dagger_{\mathbf{k},Q}=a^\star_{\mathbf{k},p},\quad a_{\mathbf{k},R}=a_{\mathbf{k},p}.$$

Then the subsidiary conditions read

$$a_{\mathbf{k},u}|n\rangle=0\,,\quad \langle n^\star|a^\star{}_{\mathbf{k},u}=0\,,$$

and we have

$$[a_{\mathbf{k},u},a^\star{}_{\mathbf{k}',u}]=[a_{\mathbf{k},p},a^\star{}_{\mathbf{k}',p}]=0\,,$$
$$[a_{\mathbf{k},u},a^\star{}_{\mathbf{k}',p}]=[a_{\mathbf{k},p},a^\star{}_{\mathbf{k}',u}]=\delta_{\mathbf{k},\mathbf{k}'}\,.$$

We shall, however, not use this latter mode of designation in this paper. $H_{0(r)}$, the Hamiltonian for noninteracting photons, can be written

$$H_{0(r)}=\sum_{\mathbf{k}} k[\sum_i a^\dagger{}_{\mathbf{k},\epsilon(i)}a_{\mathbf{k},\epsilon(i)} +a^\dagger{}_{\mathbf{k},Q}a_{\mathbf{k},Q}+a^\dagger{}_{\mathbf{k},R}a_{\mathbf{k},R}]\,. \quad (5)$$

The expectation value of $H_{0(r)}$ is $\langle n^\star|H_{0(r)}|n\rangle$. Since $|n\rangle$ contains no $a^\dagger{}_{\mathbf{k},Q}$, and $\langle n^\star|$ contains no $a_{\mathbf{k},R}$ operators, only the transverse-photon-number operator contributes to the energy. The Hamiltonian H for spinor electrodynamics consists of $H_{0(r)}$ and $H_{0(e)}$, where

$$H_{0(e)}=\int d\mathbf{x}\,\psi^\dagger(\mathbf{x})[\beta m-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}]\psi(\mathbf{x})$$

and of the interaction Hamiltonian H_1. The latter is given by

$$H_1=-\int d\mathbf{x}[\mathbf{J}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})+i\rho(\mathbf{x})A_4(\mathbf{x})]\,, \quad (6)$$

where $\mathbf{J}(\mathbf{x})=e_0\psi^\dagger(\mathbf{x})\boldsymbol{\alpha}\psi(\mathbf{x})$ and $\rho(\mathbf{x})=e_0\psi^\dagger(\mathbf{x})\psi(\mathbf{x})$. It is convenient to rewrite H_1 in terms of

$$\mathbf{J}(\mathbf{k})=\int d\mathbf{x}\,\mathbf{J}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}$$

and

$$\rho(\mathbf{k})=\int d\mathbf{x}\,\rho(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}$$

and the previously defined photon operators. This leads to

$$\begin{aligned}H_1=-\sum_{\mathbf{k}}(2k^{1/2})^{-1}\{&\sum_i[a_{\mathbf{k},\epsilon(i)}\mathbf{J}(-\mathbf{k})\cdot\boldsymbol{\varepsilon}(i)\\&+a^\dagger{}_{\mathbf{k},\epsilon(i)}\mathbf{J}(\mathbf{k})\cdot\boldsymbol{\varepsilon}(i)]\sqrt{2}+a_{\mathbf{k},R}[\rho(-\mathbf{k})-\mathbf{k}\cdot\mathbf{J}(-\mathbf{k})/k]\\&-a_{\mathbf{k},Q}[\rho(-\mathbf{k})+\mathbf{k}\cdot\mathbf{J}(-\mathbf{k})/k]\\&-a^\dagger{}_{\mathbf{k},R}[\rho(\mathbf{k})+\mathbf{k}\cdot\mathbf{J}(\mathbf{k})/k]\\&+a^\dagger{}_{\mathbf{k},Q}[\rho(\mathbf{k})-\mathbf{k}\cdot\mathbf{J}(\mathbf{k})/k]\}\,.\end{aligned} \quad (7)$$

This can be written in a more nearly manifestly covariant form by writing

$$H_1=\bar{\psi}(\mathbf{x})\mathfrak{H}\psi(\mathbf{x})\,,$$

where

$$\begin{aligned}\mathfrak{H}=-\sum_{\mathbf{k}}(2k^{3/2})^{-1}\{&\sum_j[a_{\mathbf{k},\epsilon(j)}(i\gamma\cdot\epsilon(j))e^{i\mathbf{k}\cdot\mathbf{x}}\\&+a^\dagger{}_{\mathbf{k},\epsilon(j)}(i\gamma\cdot\epsilon(j))e^{-i\mathbf{k}\cdot\mathbf{x}}]\sqrt{2}k\\&-[a_{\mathbf{k},R}(i\gamma\cdot k)e^{i\mathbf{k}\cdot\mathbf{x}}+a^\dagger{}_{\mathbf{k},Q}(i\gamma\cdot k)e^{-i\mathbf{k}\cdot\mathbf{x}}]\\&-[a_{\mathbf{k},Q}(i\gamma\cdot k^*)e^{-i\mathbf{k}\cdot\mathbf{x}}+a^\dagger{}_{\mathbf{k},R}(i\gamma\cdot k^*)e^{i\mathbf{k}\cdot\mathbf{x}}]\}\,.\end{aligned} \quad (8)$$

This form allows us to write the rules for simple vertices involving physical and unphysical photons. For transverse photons, besides the usual contributions to four-momentum conservation, etc., $\mathfrak{H}$ dictates the corner term $i\gamma\cdot\epsilon$. For physical nontransverse photons, we have $i\gamma\cdot k^*/k=i(\boldsymbol{\gamma}\cdot\mathbf{k}-\gamma_4k_4)/k$. For unphysical photons (creation of a Q-type or annihilation of an R-type photon), we have $(i\gamma\cdot k/k)$. This shows that the absence of unphysical photons in the final asymptotic wave functions for colliding systems in spinor electrodynamics implies a restricted gauge invariance; i.e., invariance of S-matrix elements in the Lorentz gauge to the transformation $A_\mu(x)\to A_\mu(x)+\partial_\mu\mathfrak{F}(x)$, where $\mathfrak{F}(x)$ is a scalar that satisfies $\Box\mathfrak{F}=0$.

III. PERSISTENCE OF THE SUBSIDIARY CONDITION

In the case of noninteracting photons the Heisenberg and interaction pictures coincide and the time-dependent subsidiary condition becomes

$$e^{iH_0t}[\boldsymbol{\nabla}\cdot\mathbf{A}^{(+)}-i\Pi_4{}^{(+)}]e^{-iH_0t}|n\rangle=0 \quad (9a)$$

or, equivalently,

$$e^{iH_0t}a_{\mathbf{k},Q}e^{-iH_0t}|n\rangle=0\,. \quad (9b)$$

Expansion of this equation leads to

$$a_{\mathbf{k},Q}+i[H_0,a_{\mathbf{k},Q}]t +\cdots(i)^n[H_0,[H_0,\cdots[H_0,a_{\mathbf{k},Q}]\cdots]]tn/n!=0\,. \quad (10)$$

Since $[H_0,a_{\mathbf{k},Q}]=-ka_{\mathbf{k},Q}$, Eq. (10) becomes

$$a_{\mathbf{k},Q}e^{-ikt}|n\rangle=0;$$

hence the subsidiary condition persists forever. This argument can be rephrased in slightly different language by noting that the time-dependent subsidiary condition can be written in the more conventional form $\chi^{(+)}(x)|n\rangle=0$. Then $\Box\chi^{(+)}(x)|n\rangle=0$ and since only positive frequencies occur in $\chi^{(+)}(x)$, the operator $(\partial/\partial t)\chi^{(+)}(x)$ is essentially the same operator as $\chi^{(+)}(x)$ (with different c-number coefficients in its Fourier decomposition) and the same sets of states obey $(\partial/\partial t)\chi^{(+)}(x)|n\rangle=0$ as obey $\chi^{(+)}(x)|n\rangle=0$. By Green's identity, therefore, $\chi^{(+)}(x)|n\rangle=0$ holds at all times, if it ever held once. It is noteworthy that an attempt to extend this argument to interacting photon–charged-fermion systems fails.

To show this, we define the Heisenberg operator

$$\bar{\chi}^{(+)}(x)=e^{iHt}\chi^{(+)}(\mathbf{x})e^{-iHt}.$$

This operator has the following significance: Suppose that at one time, chosen arbitrarily to be $t=0$, $\chi^{(+)}(\mathbf{x})|n\rangle=0$ defines the physical state $|n\rangle$. Then the question of whether states remain physical reduces to the question of whether $\chi^{(+)}(\mathbf{x})e^{-iHt}|n\rangle=0$, or, equivalently, whether $\bar{\chi}^{(+)}(x)|n\rangle=0$, where $\bar{\chi}^{(+)}(x)$ is the operator defined above. It is $\bar{\chi}^{(+)}(x)|n\rangle$ which must vanish at all times if the Gupta subsidiary condition is to persist forever. It is noteworthy in this connection that $\bar{\chi}^{(+)}(x)$ is *not* the invariant positive-frequency part of $\bar{\chi}(x)$.

It is easy to see that $\bar{\chi}^{(+)}(x)|n\rangle=0$ does not persist for all times because $\Box\bar{\chi}^{(+)}(x)|n\rangle\neq 0$. In fact, $\bar{\chi}^{(+)}(x)$ obeys the equation $\Box\bar{\chi}^{(+)}(x)=\bar{f}(x)$, where

$$\Box\bar{\chi}^{(+)}(x)=e^{iHt}\{\nabla^2\chi^{(+)}(\mathbf{x})+[H,[H,\chi^{(+)}(\mathbf{x})]]\}e^{-iHt}.$$

By using Eq. (7), we can show that $\bar{f}(x)$ is given in either of two forms: In the Schrödinger picture,

$$f(\mathbf{x})=\varepsilon_{ljn}\left\{\frac{1}{2}e_0\frac{\partial}{\partial x_l}\int d\mathbf{y}\ \mathfrak{D}(\mathbf{x}-\mathbf{y})\right.$$
$$\times\left[\frac{\partial\psi^\dagger(\mathbf{y})}{\partial y_j}\sigma_n\psi(\mathbf{y})-\psi^\dagger(\mathbf{y})\sigma_n\frac{\partial\psi(\mathbf{y})}{\partial y_j}\right]$$
$$\left.-ie_0^2\frac{\partial}{\partial x_l}\int d\mathbf{y}\,A_j(\mathbf{y})\mathfrak{D}(\mathbf{x}-\mathbf{y})\psi^\dagger(\mathbf{y})\sigma_n\psi(\mathbf{y})\right\}, \quad (11a)$$

where

$$\mathfrak{D}(\mathbf{x}-\mathbf{y})=(2\pi)^{-3}\int d\mathbf{k}\ e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}/k=2iD^{(+)}(\mathbf{x}-\mathbf{y};0)$$

or, in the Heisenberg picture,

$$\bar{f}(x)=-\tfrac{1}{2}i\Box_x\int d\mathbf{y}\ \mathfrak{D}(\mathbf{x}-\mathbf{y})\bar{\rho}(\mathbf{y},x_0); \quad (11b)$$

$\bar{f}(x)$ does not vanish.

