Sum Rules from Charge-Charge-Density and Charge-Charge Commutators*

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The spectral-function approach to the broken-$SU(3)$-symmetry sum rules has two difficulties. One is the ambiguity with respect to the Schwinger terms, and the other is the derivation of the so-called second sum rules, which are not acceptable. By using commutators involving a charge operator, our approach is free of the first difficulty. In computation, we make the following approximation for the operator V_K [which is an $SU(3)$ raising or lowering operator in the symmetry limit]: In the broken symmetry the operator V_K still acts as a generator, to a good approximation, in an appropriately chosen infinite-momenta limit. For the vector meson $\rightarrow l+\bar{l}$ couplings, we are able to derive sum rules which are essentially equivalent to the first spectral-function sum rules but not to the second ones. Instead, we obtain other sum rules which enable us to determine the first-order ω-ϕ mixing angle from the rates of $V \rightarrow l+\bar{l}$ decays. A sum rule for the $\omega \rightarrow 3\pi$, $\phi \rightarrow 3\pi$, and $K^* \rightarrow K\pi\pi$ couplings is also obtained. We also derive in our approach the Gell-Mann–Zachariasen relation and its K^* analog which favors the existence of the κ meson.

I. INTRODUCTION

In the broken $SU(3)$ symmetry there have been many interesting calculations based on the idea of current algebra. Recently, the derivation of sum rules from the spectral functions of vector and axial-vector currents has attracted much attention. Let us denote the vector and axial-vector currents by, for example, $V_\mu^{\pi^+}(x)$, $V_\mu^{K^+}(x)$, $V_\mu^{\eta}(x)$, $\cdots$, and $A_\mu^{\pi^+}(x)$, $A_\mu^{K^+}(x)$, $A_\mu^{\eta}(x)$, $\cdots$, respectively. They are normalized in a quark model as, for example,

$$V_\mu^{\pi^+}(x)=i\bar{q}(x)\gamma_\mu\tfrac{1}{2}(\lambda_1+i\lambda_2)q(x),\ V_\mu^{\eta}(x)=i\bar{q}(x)\gamma_\mu\tfrac{1}{2}\lambda_8 q(x),$$
$$A_\mu^{K^+}(x)=i\bar{q}(x)\gamma_5\gamma_\mu\tfrac{1}{2}(\lambda_4+i\lambda_5)q(x),\ \cdots. \quad (1)$$

The space integrals of the time component of these currents will be denoted by V_{π^+}, V_{K^+}, V_η, $\cdots$, and A_{π^+}, A_{K^+}, A_η, $\cdots$. We denote also the matrix elements of the $V_\mu(x)$ between the vacuum and a vector-meson state as follows [ϵ_μ^{α} is the polarization vector of the vector meson α]:

$$(2q_0)^{1/2}\langle 0|V_\mu^{\pi^+}(x)|\rho^-(\mathbf{q})\rangle=G_\rho\epsilon_\mu^{\rho},$$
$$(2q_0)^{1/2}\langle 0|V_\mu^{K^+}(x)|K^{*-}(\mathbf{q})\rangle=G_{K^*}\epsilon_\mu^{K^*},$$
$$(2q_0)^{1/2}\langle 0|V_\mu^{\eta^0}(x)|\omega^0(\mathbf{q})\rangle=G_\omega\epsilon_\mu^{\omega}, \quad (2)$$
$$(2q_0)^{1/2}\langle 0|V_\mu^{\eta^0}(x)|\phi^0(\mathbf{q})\rangle=G_\phi\epsilon_\mu^{\phi}.$$

Recently, Weinberg *et al.*,[1] and also Das *et al.*[2] from a different approach, obtained interesting sum rules for the above defined G's by using the high-energy behavior of the spectral functions of appropriately chosen combinations of the vector and axial-vector currents. This approach has the advantage that the spectral function receive contributions only from states of fixed spin and isospin. They assumed that the spectral functions are dominated by appropriate single-particle states. Their first sum rules[1,2] in the broken $SU(3)$ symmetry read

$$G_\rho^2/m_\rho^2=(G_{K^*}^2/m_{K^*}^2)+F_\kappa^2 \quad (3)$$

and

$$G_\rho^2/m_\rho^2-G_\omega^2/m_\omega^2-G_\phi^2/m_\phi^2=0. \quad (4)$$

Here, F_κ is defined by $\langle 0|V_\mu^{K^+}(x)|\kappa^-(q)\rangle=q_\mu F_\kappa$, where κ denotes the $I=\frac{1}{2}$, $Y=\pm1$ scalar resonance. The second sum rules[1] are given by

$$G_\rho^2=G_{K^*}^2 \quad (5)$$

and

$$G_\rho^2=G_\omega^2+G_\phi^2. \quad (6)$$

The second sum rules require stronger conditions than the first ones for the high-energy behavior[1,2] of the spectral functions. However, the derivation of these sum rules[1] requires a knowledge about the so-called Schwinger terms, and reference to the algebra of gauge fields[3] has often been made to make a definite statement about Schwinger terms.[1] Several works[4] have been devoted to the study of the ambiguity due to the possible existence of various types of Schwinger terms.