It is important to note that although

$$\bar{\chi}^{(+)}(x)=e^{iHt}\chi^{(+)}(\mathbf{x})e^{-iHt},$$

$\bar{\chi}^{(+)}(x)$, at times other than $t=0$, will not contain only photon annihilation operators. Because of the time evolution dictated by H, creation operators for various types of particles arise in $\bar{\chi}^{(+)}(x)$. This can be illustrated by expanding $e^{iHt}a_{\mathbf{k}Q}e^{-iHt}$ and noting that

$$[H,a_{\mathbf{k},Q}]=-\{ka_{\mathbf{k},Q}+[\rho(\mathbf{k})-\mathbf{k}\cdot\mathbf{J}(\mathbf{k})/k]\},$$

which contains fermion creation as well as annihilation operators. Subsequent commutators, like $[H,[H,a_{\mathbf{k},Q}]]$, $[H,[H,\cdots[H,a_{\mathbf{k},Q}]\cdots]]$ each generate new functionals of operators that differ from those of earlier orders so that the form of the subsidiary condition continually changes in time.

It is apparent, from Eq. (11b), that we can frame a subsidiary condition which persists in time.

If we define $\bar{\Omega}^{(+)}(x)$ by

$$\bar{\Omega}^{(+)}(x)=\bar{\chi}^{(+)}(x)+\tfrac{1}{2}i\int d\mathbf{y}\ \mathfrak{D}(\mathbf{x}-\mathbf{y})\bar{\rho}(\mathbf{y},x_0), \quad (12)$$

then from Eq. (11b) we see that $\Box\bar{\Omega}^{(+)}(x)$ does, in fact, vanish, and the condition

$$\bar{\Omega}^{(+)}(x)|\nu\rangle=0 \quad (13)$$

persists forever. This new subsidiary condition defines a new set of physical states $|\nu\rangle$.

If we write

$$\Omega^{(+)}(\mathbf{x})=i\sum_{\mathbf{k}}k^{1/2}\Omega^{(+)}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}},$$

with $\Omega^{(+)}(\mathbf{k})=a_{\mathbf{k},Q}+\frac{1}{2}\rho(\mathbf{k})k^{-3/2}$, then the new physical states can be equally well defined by $\Omega^{(+)}(\mathbf{k})|\nu\rangle=0$. To obtain the time derivative of this equation, we calculate $i[H,\Omega^{(+)}(\mathbf{k})]$ and observe that it is identically $-ik\Omega^{(+)}(\mathbf{k})$, so that $(\partial/\partial t)\bar{\Omega}^{(+)}(x)|\nu\rangle=0$ selects the same set of states as $\bar{\Omega}^{(+)}(x)|\nu\rangle=0$. We can now invoke Green's identity to prove that $\bar{\Omega}^{(+)}(x)|\nu\rangle=0$ holds at all times if it ever held at all. In fact, the time-dependent form $\bar{\Omega}^{(+)}(\mathbf{k})|\nu\rangle=0$ can be seen to be $\Omega^{(+)}(\mathbf{k})e^{-ikt}|\nu\rangle=0$. In Appendix A we also show that, in contrast to $\bar{\chi}^{(+)}(x)$, the operator $\bar{\Omega}^{(+)}(x)$ is a scalar operator and Eq. (13) is covariant.[4] The subsidiary condition $\bar{\Omega}^{(+)}(x)|\nu\rangle=0$ satisfies the requirement $\langle\nu^\star|\partial\bar{A}_\mu/\partial x_\mu|\nu\rangle=0$, since

$$\bar{\Omega}^{(+)}(x)+\bar{\Omega}^{(-)}(x)=\bar{\chi}^{(+)}(x)+\bar{\chi}^{(-)}(x)=\bar{\chi}(x).$$

Moreover, $\langle\nu^\star|(\partial\bar{A}_\mu/\partial x_\mu)^2|\nu\rangle=0$, required for the correct energy-momentum tensor, is also guaranteed by the fact that $\Omega^{(+)}(\mathbf{x})$ and $\Omega^{(-)}(\mathbf{x})=[\Omega^{(+)}(\mathbf{x})]^\star$ commute.

The preceding discussion allows us to conclude that $\bar{\Omega}^{(+)}(x)$ and $\bar{\Omega}^{(-)}(x)$ represent the invariant positive- and negative-frequency parts of $\bar{\chi}(x)$, respectively. It must be noted that $\Omega^{(+)}(\mathbf{x})$, which selects the "new" physical states according to $\Omega^{(+)}(\mathbf{x})|\nu\rangle=0$, differs from $\chi^{(+)}(\mathbf{x})$ in a way that requires an important reformulation of the scattering problem in QED. This will be discussed in detail in Sec. IV.

IV. NEW PHYSICAL STATES AND TRANSITION AMPLITUDES

In order to find the proper description of physical states, we must solve Eq. (13). It is immediately apparent that most states that we have always considered to be physical do not satisfy Eq. (13). Since $\rho(\mathbf{k})$ contains terms that are bilinear in fermion-creation operators, the vacuum is not a physical state; neither

[4] K. Bleuler (Ref. 2) suggested this form of the subsidiary condition but he apparently did not realize how sharply the physical states and the scattering formalism implied by Eq. (13) differ from those of Gupta's theory. Bleuler's work is in fact generally considered to serve as support for the application of Gupta's subsidiary condition to interacting systems, and Bleuler's paper never calls attention to any of the crucial changes required in the theory by the substitution of Eq. (13) for Eq. (1). W. Heitler (Ref. 1), calls attention to the fact that the expectation values of the operators which define the subsidiary condition must satisfy two initial conditions to guarantee that the expectation values for fields continue to obey Maxwell's equations at all times. These two conditions, in our notation, are $\langle\psi^\star|\nabla\cdot\mathbf{A}-i\Pi_4|\psi\rangle=0$ and $\langle\psi^\star|\nabla\cdot\mathbf{\Pi}-i\nabla^2A_4+\rho|\psi\rangle=0$. The states $|n\rangle$ obeying $\chi^{(+)}(\mathbf{x})|n\rangle=0$, violate the second of these equations but the states $|\nu\rangle$ obeying $\Omega^{(+)}(\mathbf{x})|\nu\rangle=0$ form expectation values that obey both of Heitler's equations. Heitler pointed out that the usual solutions of $\chi^{(+)}(\mathbf{x})|n\rangle=0$ are not the correct choice of states for interacting systems; he did not, however, attempt to construct the states we refer to as $|\nu\rangle$ or to construct any alternative dynamical formalism based on any new states.

are many-electron states, nor are the transversely polarized photon states physical. To facilitate the search for the new physical states, i.e., the solutions of Eq. (13), we note that $\chi^{(+)}(\mathbf{x})$ and $\Omega^{(+)}(\mathbf{x})$ are related by the identity

$$\Omega^{(+)}(\mathbf{x})=e^{-D}\chi^{(+)}(\mathbf{x})e^{+D}, \tag{14}$$

where D is given by

$$D=i\int d\mathbf{x}d\mathbf{y}\,\rho(\mathbf{x})[\nabla_y\cdot\mathbf{A}(\mathbf{y})+i\Pi_4(\mathbf{y})](8\pi|\mathbf{x}-\mathbf{y}|)^{-1}$$
$$=-\tfrac{1}{2}\sum_{\mathbf{k}}k^{-3/2}[a_{\mathbf{k},R}\rho(-\mathbf{k})-a^\dagger{}_{\mathbf{k},Q}\rho(\mathbf{k})]. \tag{15}$$

We can therefore solve Eq. (13) by noting that the solutions are given by

$$|\nu\rangle=e^{-D}|n\rangle. \tag{16}$$

The $|n\rangle$ states, eigenstates of the "old" subsidiary condition, are of course well known and include all electron and photon states with the exception of Q-type photons. For each $|n\rangle$-type state, there is a corresponding $|\nu\rangle$-type. For example, a physical R-type photon, to second-order in e_0, is given by

$$e^{-D}a^\dagger{}_{\mathbf{q},R}|0\rangle=a^\dagger{}_{\mathbf{q},R}|0\rangle-\tfrac{1}{2}e_0\sum_{k,\kappa,\kappa'}[a^\dagger{}_{\mathbf{k},Q}a^\dagger{}_{\mathbf{q},R}-\delta_{\mathbf{k},-\mathbf{q}}]k^{-3/2}$$
$$\times e^\dagger{}_{\kappa,j}\bar{e}^\dagger{}_{\kappa',j'}|0\rangle\bar{u}(\boldsymbol{\kappa},j)\gamma_4v(\boldsymbol{\kappa}',j')\delta(\boldsymbol{\kappa}+\boldsymbol{\kappa}'+\mathbf{k})$$
$$-\tfrac{1}{8}e_0^2\sum_{k,k',\kappa,\kappa'}[a^\dagger{}_{\mathbf{k},Q}a^\dagger{}_{\mathbf{k}',Q}a^\dagger{}_{\mathbf{q},R}+a^\dagger{}_{\mathbf{k},Q}\delta_{\mathbf{k}',-\mathbf{q}}+a^\dagger{}_{\mathbf{k}',Q}\delta_{\mathbf{k},-\mathbf{q}}]$$
$$\times\sum_{\mathbf{p},\mathbf{p}'}e^\dagger{}_{\kappa,j}\bar{e}^\dagger{}_{\kappa',j'}e^\dagger{}_{\mathbf{p},l}e^\dagger{}_{\mathbf{p}',l'}|0\rangle\bar{u}(\boldsymbol{\kappa},j)\gamma_4v(\boldsymbol{\kappa}',j')\bar{u}(\mathbf{p},l)\gamma_4v(\mathbf{p}',l')$$
$$\times\delta(\boldsymbol{\kappa}+\boldsymbol{\kappa}'+\mathbf{k})\delta(\mathbf{p}+\mathbf{p}'+\mathbf{k}'). \tag{17a}$$

Similarly, a one-electron state to first order in e_0 is given by

$$e^{-D}e^\dagger{}_{\mathbf{k},j}|0\rangle=e^\dagger{}_{\mathbf{k},j}|0\rangle$$
$$-\tfrac{1}{2}e_0\sum_{\mathbf{k}}e^\dagger{}_{\mathbf{k}-\boldsymbol{\kappa},l}a^\dagger{}_{\kappa,Q}|0\rangle\kappa^{-3/2}\bar{u}(\mathbf{k}-\boldsymbol{\kappa},l)\gamma_4u(\mathbf{k},j)$$
$$-\tfrac{1}{2}e_0\sum_{\mathbf{q},\mathbf{q}',\kappa}a^\dagger{}_{\kappa,Q}e^\dagger{}_{\mathbf{q},l}\bar{e}^\dagger{}_{\mathbf{q}',l'}e^\dagger{}_{\mathbf{k},j}|0\rangle$$
$$\times\kappa^{-3/2}\bar{u}(\mathbf{q},l)\gamma_4v(\mathbf{q}',l')\delta(\mathbf{q}+\mathbf{q}'+\boldsymbol{\kappa}). \tag{17b}$$

$u(\mathbf{k},j)$ and $v(\mathbf{k},j)$ in these expressions refer to $u_{\mathbf{k},j}(x=0)$ and $v_{\mathbf{k},j}(x=0)$, respectively, where $[m+\gamma_\mu\partial_\mu]u_{\mathbf{k},j}(x)=0$ and $[m-\gamma_\mu\partial_\mu]v_{\mathbf{k},j}(x)=0$. They are normalized to

$$\sum_j u(\mathbf{k},j)\bar{u}(\mathbf{k},j)=[m-i\gamma\cdot k](2k_0)^{-1}.$$

Similar expansions can be given for all the $|\nu\rangle$-type physical eigenstates, none of which remains in the same form that it had under the old subsidiary condition.

Note that the form of these new physical states requires a modification of the scattering theory that governs electrodynamic collision processes. Since in the usual formulation of the theory the asymptotic states[5] of the colliding system are eigenstates of H_0, the outgoing scattering states obey the integral equation

$$\zeta^{(+)}(E,n)=\phi(E,n)+(E-H_0+i\epsilon)^{-1}H_1\zeta^{(+)}(E,n),$$

where $H_0+H_1=H$ and ϕ denotes an eigenstate of H_0.

The new eigenstates $|\nu\rangle$ are, however, not eigenstates of H_0, but rather of $\mathcal{H}_0=e^{-D}H_0e^D$. We therefore write $\mathcal{H}_1=H-\mathcal{H}_0$ and the new scattering eigenstates[6] will be given by

$$\psi^{(+)}(E,\nu)=\varphi(E,\nu)+[E-\mathcal{H}_0+i\epsilon]^{-1}\mathcal{H}_1\psi^{(+)}(E,\nu), \tag{18}$$

where φ denotes an eigenstate of $\mathcal{H}_0$. It is easy to explicitly calculate $\mathcal{H}_0$ and $\mathcal{H}_1$, since the commutator $[D,H_0]$ commutes with D. We obtain

$$\mathcal{H}_1=H_1-\tfrac{1}{2}\sum_{\mathbf{k}}k^{-1/2}\{a_{\mathbf{k},R}[\rho(-\mathbf{k})-\mathbf{k}\cdot\mathbf{J}(-\mathbf{k})/k]$$
$$+a^\dagger{}_{\mathbf{k},Q}[\rho(\mathbf{k})-\mathbf{k}\cdot\mathbf{J}(\mathbf{k})/k]\}, \tag{19a}$$

and

$$\mathcal{H}_1(t)=H_1(t)+idD(t)/dt. \tag{19b}$$

We can apply Eq. (16) to the evaluation of transition amplitudes for collision processes. The most general second-order transition amplitude $T^{(2)}$ has contributions both from

$$T_a=\langle\nu_f{}^\star|\mathcal{H}_1|\nu_i\rangle$$

and from

$$T_b=\sum_\lambda\langle\nu_f{}^\star|\mathcal{H}_1|\lambda\rangle\langle\lambda'^\star|\mathcal{H}_1|\nu_i\rangle(E_i-E_\lambda+i\epsilon)^{-1},$$

where $|\lambda\rangle$ are a complete set of eigenstates[7] of $\mathcal{H}_0$ and $\Omega^{(+)}(\mathbf{x})$. T_a and T_b can be rewritten as

$$T_a=\langle n_f{}^\star|e^D\mathcal{H}_1e^{-D}|n_i\rangle$$

and

$$T_b=\sum_l\langle n_f{}^\star|e^D\mathcal{H}_1e^{-D}|l\rangle\langle l'^\star|e^D\mathcal{H}_1e^{-D}|n_i\rangle(E_i-E_l+i\epsilon)^{-1}.$$

Since $\mathcal{H}_1=H_1-[H_0,D]$, and since $[D,H_1]$ commutes with D, we have $e^D\mathcal{H}_1e^{-D}=H_1+[D,H_1]-[H_0,D]$ and

$$T_a{}^{(2)}=\langle n_f{}^\star|[D,H_1]|n_i\rangle.$$

Similarly, we have

$$T_b{}^{(2)}=\langle n_f{}^\star|H_1-[H_0,D]|l\rangle\langle l|H_1-[H_0,D]|n_i\rangle\times(E_i-E_l+i\epsilon)^{-1}.$$

[5] In this paper we are treating the entire theory as though it required no renormalization and we are ignoring the so-called Van Hove paradox of the second kind [L. Van Hove, Physica **21**, 901 (1955)] which deals with the inadequacy of eigenstates of H_0 as asymptotic states. It appears to us that the renormalization difficulties and the problem to which we are here addressing ourselves are not related and are best treated independently of each other.

[6] Although this point is perhaps very clear to the reader, we would like to emphasize that we are not here recasting the same theory into a trivially different form by making a pseudo-unitary transformation. This can be seen most easily by observing that whereas the states and the operator H_0 are transformed, H is not. $\mathcal{H}_1$ is not, for example, $e^{-D}H_1e^D$. The transform effected by e^D is simply a bookkeeping device for generating the proper physical (asymptotic) states. It is of course obvious that for $|\nu\rangle=e^{-D}|n\rangle$, $(\mathcal{H}_0-E)|\nu\rangle=0$ if $(H_0-E)|n\rangle=0$ for the same E in *both* equations.

[7] The unit operator is discussed in Appendix B.

Here $T_a^{(2)}$ and $T_b^{(2)}$ are the parts of T_a and T_b, respectively, that are the second order in the electric charge.

$T_b^{(2)}$ can be rewritten as

$$T_b^{(2)}=\frac{\langle n_f^\star|H_1|l\rangle\langle l|H_1|n_i\rangle}{E_i-E_l+i\epsilon}+\langle n_f^\star|[H_1,D]|n_i\rangle+\frac{\langle n_f^\star|D|l\rangle\langle l|H_1|n_i\rangle}{E_i-E_l+i\epsilon}(E_i-E_f)+\tfrac{1}{2}\langle n_f^\star|D^2|n_i\rangle(E_f-E_i).$$

Combining, we find that

$$T^{(2)}=\frac{\langle n_f^\star|H_1|l\rangle\langle l|H_1|n_i\rangle}{E_i-E_l+i\epsilon}+\tfrac{1}{2}\langle n_f^\star|D^2|n_i\rangle(E_f-E_i)+\frac{\langle n_f^\star|D|l\rangle\langle l|H_1|n_i\rangle}{E_i-E_l+i\epsilon}(E_i-E_f). \quad (20)$$

Examination of the various terms on the right side of Eq. (20) shows that the first one is identical to what the old perturbation theory would have led to. The second can never contribute to processes evolving from realizable situations since either Q- or R-type photon operators, neither of which refers to transverse photons, are part of every term. The third term is important; it makes contributions to the wave function which are required to keep it in the physical space *at all times;* however, both the second and third terms vanish on the energy shell. In other words, although, to this order, the old subsidiary condition $\chi^{(+)}(x)|n\rangle=0$ allows leakage of state vectors from the physical to the unphysical space, the amount of this leakage vanishes in the limit as the time elapsed in the process approaches infinity.

When the transition amplitude is examined to orders higher than the second, the same features are observed. The differences between the old and new values for the transition amplitude vanish on the mass shell.

We illustrate this feature of the theory by calculating the time-dependent wave function $\psi(t)$, which describes the scattering $e_{\mathbf{k},r}+\gamma_{\mathbf{q},t}\rightarrow e_{\mathbf{k}',r'}+\gamma_{\mathbf{q}',u}$ to second order in e_0, where $\gamma_{\mathbf{q},t}$ represents a transverse and $\gamma_{\mathbf{q}',u}$ is an unphysical photon. We let $\psi(-\infty)$ represent the state $|e_{\mathbf{k},r}\gamma_{\mathbf{q},t}\rangle$ in the infinitely remote past and observe its evolution in time.

The wave function has two parts. One part, $\psi_\chi(t)$, is the one that the use of the old subsidiary condition [Eq. (1)] leads us to. The other part, $\psi_\Omega(t)$, is a correction due to the use of the correct, persistent, subsidiary condition, Eq. (13), to define the physical states. It becomes apparent that both $\psi_\chi(t)$ and $\psi_\Omega(t)$ vanish in the limit $t\rightarrow\infty$. That $\psi_\chi(t)$ vanishes in this limit is well known and traditionally ascribed to the gauge invariance of the theory. It will, however, also be clear that $\psi_\chi(t)$ does *not* vanish at all times, and for finite values of t, $\psi_\chi(t)$ executes an excursion into the unphysical space. The sum of $\psi_\chi(t)$ and $\psi_\Omega(t)$, however, *always* remains entirely in the physical space.