As regards the practical implications of these sum rules, Sakurai[5] has pointed out that the combined use of the first and second sum rules is not consistent with experiments, so the second sum rule should be discarded.

The purpose of this paper is to show that we could also obtain sum rules for the G's from a different standpoint *without using the spectral functions* and to discuss experimental implications of these results.

We propose to use the commutation relations between *a charge-density and a charge operator* or between two charge operators. Since we always involve a charge operator, this approach can avoid a confrontation with Schwinger terms. Therefore, this approach is considerably different from the spectral-function approach so

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[1] S. L. Glashow, H. J. Schnitzer, and S. Weinberg, Phys. Rev. Letters **19**, 137 (1967); S. Weinberg, Phys. Rev. **18**, 507 (1967).
[2] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **18**, 761 (1967).
[3] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).
[4] See, for example, Y. Frishman, Phys. Rev. Letters **19**, 539 (1967); R. Perrin, *ibid.* **20**, 306 (1968).
[5] J. J. Sakurai, Phys. Rev. Letters **19**, 803 (1967).

that there is, from the outset, no guarantee that both approaches give the same results. We shall show below that our approach leads to sum rules almost equivalent to the first spectral-function sum rules but we do not obtain the problematical second spectral-function sum rules. Instead, we obtain other sum rules for the vector meson $\rightarrow l+\bar{l}$ decay couplings which enable us to predict the branching ratios of $V\rightarrow l+\bar{l}$ decays in terms of the ω-ϕ mixing angle computed in the first order of the symmetry-breaking interaction. The result is different from the one obtained by using the usual treatment of the ω-ϕ mixing. We also obtain a sum rule for the $\omega\rightarrow 3\pi$, $\phi\rightarrow 3\pi$, and $K^*\rightarrow K\pi\pi$ couplings which is also useful for the determination of the ω-ϕ mixing angle.

Finally, we also derive a Gell-Mann–Zachariasen relation, $G_{\rho^0\pi^+\pi}\cdot G_\rho = m_\rho^2$, and its K^* analog which provides some information on the κ meson.

II. $SU(3)$ APPROXIMATION

We propose to use an approximation for the vector current $V_\mu{}^K(x)$ which may be compared, in a spirit, with the asymptotic $SU(3)$ condition used in the spectral-function approach.[2] In the $SU(3)$ symmetry limit, V_K is an $SU(3)$ generator and it only connects members (A and B) of the same $SU(3)$ multiplet. In the broken $SU(3)$ symmetry, the renormalization of the value of the diagonal matrix elements $\langle A|V_\mu{}^K(x)|B\rangle$ at *zero momentum transfer* appears only in the second order [denoted as $O(\epsilon^2)$] of the $SU(3)$-symmetry-breaking interaction whose strength is symbolically denoted as ϵ.[6] Our approximation is to assume that this renormalization of the matrix elements of the $V_\mu{}^K(x)$ at zero momentum transfer is small compared with other sorts of symmetry-breaking effects. The form factors of $\langle A|V_\mu{}^K(x)|B\rangle$ which are multiplied by nonvanishing [in the $SU(3)$ limit] kinematical factors are called $SU(3)$ form factors of $V_\mu{}^K(x)$. Our approximation, therefore, corresponds to the use of the $SU(3)$ value *only* for the $SU(3)$ form factors *only at zero momentum transfer*. We call this approximation an $SU(3)$ approximation. We stress that we only make this approximation at zero momentum transfer, where the renormalization effect is expected to be minimum.

Except for this approximation for the matrix elements of V_K, we use observed values (which naturally include the effects of symmetry breaking) for other quantities such as coupling constants and masses of particles consistently with the original spirit of the current-algebra approach. In order to carry out this program in a systematic way, we always compute the matrix elements of the V_K *at the appropriately chosen infinite-momenta limit*. We then can use the $SU(3)$ value (which is at least known to the first order in the symmetry-breaking interaction) for the diagonal elements of the V_K and consistently drop the nondiagonal matrix element of V_K in this limit.[7] In other words, our approximation corresponds to assuming that, even in the broken symmetry, the operator V_K still acts as an $SU(3)$ generator, to a good approximation, in the appropriate infinite-momenta limit. So far we did not consider the cases where mixing possibility exists. We certainly need to consider the important $SU(3)$-breaking effect due to mixing which appears in the order of $O(\epsilon)$. By taking into account the effect of mixing of the order $O(\epsilon)$, we claim that our $SU(3)$ approximation is good, effectively, to the order $O(\epsilon)$.

When two $SU(3)$ states can mix in the broken $SU(3)$ symmetry, we write physical states, to the first order of symmetry breaking, in terms of the $SU(3)$ states (with which the physical states coincide in the limit $\epsilon\rightarrow 0$) and apply the same procedure whenever these states come into the matrix elements of V_K. We believe that this approach gives a more consistent way of handling the first-order mixing angle than the usual one. The difference can be seen explicitly in Sec. III. In the usual treatment, the ratio $\Gamma(\phi^0\rightarrow l+\bar{l})/\Gamma(\omega^0\rightarrow l+\bar{l})$ is given by $\cot^2\theta$, where θ is the first-order ω-ϕ mixing angle. In the present approach this ratio acquires an extra factor (m_ω/m_ϕ) which may be checked by experiment.

Our $SU(3)$ approximation in the broken $SU(3)$ symmetry must, of course, be justified.[8] First of all, the rate of K_{e3} decay indicates that the renormalization of the $SU(3)$ form factor $F_+(0)$ of the matrix element $\langle\pi^0|V_\mu{}^{K^+}(x)|K^-\rangle$ at zero momentum transfer is indeed small (2–5%).[9] Secondly, using the charge commutators typified by $A_K=[V_K,A_\pi]$ and using the above-mentioned approximation and the PCAC (partially conserved axial-vector current) hypothesis, we have, for example, derived a relation between the coupling constants for the $K^*\rightarrow K+\pi$ and $\rho\rightarrow\pi+\pi$ decays where

[6] M. Ademollo and R. Gatto, Phys. Rev. Letters **13**, 264 (1964); S. Fubini, G. Furlan, and C. Rosseti, Nuovo Cimento **40A**, 1171 (1965).

[7] The dropped nondiagonal elements of V_K are formally of first order in the symmetry-breaking interaction. One can show that the second-order renormalization of the diagonal matrix element of V_K is expressed as a second-order effect of these nondiagonal elements. Therefore, our $SU(3)$ approximation corresponds to assuming that these nondiagonal elements are, in effect, less important compared with other symmetry-breaking effects. Namely, V_K acts as a generator, to a good approximation, in the infinite-momenta limit.

[8] Application of this approximation to charge commutation relations to discuss the broken $SU(3)$ symmetry has been discussed by S. Matsuda and S. Oneda [Phys. Rev. **158**, 1594 (1967)].

[9] S. Oneda and J. Sucher, Phys. Rev. Letters **15**, 927 (1965); **15**, 1049(E) (1965). The precise measurement of the K_{e3} decay rates and the form factors $F_+(q^2)$ is important. The electromagmagnetic correction is small (less than 2%), as is to be discussed.

only one of the pions is off the mass shell,[8,10] i.e.,

$$R=\frac{2G_{K^{*+}K^-\pi^0}(m_{K^*}^2,\ m_K^2,\ m_\pi^2=0)}{G_{\rho^+\pi^-\pi^0}(m_\rho^2,\ m_\pi^2,\ m_\pi^2=0)}$$

$$=\frac{m_\rho^2+m_{K^*}^2}{2m_\rho^2}=1.19\quad(\simeq m_{K^*}/m_\rho)\,. \quad (7)$$

As Sakurai[5] often emphasized, this indeed gives a good agreement with experiment if we take $\Gamma(\rho\to\pi+\pi)\simeq128$ MeV, which seems to be a currently preferred value. We can also derive the Gell-Mann–Okubo mass formulas by using this approximation and a charge algebra as will be shown in Sec. III B.

Encouraged by these observations, we would like to apply this approximation to the charge–charge-density commutators.