The wave function $\psi(t)$ is given by

$$\psi(t)=\sum_{j,j'}\int dE_k\int dE_{k'}g(E_k,j)\times\left[\delta_{\mathbf{k},\mathbf{k}'}\delta_{j,j'}+\frac{T(E_{k'}j'|E_k,j)}{E_{k'}-E_k+i\epsilon}\right]\phi(E_k,j)e^{-iE_kt}, \quad (21)$$

where T is the transition amplitude and $g(E_k,j)$ is a packet function which describes the spectral definition of the initial state. The second order in (e_0) of $\psi_\chi(t)$ is evaluated by using $T_\chi(E_{k'},j'|E_k,j)$ which is given by

$$\sum_l\langle e_{\mathbf{k}'}\gamma_{\mathbf{q}',Q}|H_1|l\rangle\langle l|H_1|e_{\mathbf{k}}\gamma_{\mathbf{q},t}\rangle(E_l-\omega_k-q)^{-1}.$$

It is evaluated by using the vertex $\mathbf{J}(-\mathbf{k})\cdot\boldsymbol{\varepsilon}$ for the transverse and $[\rho(\mathbf{k})-\mathbf{k}\cdot\mathbf{J}(\mathbf{k})/k]$ for the unphysical photon $a^\dagger_{\mathbf{k}Q}|0\rangle$. Its value is given by

$$T_\chi(e_{\mathbf{k}',r'}\gamma_{\mathbf{q}'Q}|e_{\mathbf{k},r}\gamma_{\mathbf{q},t})$$

$$=\frac{\bar{u}(\mathbf{k}',r)}{(2)^{3/2}(q')^{3/2}(q)^{1/2}}\Bigg\{\gamma_4\Bigg[\sum_j u(\mathbf{k}+\mathbf{q},j)\bar{u}(\mathbf{k}+\mathbf{q},j)\frac{(q'+\omega_{k'}-\omega_{|\mathbf{k}+\mathbf{q}|})}{(q+\omega_k-\omega_{|\mathbf{k}+\mathbf{q}|})}+\sum_j v(-(\mathbf{k}+\mathbf{q}),j)\bar{v}(-(\mathbf{k}+\mathbf{q}),j)\Bigg]\gamma\cdot\epsilon$$

$$-\gamma\cdot\epsilon\Bigg[\sum_j u(\mathbf{k}-\mathbf{q}',j)\bar{u}(\mathbf{k}-\mathbf{q}',j)+\sum_j v(\mathbf{q}'-\mathbf{k},j)\bar{v}(\mathbf{q}'-\mathbf{k},j)\frac{(q'-\omega_k-\omega_{|\mathbf{k}-\mathbf{q}'|})}{(q-\omega_{k'}-\omega_{|\mathbf{k}-\mathbf{q}'|})}\Bigg]\gamma_4\Bigg\}u(\mathbf{k},r)$$

$$=\frac{\bar{u}(\mathbf{k}',r)}{(2)^{3/2}(q')^{3/2}(q)^{1/2}}\Bigg\{\gamma_4\Bigg[\sum_j\frac{u(\mathbf{k}+\mathbf{q},j)\bar{u}(\mathbf{k}+\mathbf{q},j)}{(q+\omega_k-\omega_{|\mathbf{k}+\mathbf{q}|})}\Bigg]\gamma\cdot\epsilon-\gamma\cdot\epsilon\Bigg[\sum_j\frac{v(\mathbf{q}'-\mathbf{k},j)\bar{v}(\mathbf{q}'-\mathbf{k},j)}{(q-\omega_{k'}-\omega_{|\mathbf{k}-\mathbf{q}'|})}\Bigg]\gamma_4\Bigg\}u(\mathbf{k},r)(q'+\omega_{k'}-q-\omega_k). \quad (22)$$

It is easy to see that $T_\chi(e_{\mathbf{k}',r'}\gamma_{\mathbf{q}'Q}|e_{\mathbf{k},r}\gamma_{\mathbf{q},t})$ vanishes on the energy shell when $\omega_k+q=\omega_{k'}+q'$. However, at other times it does not vanish.

At finite times t, $\psi_\chi(t)$ has components involving unphysical photons, and the probability of observing them is not zero. If, however, we evaluate

$$T_\Omega=\langle e_{\mathbf{k}'r'}\gamma_{\mathbf{q}'Q}|D|l\rangle\langle l|H_1|e_{\mathbf{k},r}\gamma_{\mathbf{q},t}\rangle \times(E_l-\omega_k-q)^{-1}(\omega_k+q-\omega_{k'}-q'),$$

then we observe that even *off* the energy shell $T_\Omega=-T_\chi$ and the total wave function $\psi(t)$ *never* enters the unphysical space at all.

Note, incidentally, that the probability amplitude for the existence of an R-type photon in $\psi_\chi(t)$ does not vanish even on the energy shell. The corner term for this photon type in the S matrix is $i\gamma\cdot q^*=(i\boldsymbol{\gamma}\cdot\mathbf{q}+\gamma_4 q)$; or since the amplitude involving an unphysical photon is $i\gamma\cdot q$, and leads to a vanishing S-matrix element, $i\gamma\cdot q^*$ can be written $2\gamma_4 q$. The R-type photon of course is physical though not transverse and its presence is entirely consistent with the subsidiary condition. Its presence can never affect probabilities since $\langle a_{\mathbf{q}R}^\star|a_{\mathbf{q}R}\rangle=0$; the matrix element $\langle a_{\mathbf{q}Q}^\star|a_{\mathbf{q}R}\rangle\neq0$, but such can never arise in the evaluation of probabilities since the Q-type photon may never appear. The crucial distinction between Q- and R-type photons in ket vectors has not always been made in the literature.[8] The appearance of a Q-type photon in the asymptotic state $t\rightarrow\infty$ is a catastrophe; the appearance of R-type photons is perfectly all right.

V. SCATTERING MATRIX

The scattering matrix for stable particles vanishes except on the mass shells of participating particles, and in the energy continuum it is possible to prove that this quantity, when evaluated from the viewpoint of the proper, new, subsidiary condition [Eq. (13)], remains identical to the conventional S matrix. The expression for the S matrix for a transition from an initial state $|\nu_i\rangle$ to a final state $|\nu_f\rangle$, for particles having only electromagnetic interactions, is

$$\tilde{S}(f,i)=\langle\nu_f^\star|\mathcal{T}\left\{\exp\left[-i\int d_4x\,\tilde{\mathcal{H}}_1(x)\right]\right\}|\nu_i\rangle. \quad (23)$$

Here $\mathcal{T}$ denotes the time-ordered product and $\mathcal{H}_1(\mathbf{x})$ denotes the interaction Hamiltonian density $[\mathcal{H}_1=\int d\mathbf{x}\,\mathcal{H}_1(\mathbf{x})]$ in the interaction picture. It is crucial in this connection to be precise about what the time-evolution operator in the interaction picture is. It is given by

$$\tilde{\mathcal{H}}_1(t)=\exp(i\mathcal{H}_0t)\mathcal{H}_1\exp(-i\mathcal{H}_0t),$$

where $\mathcal{H}_0$ is that "free field" Hamiltonian of which the noninteracting particle configuration is an eigenstate. It is important to recognize the difference between this and the interaction picture $\exp(iH_0t)\mathcal{H}_1\exp(-iH_0t)$. We denote the operator $\exp(iH_0t)\mathcal{H}_1\exp(-iH_0t)$ by $\mathcal{H}_1(t)$ and $\exp(i\mathcal{H}_0t)\mathcal{H}_1\exp(-i\mathcal{H}_0t)$ by $\tilde{\mathcal{H}}_1(t)$.

In order to prove the main theorem, it is necessary to prove the following lemmas[9]:

Lemma 1.

$$\langle\beta^\star|\mathcal{T}\left\{\exp\left[-i\int_{t_1}^{t_2}dt\left(\tilde{H}_1(t)+i\frac{dD}{dt}\right)\right]\right\}|\alpha\rangle =\langle\beta^\star|e^{D(t_2)}\mathcal{T}\left\{\exp\left(-i\int_{t_1}^{t_2}dt\,\tilde{K}(t)\right)\right\}e^{-D(t_1)}|\alpha\rangle,$$

where

$$\tilde{K}(t)=e^{-D(t)}\tilde{H}_1(t)e^{D(t)}.$$

If we write

$$\mathcal{T}\left\{\exp\left[-i\int_{t_1}^{t_2}dt\left(\tilde{H}_1(t)+i\frac{dD}{dt}\right)\right]\right\}=\Sigma(t_2,t_1)$$

and let

$$\sigma(t_2,t_1)=e^{-D(t_2)}\Sigma(t_2,t_1)e^{D(t_1)},$$

straightforward differentiation leads us to

$$d\sigma(t_2,t_1)/dt_1=i\sigma(t_2,t_1)\tilde{K}(t_1)$$

and

$$d\sigma(t_2,t_1)/dt_2=-i\tilde{K}(t_2)\sigma(t_2,t_1),$$

which can be integrated to give

$$\sigma(t_2,t_1)=\mathcal{T}\left\{\exp\left(-i\int_{t_1}^{t_2}dt\,\tilde{K}(t)\right)\right\}.$$

This proves the lemma.

In our case, $\mathcal{H}_1=H_1-[H_0,D]$ and

$$\tilde{\mathcal{H}}_1(t)=\tilde{H}_1(t)-\exp[i\mathcal{H}_0t][H_0,D]\exp[-i\mathcal{H}_0t].$$

D commutes with $[H_0,D]$, $\tilde{D}(t)=D(t)$, and

$$e^{iH_0t}[H_0,D]e^{-iH_0t}=idD(t)/dt=H_1(t)-\mathcal{H}_1(t).$$

Hence we have that

$$\tilde{S}(f,i)=\lim_{t_2\to+\infty,\,t_1\to-\infty}\langle\nu_f^\star|e^{D(t_2)} \times\mathcal{T}\left\{\exp\left(-i\int_{t_1}^{t_2}dt\,\tilde{K}(t)\right)\right\}e^{-D(t_1)}|\nu_i\rangle. \quad (24)$$

We now turn to the second lemma, which has to do with the time evolution of the operator $e^{D(t)}$.

Lemma 2.

$$\lim_{t\to\pm\infty}e^{\pm D(t)}|\nu\rangle=|\nu\rangle.$$

[8] I. Bialynicki-Birula, Phys. Rev. 155, 1414 (1967).

[9] An incorrect version of this theorem is given in M. Zulauf, Helv. Phys. Acta 39, 439 (1966) [Eqs. (40), (41)]; H_1 appears instead of K in this reference.