III. SUM RULES FOR THE VECTOR MESON $\to l+\bar{l}$ DECAYS

A. Derivation and Experimental Implications

Consider the following charge–charge-density commutators:

$$[V_{\bar{K}^0},V_0^{\pi^-}(x)]=V_0^{K^-}(x)\,. \quad (8)$$

If we take the matrix element of this commutator between the vacuum and the K^*-meson state with infinite momentum, we obtain in our approximation (in a symbolic notation),

$$\langle0|V_{\bar{K}^0}|\kappa^0\rangle\langle\kappa^0|V_0^{\pi^-}(x)|K^{*+}(\mathbf{q})\rangle - \langle0|V_0^{\pi^-}(x)|\rho^+\rangle\langle\rho^+|V_{\bar{K}^0}|K^{*+}(\mathbf{q})\rangle = \langle0|V_0^{K^-}(x)|K^{*+}(\mathbf{q})\rangle,\quad |\mathbf{q}|=\infty\,.$$

For the matrix elements of *charge operators* involving vacuum states, such as $\langle0|V|n\rangle$ or $\langle0|A|n\rangle$, the only *one-particle states* which can contribute are the spin-zero mesons. Therefore, in the above equation, for the element $\langle0|V_{\bar{K}^0}|n\rangle$, we only need to consider the κ-meson state ($n=\kappa$). For the matrix elements $\langle n|V_{\bar{K}^0}|K^{*+}(\mathbf{q})\rangle$ [according to our $SU(3)$ approximation] we only need to keep the state $n=\rho^+$ in the limit $|\mathbf{q}|=\infty$. We shall show in Appendix A that the contribution of the κ term in the above equation is at least of second order in the symmetry-breaking interaction. Therefore, we drop this contribution consistently with our $SU(3)$ approximation. We then obtain, after summing over the ρ-meson spin states (the computation is similar to the one given in Ref. 8),

$$G_{K^*}=\left(\frac{m_\rho^2+m_{K^*}^2}{2m_\rho^2}\right)G_\rho\,. \quad (9)$$

In the same way, taking the matrix elements of the commutator $[V_{K^-},V_0^{\eta}(x)]=-\frac{1}{2}\sqrt{3}V_0^{K^-}(x)$, between the vacuum and the $K^*(\mathbf{q})$ state with $|\mathbf{q}|=\infty$, we also obtain

$$G_\omega\sin\theta\left(\frac{m_\omega^2+m_{K^*}^2}{2m_\omega^2}\right) - G_\phi\cos\theta\left(\frac{m_\phi^2+m_{K^*}^2}{2m_\phi^2}\right)=\frac{1}{\sqrt2}G_{K^*}\,, \quad (10)$$

where we have written, to the order $O(\epsilon)$,

$$\omega=\cos\theta\ \omega_1+\sin\theta\ \omega_8,\quad \phi=\cos\theta\ \omega_8-\sin\theta\ \omega_1\,. \quad (11)$$

$\omega\to\omega_1$ and $\phi\to\omega_8$ in the limit $\epsilon\to0$. Since we are interested in the mixing of the order $O(\epsilon)$, this description in terms of θ is sufficient for our purpose.

If we, instead, choose the commutator $[V_{K^+},V_0^{K^-}]=V_\mu^{\pi^0}+\sqrt3V_0^{\eta^0}$ and insert it between the vacuum state and the $\phi(\mathbf{q})$ or $\omega(\mathbf{q})$ states with $|\mathbf{q}|=\infty$, we obtain

$$G_\phi=\frac{1}{\sqrt2}G_{K^*}\cos\theta\left(\frac{m_{K^*}^2+m_\phi^2}{2m_{K^*}^2}\right)$$

$$=\frac{1}{\sqrt2}G_\rho\cos\theta\left(\frac{m_\rho^2+m_{K^*}^2}{2m_\rho^2}\right)\left(\frac{m_{K^*}^2+m_\phi^2}{2m_{K^*}^2}\right). \quad (12a)$$

As will be shown in Sec. IV, this relation is numerically equivalent (within 1.5%) to

$$G_\phi\simeq\left(\frac{1}{\sqrt2}\right)G_{K^*}\cos\theta\left(\frac{m_\phi}{m_{K^*}}\right)\simeq\left(\frac{1}{\sqrt2}\right)G_\rho\cos\theta\left(\frac{m_\phi}{m_\rho}\right). \quad (12b)$$

We also obtain, for G_ω,

$$G_\omega=\frac{1}{\sqrt2}G_{K^*}\sin\theta\left(\frac{m_{K^*}^2+m_\omega^2}{2m_{K^*}^2}\right)$$

$$=\frac{1}{\sqrt2}G_\rho\sin\theta\left(\frac{m_\rho^2+m_{K^*}^2}{2m_\rho^2}\right)\left(\frac{m_{K^*}^2+m_\omega^2}{2m_{K^*}^2}\right), \quad (13a)$$

which is also numerically equivalent to

$$G_\omega\simeq\left(\frac{1}{\sqrt2}\right)G_{K^*}\sin\theta\left(\frac{m_\omega}{m_{K^*}}\right)\simeq\left(\frac{1}{\sqrt2}\right)G_\rho\sin\theta\left(\frac{m_\omega}{m_\rho}\right). \quad (13b)$$

From (9), (12), and (13), we have established relations between G's. In particular, (12) and (13) give

$$\frac{\gamma_\phi}{\gamma_\omega}=\frac{G_\phi}{G_\omega}=\left(\frac{m_{K^*}^2+m_\phi^2}{m_{K^*}^2+m_\omega^2}\right)\cot\theta\simeq\left(\frac{m_\phi}{m_\omega}\right)\cot\theta\,, \quad (14a)$$

where γ_ω and γ_ϕ are the photon-ω and photon-ϕ coupling constants, respectively. For the rates of $\phi\to l+\bar{l}$ and $\omega\to l+\bar{l}$ decay, Γ_ϕ, and Γ_ω, we therefore predict in terms of the first-order ω-ϕ mixing angle θ

[10] $R=m_{K^*}/m_\rho$ has been inferred by many people. For example, see H. T. Nieh, Phys. Rev. Letters **15**, 902 (1965); Phys. Rev. **146**, 1012 (1966); Riazuddin and Fayyazuddin, *ibid.* **147**, 1071 (1966). Our derivation has the advantage that the off-mass-shell extrapolation can be clearly stated, and no model is used except for the use of charge commutation relations.

from (14a):

$$\left(\frac{\Gamma_\phi}{\Gamma_\omega}\right)^{1/2} \simeq \left(\frac{m_\omega}{m_\phi}\right)^{3/2} \left(\frac{m_{K^*}{}^2+m_\phi{}^2}{m_{K^*}{}^2+m_\omega{}^2}\right)\cot\theta \simeq \left(\frac{m_\omega}{m_\phi}\right)^{1/2} \cot\theta. \quad (14b)$$

Equation (14b) may be useful to determine the first-order ω-ϕ mixing angle θ from the precise measurement of $\phi(\omega) \to l+\bar{l}$ decays. Note that the usual approach gives $G_\phi/G_\omega = \cot\theta$. Notice the extra factor (m_ϕ/m_ω) in (14a), which arises from our way of handling the lowest-order mixing. The measurements G_ρ/G_ω from the ratio of Γ_ω to Γ_ρ will also be useful in testing our approach and approximation for the broken $SU(3)$ symmetry. We predict, in terms of θ,

$$\left(\frac{3\Gamma_\omega}{\Gamma_\rho}\right)^{1/2} \simeq \left(\frac{m_\rho}{m_\omega}\right)^{3/2} \left(\frac{m_\rho{}^2+m_{K^*}{}^2}{2m_\rho{}^2}\right)\left(\frac{m_{K^*}{}^2+m_\omega{}^2}{2m_{K^*}{}^2}\right)\sin\theta \simeq \left(\frac{m_\rho}{m_\omega}\right)^{1/2} \sin\theta, \quad (14c)$$

$$\left(\frac{3\Gamma_\phi}{\Gamma_\rho}\right)^{1/2} \simeq \left(\frac{m_\rho}{m_\phi}\right)^{3/2} \left(\frac{m_\rho{}^2+m_{K^*}{}^2}{2m_\rho{}^2}\right)\left(\frac{m_{K^*}{}^2+m_\omega{}^2}{2m_{K^*}{}^2}\right)\cos\theta \simeq \left(\frac{m_\rho}{m_\phi}\right)^{1/2} \cos\theta. \quad (14d)$$

Upon eliminating θ, (14c) and (14d) lead to

$$\tfrac{1}{3}m_\rho\Gamma_\rho \simeq m_\omega\Gamma_\omega + m_\phi\Gamma_\phi. \quad (14e)$$

B. Remarks on the ω-φ Mixing Angle

We have thus shown that Eqs. (14b), (14c), and (14d) may serve to determine the first-order ω-ϕ mixing angle θ. The derivations of these formulas are general since we do not make a specific assumption about the nature of the symmetry-breaking interaction. It is therefore, of great interest to study whether the value of θ determined from the $V \to l+\bar{l}$ decays coincides with the value $\theta_m \simeq 40°$ derived from the first-order vector-meson mass formulas based on the usual assumption of the symmetry-breaking interaction. In order to further illustrate this point and also the applicability of our $SU(3)$ approximation, we show below a new derivation of the θ_m by utilizing a current algebra and our $SU(3)$ approximation.