We can write

$$e^{D(t)}|\nu\rangle=\exp(i\mathcal{H}_0t)e^{D(0)}\exp(-i\mathcal{H}_0t)|\nu\rangle,$$

where

$$(\mathcal{H}_0-E)|\nu\rangle=0.$$

If this expression approaches a limit as $t\rightarrow+\infty$, this limit is given by[10]

$$e^{D(\infty)}|\nu\rangle=\lim_{t\to\infty}e^{D(t)}|\nu\rangle=\lim_{\epsilon\to 0}\epsilon\int_0^\infty dt\,e^{i[D(t)-\epsilon t]}|\nu\rangle$$

and

$$e^{D(\infty)}|\nu\rangle=[1-(\mathcal{H}_0-E+i\epsilon)^{-1}(\mathcal{H}_0-E)]e^{D(0)}|\nu\rangle.$$

Since $e^{D(0)}|\nu\rangle=|n\rangle$, and since $(H_0-E)|n\rangle=0$, we have

$$e^{D(\infty)}|\nu\rangle=[1+(\mathcal{H}_0-E+i\epsilon)^{-1}U]|n\rangle,$$

where

$$\mathcal{H}_0=H_0+U,\quad U=[H_0,D].$$

We now recognize that $[1+(\mathcal{H}_0-E+i\epsilon)^{-1}(\mathcal{H}_0-H_0)]$ is an operator (the Møller operator) which forms an eigenstate of $\mathcal{H}_0$ from an eigenstate of H_0, where both have the same eigenvalue of the operators $\mathcal{H}_0$ and H_0, respectively (the sign of $i\epsilon$ in this case does not matter since both states are plane-wave and no real scattering is mediated by U). We therefore have $\exp[D(\infty)|\nu\rangle=|\nu\rangle$ and the lemma is proven; the case of $\lim t\rightarrow-\infty$ is proven the same way.

It is possible to prove this identity in a second, very direct, way. We expand the Møller operator[10a]

$$\{1-(\mathcal{H}_0-E+i\epsilon)^{-1}[H_0,D]\}=1+\sum_{l=1}^{\infty}\mathfrak{M}_l,$$

where

$$\mathfrak{M}_l=G_0[H_0,D]\cdots G_0[H_0,D].$$

Here $G_0=(E-H_0)^{-1}$, and G_0 and $[H_0,D]$ each appear l times. We shall prove that $\mathfrak{M}_l=[-D]^l/l!$. For $l=1$ we have

$$(E-H_0)^{-1}[H_0,D]|n\rangle=-D|n\rangle.$$

If we assume that $\mathfrak{M}_l=[-D]^l/l!$, we then have

$$\begin{aligned}\mathfrak{M}_{(l+1)}&=G_0[H_0,D]\mathfrak{M}_l=(-1)^lG_0[H_0,D]D^l/l!\\&=G_0(-1)^l(H_0-E)D^{l+1}/l!\\&\qquad+G_0\{(-1)^l/l!\}[D^l,H_0]D.\end{aligned}$$

Since $[D,H_0]$ commutes with D, we have

$$[D^l,H_0]=lD^{l-1}[D,H_0]$$

and

$$\mathfrak{M}_{(l+1)}=(-1)^{l+1}D^{l+1}/l!+(-1)^l\{G_0/(l-1)!\}D^l[D,H_0].$$

The second term on the right side can be rewritten

$$\{(-1)^lG_0D^l(E-H_0)D/(l-1)!\}.$$

Since $[H_0,D]$ and D commute, $\mathfrak{M}_{(l+1)}$ can be rewritten as

$$\mathfrak{M}_{(l+1)}=(-1)^lG_0D^l(H_0-E)D/l!$$

or

$$\mathfrak{M}_{(l+1)}=(-1)^{l+1}D^{l+1}/l!-l\mathfrak{M}_{(l+1)}.$$

This leads to

$$\mathfrak{M}_{(l+1)}=(-1)^{l+1}D^{l+1}/(l+1)!$$

and proves that

$$1+\sum_{l=1}^{\infty}\mathfrak{M}_l=e^{-D}.$$

Returning to Eq. (24), we now have

$$\tilde{S}(f,i)=\langle\nu_f{}^\star|\mathcal{T}\left\{\exp\left(-i\int_{-\infty}^{+\infty}dt\,\tilde{K}(t)\right)\right\}|\nu_i\rangle \qquad (25a)$$

or

$$\tilde{S}_l(f,i)=\frac{(-i)^l}{l!}\mathcal{T}\int_{-\infty}^{+\infty}dt_1\cdots dt_l\{\tilde{K}(t_1)\cdots\tilde{K}(t_l)\}. \qquad (25b)$$

We must now write $\tilde{K}(t)$ in terms of $K(t)$ in order to bring this expression into the familiar form in which the interaction operators are written in the interaction picture $\exp[iH_0t]\cdots\exp[-iH_0t]$. For this purpose we note that

$$\tilde{K}(t)=e^{-D(t)}\tilde{H}_1(t)e^{D(t)}, \qquad (26)$$

and since

$$e^{-D(t)}=\exp(i\mathcal{H}_0t)e^{-D(0)}\exp(-i\mathcal{H}_0t),$$

we have

$$\tilde{K}(t)=\exp(i\mathcal{H}_0t)e^{-D(0)}H_1e^{+D(0)}\exp(-i\mathcal{H}_0t). \qquad (27)$$

Since, moreover, $\mathcal{H}_0e^{-D(0)}=e^{-D(0)}H_0$, we have

$$\tilde{K}(t)=e^{-D(0)}e^{iH_0t}H_1e^{-iH_0t}e^{D(0)}$$

and

$$\tilde{K}(t)=e^{-D(0)}H_1(t)e^{D(0)}. \qquad (28)$$

Substituting Eq. (28) into Eq. (25b), we have

$$\tilde{S}(f,i)=\langle\nu_f{}^\star|e^{-D(0)}\mathcal{T}\left\{\exp\left(-i\int_{-\infty}^{+\infty}d_4x\,H_1(x)\right)\right\}e^{D(0)}|\nu_i\rangle.$$

Since $e^{D(0)}|\nu\rangle=|n\rangle$, we can now prove the relation

$$\tilde{S}(f,i)=\langle n_f{}^\star|\mathcal{T}\left\{\exp\left(-i\int_{-\infty}^{+\infty}d_4x\,H_1(x)\right)\right\}|n_i\rangle. \qquad (29)$$

In the energy continuum, $\tilde{S}(f,i)$ is therefore equal to $S(f,i)$, the S-matrix element for this transition derivable from the old subsidiary condition.

VI. PHYSICAL CONSEQUENCES OF THE NEW SUBSIDIARY CONDITION

In this section, we address ourselves to the question of what observable consequences we can expect from

[10] M. Gell-Mann and M. L. Goldberger, Phys. Rev. 91, 398 (1953).

[10a] *Note added in proof.* The following argument assumes that the various operators act on an eigenstate $|n\rangle$ for which $(H_0-E)|n\rangle=0$.

the nonpersistence of the old subsidiary condition, Eq. (1).

One source of such effects is in the electromagnetic decay of short-lived particles. The spread of the final-state energies over a resonant spectral curve is due to the detection of the "final" state after a finite time and should show the effects of any leakage of state vectors into the unphysical subspace. Specifically, the term

$$\langle n_f^\star | D | l\rangle\langle l | H_1 | n_i\rangle (E_i - E_l)^{-1}(E_i - E_f),$$

which is a contribution to the second-order transition amplitude connects states $|n_i\rangle$ and $|n_f\rangle$ which are both in the physical subspace (and represent detectable particles). This term does not vanish when the spectrum of energies of final states properly represents the finite lifetime of the decaying particle, though it is an open question whether it will have a measurable effect in any actual decays.

Another direction along which to look for possible physical effects of the new subsidiary condition is the scattering or decay, via electromagnetic forces, of systems held together by the strong interactions. In that case we derive a final-state theorem, if we wish the transition amplitude to be exact in the strong interaction and wish to evaluate it to some definite order in e_0. Let us consider a Hamiltonian $\mathbf{H} = \mathcal{H}_0 + \mathcal{H}_1 + V$, where V is some strong-interaction Hamiltonian. Consider an asymptotic state $|\nu_i\rangle$ corresponding to a scattering state Ψ_i which is an eigenstate $(\mathbf{H} - E)\Psi_i = 0$. The relation between these two quantities is given by

$$\Psi_i = [1 + (E - \mathbf{H} + i\epsilon)^{-1}(\mathbf{H} - E)]|\nu_i\rangle. \quad (30)$$

If we choose the state $|\nu_i\rangle$ to be $\exp[-D]|n_i\rangle$, then the transition amplitude to a state $|\nu_f\rangle$ is given by

$$\mathbf{T}(f,i) = \langle n_f^\star | e^D(\mathbf{H} - E) \times [1 + (E - \mathbf{H} + i\epsilon)^{-1}(\mathbf{H} - E)]e^{-D}|n_i\rangle. \quad (31)$$

Since

$$\mathcal{H}_0 e^{-D} = e^{-D} H_0$$

and

$$\mathcal{H}_1 e^{-D} = e^{-D}\{\mathcal{H}_1 + \mathcal{F}\},$$

where $\mathcal{F} = [D, \mathcal{H}_1]$, we have

$$\mathbf{T}(f,i) = \langle n_f^\star | e^D(\mathbf{H} - E)\{e^{-D} + (E - \mathbf{H} + i\epsilon)^{-1} \times [e^{-D}(H_0 + \mathcal{H}_1 + \mathcal{F} - E) + Ve^{-D}]\}|n_i\rangle. \quad (32a)$$

If we write

$$e^D V e^{-D} = V + \theta + \alpha,$$

where

$$\theta = [D, V]$$

and

$$\alpha = \tfrac{1}{2}[D,[D,V]] + (1/3!)[D,[D,[D,V]]] + \cdots + (1/n!)[D, \cdots [D,V]\cdots] + \cdots,$$

then we have

$$Ve^{-D} = e^{-D}[V + \theta + \alpha]$$

and

$$\mathbf{T}_{f,i} = \langle n_f^\star | e^D(\mathbf{H} - E)\{e^{-D} + (E - \mathbf{H} + i\epsilon)^{-1} \times e^{-D}[H_0 + \mathcal{H}_1 + \mathcal{F} + V + \theta + \alpha - E]\}|n_i\rangle. \quad (32b)$$

Similarly,

$$(E - \mathbf{H} + i\epsilon)^{-1}e^{-D} = e^{-D}(E - H_0 - \mathcal{H}_1 - \mathcal{F} - \theta - V - \alpha + i\epsilon)^{-1},$$

and if we let

$$H_0 + V = H_S,$$

we have (since $[D,H_1] = [D,\mathcal{H}_1]$)

$$\mathbf{T}_{f,i} = \langle n_f^\star | e^D(\mathbf{H} - E)e^{-D} \times \{1 + (E - H_S - \mathcal{H}_1 - [D,H_1] - [D,V] - \alpha + i\epsilon)^{-1} \times (H_S - E + \mathcal{H}_1 + [D,H_1] + [D,V] + \alpha)\}|n_i\rangle. \quad (32c)$$