Let us now assume that the following commutation relation holds:

$$[V_{K^0}, \dot{V}_{K^0}] = 0.$$

Compared with other commutators used in this paper, the validity of this commutator is model-dependent.[6] If, for example, the $SU(3)$ symmetry-breaking interaction transforms like $\int \bar{q}(x)\lambda_8 q(x) d^3x$, the above commutator is satisfied. Consider the equation $\langle K^{*0}(\mathbf{q})| \times [V_{K^0}, \dot{V}_{K^0}] | \bar{K}^{*0}(\mathbf{q})\rangle = 0$ with $|\mathbf{q}| = \infty$ and use our $SU(3)$ approximation. By expressing the ω and ϕ states in terms of ω_1 and ω_8 states using Eq. (11), we then obtain from the above equation directly the first-order ω-ϕ mixing angle θ_m given by

$$\sin^2\theta_m = (3m_\phi{}^2 - 4m_{K^*}{}^2 + m_\rho{}^2)/3(m_\phi{}^2 - m_\omega{}^2).$$

If we substitute this value of θ_m for θ in Eqs. (12b) and (13b), we obtain

$$G_\phi{}^2 \simeq \frac{1}{2}\left(\frac{m_\phi{}^2}{m_\rho{}^2}\right)\frac{(4m_{K^*}{}^2 - m_\rho{}^2 - 3m_\omega{}^2)}{3(m_\phi{}^2 - m_\omega{}^2)} G_\rho{}^2$$

and

$$G_\omega{}^2 \simeq \frac{1}{2}\left(\frac{m_\omega{}^2}{m_\rho{}^2}\right)\frac{(3m_\phi{}^2 - 4m_{K^*}{}^2 + m_\rho{}^2)}{3(m_\phi{}^2 - m_\omega{}^2)} G_\rho{}^2.$$

These results are the same as the ones obtained by Das *et al.*[11] Therefore, if we assume $\theta_m = \theta$ (which follows, as shown above, if we assume further, for example, the validity of the commutation relation $[V_{K^0}, \dot{V}_{K^0}] = 0$) our results agree with those of Das *et al.*[11] obtained in a spectral-function approach by further modifying the second sum rules (5) and (6) by introducing specific $SU(3)$-violating effects. Recently Sakurai and Oakes[12] have also attempted a similar modification of the second spectral-function sum rules. As mentioned in the Introduction, we do not need to introduce such a procedure to derive Eqs. (12), (13), and (14). Our $SU(3)$ approximation directly gives a kind of result which Das *et al.*[11] and Oakes and Sakurai[12] wanted to derive by patching up the second spectral-function sum rules by imposing further a specific model of symmetry breaking. For the sake of simplicity, we, of course, hope that experiments will turn out to be consistent with having $\theta_m = \theta$.

IV. COMPARISON WITH THE SPECTRAL-FUNCTION SUM RULES

We now wish to discuss the relation between our sum rules and the spectral-function sum rules. We first note that the function $f(m_A/m_B) = (m_A/m_B) - 2m_A{}^2/(m_A{}^2 + m_B{}^2)$ has an extreme value at $m_A/m_B = 1$. Therefore, the relations

$$\frac{m_\rho{}^2 + m_{K^*}{}^2}{2m_\rho{}^2} = \frac{m_{K^*}}{m_\rho}, \quad \frac{m_{K^*}{}^2 + m_\phi{}^2}{2m_{K^*}{}^2} = \frac{m_\phi}{m_{K^*}},$$

and

$$\frac{m_{K^*}{}^2 + m_\omega{}^2}{2m_{K^*}{}^2} = \frac{m_\omega}{m_{K^*}} \quad (15)$$

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[11] Similar problems have also been discussed recently from a different standpoint by T. Das, V. S. Mathur, and S. Okubo [Phys. Rev. Letters 19, 470 (1967)].

[12] After we submitted this paper, the work of Oakes and Sakurai [R. J. Oakes and J. J. Sakurai, Phys. Rev. Letters 19, 1266 (1967)] has been published. If $\theta = \theta_m$ (which we prefer), our result on $V \to l+\bar{l}$ decays is different from theirs.

hold within 1.4%, 0.9%, and 0.9%, respectively. We have already used this in deriving Eqs. (12b), (13b), (14b), (14c), and (14d). By noting this fact, we can rewrite (9) as

$$G_\rho^2/m_\rho^2 = G_{K^*}^2/m_{K^*}^2 \text{ (within 1.4\%)}. \tag{16}$$

In Eq. (3), F_κ^2 is of the order $O(\epsilon^2)$ which should be neglected in our $SU(3)$ approximation.

We can easily verify that Eqs. (10), (12), and (13) are mutually consistent within 1.0%. Moreover, from (12) and (13) we also obtain, with a good accuracy [compare with Eq. (4)],

$$\frac{G_\phi^2}{m_\phi^2}+\frac{G_\omega^2}{m_\omega^2}=\frac{G_{K^*}^2}{m_{K^*}^2}=\frac{G_\rho^2}{m_\rho^2}. \tag{17}$$

Therefore, it is demonstrated that our approximation and use of charge–charge-density commutation relations lead essentially to the first sum rules, (3) and (4), of the spectral-function approaches. However, our $SU(3)$ approximation *does not lead us to the second sum rules*, (5) and (6), which are not acceptable. The $SU(3)$ symmetry is certainly broken. We have, however, assumed that the charge V_K could remain, to a good approximation, to be an $SU(3)$ generator in the *infinite-momenta limit* discussed above. We, nevertheless, have obtained results with an appreciable $SU(3)$ breaking. In the spectral-function approach with the asymptotic $SU(3)$ condition, we have to introduce another criterion which prevents us from obtaining strict symmetry results.[11,12] In our approach, this criterion seems to be already built-in by our $SU(3)$ approximation. This seems to be the reason why we do not encounter the second spectral-function sum rules in our method.

V. SUM RULES FOR THE $\omega\to 3\pi$, $\phi\to 3\pi$, AND $K^*\to K\pi\pi$ DECAYS

In addition to Eq. (14), we would like to present another way of measuring ω-ϕ mixing angle defined by (11), which may be feasible in the future. Usually the $\phi\to K+\bar{K}$ decay is taken to be the best place to measure the mixing angle. However, theoretically this is not the case.[8,11] In order to relate the $\phi\to K+\bar{K}$ coupling to, for instance, the $\rho\to\pi+\pi$ coupling, the usual prescription is to write a pure $SU(3)$ F-type VPP coupling and then re-express the ω_8 field in the ω_8 $K\bar{K}$ interaction in terms of the physical fields ϕ and ω. However, we must note that once the $SU(3)$ symmetry is broken, the $SU(3)$ coupling relations must also be affected. This is not taken into account in the usual determination of the mixing angle from the $\phi\to K+\bar{K}$ decay. [Note that in the $K^*K\pi$ and $\rho\pi\pi$ coupling constants we have already seen that this effect is around 20% in the amplitude. See Eq. (7).] We have shown in Ref. 8 that we can make some estimate of this effect if we tolerate the kaon PCAC which requires an off-mass-shell extrapolation $m_K^2\to 0$.

We wish to point out that in the comparison of the rates of $K^*\to K+\pi+\pi$ and $\omega(\phi)\to 3\pi$ decays we can avoid this difficulty and make a consistent determination of the mixing angle. Consider the following relation between *charge commutators*:

$$2[V_{K^-},A_{\pi^0}]=[V_{\bar{K}^0},A_{\pi^-}] \quad (=A_{K^-}). \tag{18}$$

Take the matrix element of the above commutators between the $I=1$, $(\pi^+(\mathbf{p}^+),\pi^-(\mathbf{p}^-))$ state and the $K^{*+}(\mathbf{q})$ state with $|\mathbf{q}|=\infty$. Taking the $SU(3)$ approximation for the matrix elements of V_K and using pion PCAC for the A_π's, the computation in the frame $\mathbf{p}^+=\mathbf{p}^-=\frac{1}{2}\mathbf{q}$ yields *a broken-$SU(3)$ sum rule:*

$$(\sin\theta\, G_{\omega^0\to 3\pi}-\cos\theta\, G_{\phi^0\to 3\pi})=\frac{1}{\sqrt{3}}G_{K^{*+}\to K^+\pi^+\pi^-}, \tag{19}$$

where each coupling constant[13] G is defined with one of the pions off the mass-shell ($m_\pi^2\to 0$).

In (19), we are justified in using the physical masses for the particles involved.[14] Thus if our approximation for V_K is good and the off-mass-shell effect ($m_\pi\to 0$) is negligible, as usually assumed, Eq. (19) can be used to determine rather precisely the value of θ.

In (19), since $\Gamma(\phi\to\rho+\pi \text{ and } 3\pi)\simeq 0.05\,\Gamma(\omega\to 3\pi)$, $G_{\omega\to 3\pi}\gg G_{\phi\to 3\pi}$, considering that the phase space of the $\phi\to 3\pi$ decay is larger than that of the $\omega\to 3\pi$ decay. Therefore, in practice we may neglect the $G_{\phi\to 3\pi}$ term in Eq. (19). We think that a precise determination of the rate of $K^*\to K+\pi+\pi$ decay is useful in this respect.