This can be rewritten as

$$\mathbf{T}_{f,i} = \langle n_f^\star | \{H_S - E + H_1 + [D,H_S] + [D,H_1] + \alpha\} \times \{1 + (E - H_S - H_1 - [D,H_S] - [D,H_1] - \alpha + i\epsilon)^{-1} \times (H_S - E + H_1 + [D,H_S] + [D,H_1] + \alpha)\}|n_i\rangle. \quad (32d)$$

The on-the-energy-shell value of $\mathbf{T}_{f,i}$ to second order in the electric charge is denoted by $\mathbf{T}_{f,i}^{(2)}$, which is given by

$$\mathbf{T}_{f,i}^{(2)} = \langle \xi_f^{(-)} | [D,H_1] | \xi_i^{(+)}\rangle + \langle \xi_f^{(-)} | \{H_1 + [D,H_S]\}(E - H_S + i\epsilon)^{-1} \times \{H_1 + [D,H_S]\} | \xi_i^{(+)}\rangle + \tfrac{1}{2}\langle \xi_f^{(-)} | [D,[D,V]] | \xi_i^{(+)}\rangle, \quad (33)$$

where

$$\xi_i^{(+)} = [1 + (E - H_S + i\epsilon)^{-1}V]|n_i\rangle$$

and

$$\xi_f^{(-)} = [1 + (E - H_S - i\epsilon)^{-1}V]|n_f\rangle.$$

The first two terms on the right side of Eq. (33) lead to identical results as the right side of Eq. (20). This means that the on-the-energy-shell value of $\mathbf{T}_{f,i}^{(2)}$ is given by

$$\mathbf{T}_{f,i}^{(2)} = T_{f,i}^{(2)} + \Delta_{f,i}^{(2)}, \quad (34)$$

where

$$T_{f,i}^{(2)} = \langle \xi_f^{(-)} | H_1(E - H + i\epsilon)^{-1}H_1 | \xi_i^{(+)}\rangle$$

and

$$\Delta_{f,i}^{(2)} = \tfrac{1}{2}\langle \xi_f^{(-)} | [D,[D,V]] | \xi_i^{(+)}\rangle.$$

$T_{f,i}^{(2)}$ is identical to what the old formulation of QED in the Lorentz gauge would have given; $\Delta_{f,i}^{(2)}$ is an additional on-the-energy-shell term which appears in this new formulation only. Since V is independent of the photon field, we can write

$$\Delta_{f,i}^{(2)} = \tfrac{1}{8}\sum_{\mathbf{k},\mathbf{k}'}(kk')^{-3/2} \times \langle \xi_f^{(-)\star} | a^\dagger_{\mathbf{k},Q} a^\dagger_{\mathbf{k}',Q}[\rho(\mathbf{k}),[\rho(\mathbf{k}'),V]] | \xi_i^{(+)}\rangle. \quad (35)$$

To this order, the asymptotic state, as $t \to +\infty$, will include

$$\psi_\Delta(+\infty) = -2\pi i \int dE_f dE_i \delta(E_f - E_i) g(E_i)\Delta_{f,i}^{(2)} \times \langle \xi_f^{(-)\star} | \xi_f^{(-)}\rangle | \xi_f^{(-)}\rangle e^{-iE_f t},$$

which can be written as

$$\psi_\Delta(+\infty)=-2\pi i\int dE_f dE_i \delta(E_f-E_i)g(E_i)\times\tfrac{1}{8}\sum_{\mathbf{k},\mathbf{k}'}(kk')^{-3/2}$$
$$\times\langle\xi_f^{(-)}|a^\dagger{}_{\mathbf{k}Q}a^\dagger{}_{\mathbf{k}'Q}[\rho(\mathbf{k}),[\rho(\mathbf{k}'),V]]|\xi_i^{(+)}\rangle|\xi_f^{(-)}\rangle e^{-iE_f t}.$$

In contrast to the case of pure electromagnetic interactions, in this case there are differences between the final asymptotic states of colliding systems as evaluated on the basis of the old and the new formulation of the theory. In this case these differences will appear when the final-state ket vector $|\xi_f^{(-)}\rangle$ contains two Q-type photons and provided of course that the operator $[\rho(\mathbf{k}),[\rho(\mathbf{k}),V]]$ does not vanish. It is only for unusual interaction Hamiltonians H_S for which this double commutator is not zero; in particular, in the case of strong interactions that are derivative-free and local, $[\rho(\mathbf{k}),[\rho(\mathbf{k}),V]]$ does disappear.

These terms [i.e., $\psi_\Delta(+\infty)$], when they do not vanish, presumably mean that the subsidiary condition $\chi^{(+)}(\mathbf{x})|n\rangle=0$ is not succeeding in forcing the state vector back into the physical space of the theory. This circumstance therefore raises serious questions not only about the physical interpretability of the Gupta theory for this case, but also about the gauge invariance of transition amplitudes in the Gupta theory for this type of process. None of this, however, should be taken as a criticism of the strong-interaction Hamiltonian involved, since presumably the use of $\Omega^{(+)}(\mathbf{x})|\nu\rangle=0$ as a subsidiary condition should lead to consistent and gauge-invariant results in these cases too.

The authors also believe that the effect of the new subsidiary condition on field-theoretic corrections to bound states should be reexamined to determine whether these are in any way affected by the change in subsidiary condition.[11]

APPENDIX A: COVARIANCE OF THE SUBSIDIARY CONDITION

An operator $\mathcal{O}(\mathbf{x})$ in the Schrödinger picture is defined to be a four-dimensional scalar operator if its expectation values (when taken with state vectors that obey the equations of motion) are four-dimensional scalar functions. When this criterion is applied, $\Omega^{(+)}(\mathbf{x})$ can be seen to be such a scalar operator, though $\chi^{(+)}(\mathbf{x})$ is not.

The commutation relations that $\mathcal{O}(\mathbf{x})$ must satisfy with the generators of the Lorentz group, in order to satisfy the preceding criterion for a scalar operator, are

$$[P_k,\mathcal{O}(\mathbf{x})]=-i\partial\mathcal{O}(\mathbf{x})/\partial x_k, \tag{A1}$$

$$[J_k,\mathcal{O}(\mathbf{x})]=-i\varepsilon_{kln}x_l\partial\mathcal{O}(\mathbf{x})/\partial x_n, \tag{A2}$$

and

$$[M_k,\mathcal{O}(\mathbf{x})]=x_k[H,\mathcal{O}(\mathbf{x})]. \tag{A3}$$

The first two conditions are usually trivial and can be verified by inspection. The only condition that requires careful scrutiny is (A3). It obviously arises from the pure Lorentz (velocity) transformation. In the case of QED, M_k is given by[12]

$$\int d\mathbf{x}\{x_k H(\mathbf{x})-i[\Pi_4(\mathbf{x})A_k(\mathbf{x})-\Pi_k(\mathbf{x})A_4(\mathbf{x})]+\tfrac{1}{2}\bar{\psi}(\mathbf{x})\gamma_k\psi(\mathbf{x})\}. \tag{A4}$$

By making use of

$$[A_\mu^{(+)}(\mathbf{x}),A_\nu^{(-)}(\mathbf{y})]=\tfrac{1}{2}\delta_{\mu,\nu}\mathfrak{D}(\mathbf{x}-\mathbf{y}), \tag{A5a}$$

$$[\Pi_\mu^{(+)}(\mathbf{x}),\Pi_\nu^{(-)}(\mathbf{y})]=-\tfrac{1}{2}\delta_{\mu,\nu}\nabla^2\mathfrak{D}(\mathbf{x}-\mathbf{y}), \tag{A5b}$$

$$[A_\mu^{(+)}(\mathbf{x}),\Pi_\nu^{(-)}(\mathbf{y})]=\tfrac{1}{2}i\delta_{\mu,\nu}\delta(\mathbf{x}-\mathbf{y}), \tag{A5c}$$

and

$$\Pi_\mu^{(+)}(\mathbf{x})=\frac{1}{2}\left\{\Pi_\mu(\mathbf{x})+i\int d\mathbf{y}\;\mathfrak{D}(\mathbf{x}-\mathbf{y})\nabla^2 A(\mathbf{y})\right\}, \tag{A6a}$$

$$A_\mu^{(+)}(\mathbf{x})=\frac{1}{2}\left\{A_\mu(\mathbf{x})+i\int d\mathbf{y}\;\mathfrak{D}(\mathbf{x}-\mathbf{y})\Pi_\mu(\mathbf{y})\right\}, \tag{A6b}$$

and the properties of $\mathfrak{D}(\mathbf{x}-\mathbf{y})$, we obtain

$$\begin{aligned}[M_l,\chi^{(+)}(\mathbf{x})]&=-ix_l\nabla\cdot\mathbf{\Pi}^{(+)}(\mathbf{x})-x_l\nabla^2A_4^{(+)}(\mathbf{x})-\tfrac{1}{2}ix_l\rho(\mathbf{x})\\&\quad+\frac{1}{2}\frac{\partial}{\partial x_n}\int d\mathbf{y}\,y_lJ_n(\mathbf{y})\mathfrak{D}(\mathbf{x}-\mathbf{y})\\&=x_l\left\{-i\nabla\cdot\mathbf{\Pi}^{(+)}(\mathbf{x})-\nabla^2A_4^{(+)}(\mathbf{x})-\tfrac{1}{2}i\rho(\mathbf{x})\right.\\&\quad\left.+\frac{1}{2}\int d\mathbf{y}\;\mathfrak{D}(\mathbf{x}-\mathbf{y})\nabla\cdot\mathbf{J}(\mathbf{y})\right\}\\&\quad+\frac{1}{2}\frac{\partial}{\partial x_l}\int d\mathbf{y}\;\mathcal{G}(\mathbf{x}-\mathbf{y})\nabla\cdot\mathbf{J}(\mathbf{y})\\&\quad+\frac{1}{2}\int d\mathbf{y}\;\mathfrak{D}(\mathbf{x}-\mathbf{y})J_l(\mathbf{y}),\end{aligned} \tag{A7}$$

where

$$\mathcal{G}(x-y)=(2\pi)^{-3}\int d\mathbf{k}k^{-3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}.$$

It can easily be shown that

$$[H,\chi^{(+)}(\mathbf{x})]=-i\nabla\cdot\mathbf{\Pi}^{(+)}(\mathbf{x})-\nabla^2A_4^{(+)}(\mathbf{x})-\tfrac{1}{2}i\rho(\mathbf{x})+\frac{1}{2}\int d\mathbf{y}\;\mathfrak{D}(\mathbf{x}-\mathbf{y})\nabla\cdot\mathbf{J}(\mathbf{y}) \tag{A8}$$

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[11] W. Heitler (Ref. 1, Appendix, Sec. 3) commented on the likelihood that the use of the $|n\rangle$ states for bound-state problems in quantum electrodynamics was incorrect and would lead to difficulties.

[12] A. I. Akhiezer and V. B. Berestetskii (Ref. 1, pp. 226 and 278).

and that

$$[M_l,\chi^{(+)}(\mathbf{x})]=x_l[H,\chi^{(+)}(\mathbf{x})]+\frac{1}{2}\frac{\partial}{\partial x_l}\int d\mathbf{y}\,\mathcal{G}(\mathbf{x}-\mathbf{y})\nabla\cdot\mathbf{J}(\mathbf{y})+\frac{1}{2}\int d\mathbf{y}\,\mathfrak{D}(\mathbf{x}-\mathbf{y})J_l(\mathbf{y}). \quad \text{(A9)}$$

This shows that $\chi^{(+)}(\mathbf{x})$ is *not* a four-dimensional scalar operator. Since

$$[M_l,\Omega^{(+)}(\mathbf{x})]=x_l\{-i\nabla\cdot\mathbf{\Pi}^{(+)}(\mathbf{x})-\nabla^2A_4{}^{(+)}(\mathbf{x})-\tfrac{1}{2}i\rho(\mathbf{x})\},$$

we find that

$$[M_l,\Omega^{(+)}(\mathbf{x})]=x_l[H,\Omega^{(+)}(\mathbf{x})], \quad \text{(A10)}$$

and $\Omega^{(+)}(\mathbf{x})$ is a four-dimensional scalar operator. Note that if H_0, rather than H, had governed the time evolution of the system, $\chi^{(+)}(\mathbf{x})$ would have satisfied the criterion for a four-dimensional scalar operator.

APPENDIX B: UNIT OPERATOR

In this Appendix, we first address ourselves to the problem of specifying the unit operator in the indefinite metric space. In the representation in which we refer to the photons as transverse, longitudinal, and timelike, (i.e., $a^\dagger{}_{\mathbf{k},\epsilon,i)},a^\dagger{}_{\mathbf{k},L},a^\dagger{}_{\mathbf{k},4}$) we find that the matrix elements $\langle 0|a_{\mathbf{k}',\alpha'}a^\dagger{}_{\mathbf{k},\alpha}|0\rangle=\delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha,\alpha'}$ and that $1=|l\rangle\langle l|=|l\rangle\langle l^\star|\eta_l$, where $\eta_l=\langle l|\eta|l\rangle=\langle l^\star|l\rangle$. This can easily be seen to be identical to $1=|l\rangle\langle l'^\star|$ in the representation in which the designation R- or Q-type photon is used; in the form $1=|l\rangle\langle l'^\star|$, the l and l' states are chosen so that for every Q-type photon operator in l, the corresponding R-type appears in l', and vice versa.

In the case of the $|\nu\rangle$ type states, in order to guarantee that orthogonal states appear in the unit operator, the form $1=|\lambda\rangle\langle\lambda'^\star|$ must be used since

$$\langle\alpha_{\mathbf{k}}'^\star|\alpha_{\mathbf{k}'}\rangle=\langle a_{\mathbf{k}}'^\star|\exp[D]\exp[-D]|a_{\mathbf{k}'}\rangle=\delta_{\mathbf{k},\mathbf{k}'}$$

but

$$\langle\alpha_{\mathbf{k},\beta}|\alpha_{\mathbf{k}',\beta'}\rangle=\langle a_{\mathbf{k},\beta}|\exp[-D^\dagger]\exp[-D]|a_{\mathbf{k}',\beta'}\rangle\neq\delta_{\mathbf{k},\mathbf{k}'}\delta_{\beta,\beta'}.$$

APPENDIX C: ALTERNATIVE TRANSFORMATIONS TO THE "FREE FIELD" SUBSIDIARY CONDITION

We have used the pseudo-unitary transformation e^D to effect a simplification of the set of states satisfying the subsidiary condition. In this Appendix, we consider some alternative transformations and compare them with e^D.

First let us consider $\mathfrak{U}=e^{iF}$, where

$$F=-i(4\pi)^{-1}\int d\mathbf{x}d\mathbf{y}\,\rho(\mathbf{x})(|\mathbf{x}-\mathbf{y}|)^{-1}\Pi_4(\mathbf{y}).$$

Note that $F^\dagger=-F$ but $F^\star=F$, so that the transformation is pseudo-unitary. Now, since

$$\mathfrak{U}^{-1}\Pi_4{}^{(+)}(\mathbf{x})\mathfrak{U}=\Pi_4{}^{(+)}(\mathbf{x})+\frac{1}{2}\int d\mathbf{y}\,\mathfrak{D}(\mathbf{x}-\mathbf{y})\rho(\mathbf{y}), \quad \text{(C1a)}$$

$$\mathfrak{U}^{-1}A_4{}^{(+)}(\mathbf{x})\mathfrak{U}=A_4{}^{(+)}(\mathbf{x})+i(8\pi)^{-1}\int d\mathbf{y}\,|\mathbf{x}-\mathbf{y}|^{-1}\rho(\mathbf{y}), \quad \text{(C1b)}$$

and

$$\mathfrak{U}^{-1}A_i{}^{(+)}(\mathbf{x})\mathfrak{U}=A_i{}^{(+)}(\mathbf{x}), \quad \text{(C1c)}$$

we have

$$\mathfrak{U}^{-1}\Omega^{(+)}(\mathbf{x})\mathfrak{U}=\chi^{(+)}(\mathbf{x}), \quad \text{(C2a)}$$

and

$$\mathfrak{U}^{-1}[H,\Omega^{(+)}(\mathbf{x})]\mathfrak{U}=-i\nabla\cdot\mathbf{\Pi}^{(+)}(\mathbf{x})-\nabla^2A_4{}^{(+)}(\mathbf{x})=[H_0,\chi^{(+)}(\mathbf{x})]. \quad \text{(C2b)}$$

Since the transformation $\mathfrak{U}$ is pseudo-unitary it is clear that if $|\bar{\nu}\rangle=\mathfrak{U}|n\rangle$, when the $|n\rangle$ are normalized, the $|\bar{\nu}\rangle$ are too. However, we can ask whether the states $|\bar{\nu}\rangle$ are expandable in terms of the states $|n\rangle$, i.e., whether the inner product $\langle n^\star|\bar{\nu}'\rangle=\langle n|\eta\mathfrak{U}|n'\rangle$ exists, where $|n\rangle$, $|n'\rangle$ are two states in the set satisfying the "free" Lorentz condition [Eq. (1)].

Let us write

$$F=F^{(+)}+F^{(-)},$$

where

$$F^{(\pm)}=-i(4\pi)^{-1}\int d\mathbf{x}d\mathbf{y}\,\rho(\mathbf{x})(|\mathbf{x}-\mathbf{y}|)^{-1}\pi_4{}^{(\pm)}(\mathbf{y})$$

and

$$[F^{(-)},F^{(+)}]=\frac{1}{2}\int d\mathbf{x}d\mathbf{y}\,\rho(\mathbf{x})\mathcal{G}(\mathbf{x}-\mathbf{y})\rho(\mathbf{y}),$$

where

$$\mathcal{G}(\mathbf{x}-\mathbf{y})=(2\pi)^{-3}\int d\mathbf{k}\,k^{-3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}.$$

Clearly,

$$[F^{(\pm)},[F^{(-)},F^{(+)}]]=0$$

and, therefore,

$$\exp iF=\exp[iF^{(-)}]\exp[iF^{(+)}]\exp\tfrac{1}{2}[F^{(-)},F^{(+)}].$$

Hence we have

$$\langle n^\star|e^{iF}|n'\rangle=\langle n|e^{\frac{1}{2}[F^{(-)},F^{(+)}]}|n'\rangle$$

if there are no timelike photons in $|n\rangle$, $|n'\rangle$. Thus

$$\langle n^\star|\bar{\nu}'\rangle=\langle n|\exp\left\{\frac{1}{4}\int d\mathbf{x}d\mathbf{y}\,\rho(\mathbf{x})\mathcal{G}(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})\right\}|n'\rangle.$$

Note that $\int d\mathbf{x}d\mathbf{y}\rho(\mathbf{x})\mathcal{G}(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})$ is a positive semidefinite operator and is, in fact, infinite; e.g., for the vacuum state,

$$\begin{aligned}\langle 0|\int d\mathbf{x}d\mathbf{y}\,\rho(\mathbf{x})\mathcal{G}(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})|0\rangle&=\sum_l\int d\mathbf{x}d\mathbf{y}\langle 0|\rho(\mathbf{x})|l\rangle\mathcal{G}(\mathbf{x}-\mathbf{y})\langle l|\rho(\mathbf{y})|0\rangle\\&=\sum_l\int d\mathbf{x}d\mathbf{y}\,e^{i\mathbf{P}_l\cdot(\mathbf{x}-\mathbf{y})}\mathcal{G}(\mathbf{x}-\mathbf{y})|\langle 0|\rho(0)|l\rangle|^2\\&=\sum_l\int d\mathbf{R}d\mathbf{r}\,e^{i\mathbf{P}_l\cdot\mathbf{r}}\mathcal{G}(\mathbf{r})|\langle 0|\rho(0)|l\rangle|^2=\infty.\end{aligned}$$

Thus $\langle n^\star|\bar{\nu}'\rangle=\infty$.

Another transformation which leads from the interacting to free Lorentz condition has been used by Bleuler.[13] Let

$$\mathcal{W}=e^G,$$

where

$$G=-i(4\pi)^{-1}\int d\mathbf{x}d\mathbf{y}\,\rho(\mathbf{x})|\mathbf{x}-\mathbf{y}|^{-1}\nabla\cdot\mathbf{A}(\mathbf{y}).$$

Note that here $G^\dagger=G^\star=-G$ and e^G is both unitary and pseudo-unitary. From the relations

$$\mathcal{W}^{-1}\Pi_4^{(+)}(\mathbf{x})\mathcal{W}=\Pi_4^{(+)}(\mathbf{x}),\quad \mathcal{W}^{-1}A_4^{(+)}(\mathbf{x})\mathcal{W}=A_4^{(+)}(\mathbf{x}),$$

$$\mathcal{W}^{-1}A_j^{(+)}(\mathbf{x})\mathcal{W}=A_j^{(+)}(\mathbf{x})+i(8\pi)^{-1}\int d\mathbf{x}'d\mathbf{y}'\rho(\mathbf{x}')\frac{\partial}{\partial x_j}\times\{|\mathbf{x}'-\mathbf{y}'|^{-1}\mathfrak{D}(\mathbf{x}-\mathbf{y}')\},$$

$$\mathcal{W}^{-1}\Pi_j^{(+)}(\mathbf{x})\mathcal{W}=\Pi_j^{(+)}(\mathbf{x})-(8\pi)^{-1}\frac{\partial}{\partial x_j}\int d\mathbf{y}\,\rho(\mathbf{y})|\mathbf{x}-\mathbf{y}|^{-1},$$

we find, again,

$$\mathcal{W}^{-1}\Omega^{(+)}(\mathbf{x})\mathcal{W}=\chi^{(+)}(\mathbf{x}),$$
$$\mathcal{W}^{-1}[H,\Omega^{(+)}(\mathbf{x})]\mathcal{W}=[H_0,\chi^{(+)}(\mathbf{x})]$$

so that the states satisfying the free Lorentz condition $|n\rangle$ are pseudo-unitarily equivalent to the states satisfying the interacting Lorentz condition. Once again, if the states $|n\rangle$ are normalized, so are the states $|\bar{\nu}\rangle=\mathcal{W}|n\rangle$. But, once again, let us compute the inner product of a free and interacting Lorentz state

$$\langle n^\star|\bar{\nu}'\rangle=\langle n^\star|e^G|n'\rangle.$$

We use the same trick that we used previously. Let

$$G=G^{(+)}+G^{(-)},$$

where

$$G^{(\pm)}=-i(4\pi)^{-1}\int d\mathbf{x}d\mathbf{y}\,\rho(\mathbf{x})|\mathbf{x}-\mathbf{y}|^{-1}\nabla\cdot\mathbf{A}^{(\pm)}(\mathbf{y}).$$

We now find

$$[G^{(-)},G^{(+)}]=\frac{1}{2}\int d\mathbf{x}d\mathbf{y}\,\rho(\mathbf{x})\mathcal{G}(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})$$

and $[G^{(\pm)},[G^{(-)},G^{(+)}]]=0$. Hence

$$\langle n^\star|\bar{\nu}\rangle=0.$$

The characteristic of D that makes it so useful is that $[D^{(+)},D^{(-)}]=0$ and hence $\langle n^\star|\nu'\rangle=\delta_{n,n'}$.

APPENDIX D: RELATION BETWEEN THE LORENTZ AND THE COULOMB GAUGE

In the body of this paper we have related state vectors $|\nu\rangle$ which obey Eq. (13) to others, $|n\rangle$, which obey Eq. (1), by the pseudo-unitary transformation $|\nu\rangle=e^{-D}|n\rangle$. The expectation value of H taken with the physical states $|\nu\rangle$ is $\langle\nu^\star|H|\nu\rangle$ and this can also be regarded as $\langle n^\star|\hat{H}|n\rangle$, where $\hat{H}=e^{-D}He^D$; $\hat{H}$ is given by

$$\begin{aligned}\hat{H}=H+(8\pi)^{-1}\int d\mathbf{x}d\mathbf{y}&\times[\nabla\cdot\mathbf{A}(\mathbf{y})+i\Pi_4(\mathbf{y})]|\mathbf{x}-\mathbf{y}|^{-1}\nabla\cdot\mathbf{J}(\mathbf{x})\\&+i(8\pi)^{-1}\int d\mathbf{x}d\mathbf{y}\,\rho(\mathbf{y})\left\{\frac{\partial}{\partial x_k}|\mathbf{x}-\mathbf{y}|^{-1}\right\}\\&\times\left[\frac{\partial}{\partial x_k}A_4(\mathbf{x})-i\Pi_k(\mathbf{x})\right]\\&+(8\pi)^{-1}\int d\mathbf{x}d\mathbf{y}\,\rho(\mathbf{x})|\mathbf{x}-\mathbf{y}|^{-1}\rho(\mathbf{y}).\quad\text{(D1)}\end{aligned}$$

The meaning of $\hat{H}$ can be clarified if we introduce the new fields

$$\mathfrak{A}_i(\mathbf{x})=A_i(\mathbf{x})+(8\pi)^{-1}\frac{\partial}{\partial x_i}\int d\mathbf{y}\times|\mathbf{x}-\mathbf{y}|^{-1}[\nabla\cdot\mathbf{A}(\mathbf{y})+i\Pi_4(\mathbf{y})],\quad\text{(D2a)}$$

[13] K. Bleuler (Ref. 2).

$$\mathfrak{A}_4(\mathbf{x})=A_4(\mathbf{x})-(8\pi)^{-1}\int d\mathbf{y}\times\left\{\frac{\partial}{\partial y_k}|\mathbf{x}-\mathbf{y}|^{-1}\right\}\left[\frac{\partial}{\partial x_k}A_4(\mathbf{x})-i\Pi_k(\mathbf{y})\right] \quad \text{(D2b)}$$

$$=\tfrac{1}{2}A_4(\mathbf{x})-i(8\pi)^{-1}\int d\mathbf{y}|\mathbf{x}-\mathbf{y}|^{-1}\nabla\cdot\mathbf{\Pi}(\mathbf{y}).$$

Also,

$$\nabla\cdot\mathfrak{A}=\nabla\cdot\mathbf{A}-\tfrac{1}{2}(\nabla\cdot\mathbf{A}+i\Pi_4)=\tfrac{1}{2}(\nabla\cdot\mathbf{A}-i\Pi_4). \quad \text{(D3)}$$

Therefore, we have

$$\langle n^\star|(\partial\mathfrak{A}_i(\mathbf{x})/\partial x_i)|n\rangle=0 \quad \text{(D4)}$$

and

$$\langle n^\star|(\partial\mathfrak{A}_i(\mathbf{x})/\partial x_i)^2|n\rangle=0. \quad \text{(D5)}$$

Moreover, we have

$$\langle n^\star|\mathfrak{A}_4(x)|n\rangle=\tfrac{1}{2}\langle n^\star|\left\{A_4(x)-i(4\pi)^{-1}\int d\mathbf{y}\times|\mathbf{x}-\mathbf{y}|^{-1}\nabla\cdot\mathbf{\Pi}(\mathbf{y})\right\}|n\rangle, \quad \text{(D6)}$$

and since

$$\langle n^\star|\{\nabla\cdot\mathbf{\Pi}(\mathbf{x})-i\nabla^2A_4(\mathbf{x})\}|n\rangle=0$$

we also have

$$\langle n^\star|\mathfrak{A}_4(\mathbf{x})|n\rangle=0. \quad \text{(D7)}$$

The potentials $\mathfrak{A}_\mu(\mathbf{x})$ act on the $|n\rangle$ states as though they were in the Coulomb gauge. Note that in terms of the $\mathfrak{A}_\mu(\mathbf{x})$

$$\hat{H}=H_0-\int d\mathbf{x}\,J_\mu(\mathbf{x})\mathfrak{A}_\mu(\mathbf{x})+(8\pi)^{-1}\int d\mathbf{x}d\mathbf{y}\times\rho(\mathbf{x})|\mathbf{x}-\mathbf{y}|^{-1}\rho(\mathbf{y}). \quad \text{(D8)}$$

Let us now find the equations of motion for the $\mathfrak{A}_\mu$. The following are easily shown to be the case:

$$[\mathfrak{A}_i(\mathbf{x}),\mathfrak{A}_j(\mathbf{y})]=0,$$

$$[\mathfrak{A}_i(\mathbf{x}),A_j(\mathbf{y})]=\delta_{i,j}(8\pi)^{-1}\frac{\partial}{\partial x_i}|\mathbf{x}-\mathbf{y}|^{-1},$$

$$[\mathfrak{A}_i(\mathbf{x}),\Pi_j(\mathbf{y})]=i\left\{\delta_{i,j}\delta(\mathbf{x}-\mathbf{y})+(8\pi)^{-1}\frac{\partial^2}{\partial x_i\partial x_j}|\mathbf{x}-\mathbf{y}|^{-1}\right\},$$

$$[\mathfrak{A}_4(\mathbf{x}),A_i(\mathbf{y})]=-(8\pi)^{-1}\frac{\partial}{\partial x_i}|\mathbf{x}-\mathbf{y}|^{-1},$$

$$[\mathfrak{A}_i(\mathbf{x}),\Pi_4(\mathbf{y})]=0,$$

$$[\mathfrak{A}_4(\mathbf{x}),\Pi_4(\mathbf{y})]=\tfrac{1}{2}i\delta(\mathbf{x}-\mathbf{y}),$$

$$[\mathfrak{A}_4(\mathbf{x}),\Pi_i(\mathbf{y})]=0,$$

and

$$[\mathfrak{A}_4(\mathbf{x}),\mathfrak{A}_i(\mathbf{y})]=0. \quad \text{(D9)}$$

Using these commutation rules we find that

$$\nabla^2\mathfrak{A}_i(\mathbf{x})+[H,[H,\mathfrak{A}_i(\mathbf{x})]]=J_i(\mathbf{x})+(4\pi)^{-1}\frac{\partial}{\partial x_i}\int d\mathbf{y}|\mathbf{x}-\mathbf{y}|^{-1}\nabla\cdot\mathbf{J}(\mathbf{y}) \quad \text{(D10)}$$

and the $\mathfrak{A}_i$ are only coupled to the transverse parts of the current, which is consistent with

$$\langle n^\star|(\partial\mathfrak{A}_i(\mathbf{x})/\partial x_i)|n\rangle=0.$$

Furthermore, $\nabla^2\mathfrak{A}_4(\mathbf{x})+[H,[H,\mathfrak{A}_4(\mathbf{x})]]=0$ and $\mathfrak{A}_4$ is a free field. Since $\langle n^\star|\mathfrak{A}_4|n\rangle=0$, we can take $\mathfrak{A}_4(\mathbf{x})=0$. Then the expectation values $\langle n^\star|\mathfrak{A}_\mu|n\rangle$ are unaffected by the gauge transformations still permitted within the free Lorentz gauge (which characterizes the states $|n\rangle$). e^{-D} is *not* simply a gauge transformation, however, since $e^{-D}A_\mu e^D\neq\mathfrak{A}_\mu$.