VI. DERIVATION OF THE GELL-MANN–ZACHARIASEN RELATION AND ITS K^* ANALOG AND POSSIBLE EXISTENCE OF THE κ MESON

We wish to add an application of our procedure which now involves the axial charge A_π instead of V_K. We show that by using a reasonable approximation we can derive the Gell-Mann–Zachariasen relation.[15] We make use of the commutator

$$[V_0^{\pi^0}(x),A_{\pi^+}]=A_0^{\pi^+}(x). \tag{20}$$

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[13] $G_{\omega^0\pi^+\pi^-\pi^0}$, for example, is defined in such a way that the $\omega^0\to\pi^+\pi^-\pi^0$ decay matrix element is given by $(16E_\omega E_+E_-E_0)^{-1/2}\times G_{\omega^0\pi^+\pi^-\pi^0}\epsilon_{\alpha\beta\gamma\delta}\epsilon_\alpha^{\omega^0}p_\beta^{\pi^+}p_\gamma^{\pi^-}p_\delta^{\pi^0}$.

[14] In a similar approximation we have also shown (see Ref. 8) that we can derive a set of sum rules between the axial-vector coupling constants of hyperon β decay at zero momentum transfer. These sum rules coincide with those obtained by eliminating the ratio of d-type and f-type coupling in Cabibbo's original pure $SU(3)$ analysis. This demonstrates that the usual way of determining Cabibbo's angle is justified rather well even in the broken $SU(3)$ symmetry if our approximation is good. The same approach was also applied to the direct calculation of the pion-pion scattering length from the K_{e4} decay. Results similar to those of the usual current-algebra calculation have been obtained [S. Matsuda and S. Oneda, Phys. Rev. **165**, 1749 (1968)].

[15] M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 953 (1961).

Take the matrix element between the vacuum and the $\pi^-(\mathbf{q})$ state with $|\mathbf{q}|=\infty$. We assume that single-particle states dominate the intermediate states. We then obtain

$$\langle 0|V_\mu^{\pi^0}(x)|\rho^0\rangle\langle\rho^0|A_{\pi^+}|\pi^-\rangle-\langle 0|A_{\pi^+}|\pi^-\rangle \times\langle\pi^-|V_\mu^{\pi^0}(x)|\pi^-\rangle=\langle 0|A_\mu^{\pi^+}(x)|\pi^-\rangle. \quad (21)$$

We again note that for the matrix element of the *charges*, $\langle 0|A|n\rangle$ or $\langle 0|V|n\rangle$, the only one-particle states which contribute are the spin-zero mesons. Therefore, together with the restriction on the state n in the matrix elements involving the current density, $\langle 0|V_0(x)$ or $A_0(x)|n\rangle$, we see that the restriction on the intermediate states n is stringent as in the case of spectral-function sum rules and we may expect a quick saturation.[16]

If the pion electromagnetic form factor satisfies an unsubtracted dispersion relation, as usually assumed (see also the discussion given in Appendix B), the second term of the left-hand side of Eq. (21) will drop. [$F(s)\to 0$ as $s\to\infty$, where $F(s)$ is the pion form factor.] Then at $|\mathbf{q}|\to\infty$, if we use PCAC for A_{π^+}, Eq. (21) yields

$$G_{\rho^0\pi^+\pi^-}G_\rho=m_\rho^2, \quad (22)$$

which is the Gell-Mann–Zachariasen relation except for the fact that the $G_{\rho\pi\pi}$ coupling is now defined with one of the pions off the mass shell ($m_\pi\to 0$). However, G_ρ is on the mass shell.

If we, instead of (20), use the commutator

$$[V_0^{K^0}(x),A_{\pi^+}]=A_0^{K^+}(x) \quad (23)$$

and take the matrix element between the vacuum and the $K^-(\mathbf{q})$ state with $|\mathbf{q}|=\infty$, we obtain the relation, assuming that the K_{l3}-decay form factor satisfies an unsubtracted dispersion relation,

$$G_{K^*}G_{K^{*0}K^+\pi^-}=m_{K^*}^2\left(\frac{F_K}{F_\pi}-\frac{F_\kappa G_{\kappa^0K^-\pi^+}}{m_\kappa^2-m_K^2}\right), \quad (24)$$

where F_π and F_K are the amplitudes of the $\pi\to\mu+\nu$ and $K\to\mu+\nu$ decays and $G_{\kappa K\pi}$ is the coupling constant of the $\kappa\to K+\pi$ decay defined with a pion off the mass shell. It is very interesting to observe that the K^* meson cannot completely saturate the commutator (23) taken between the vacuum and the $K^-(\mathbf{q})$ state with $|\mathbf{q}|=\infty$, although we have seen that the ρ meson can saturate the corresponding commutation relation (20) taken between the vacuum and the $\pi(\mathbf{q})$ state with $|\mathbf{q}|=\infty$. To show this, let us, for the time being, neglect the κ meson term in Eq. (24). We obtain

$$G_{K^*}G_{K^{*0}K^+\pi^-}=(F_K/F_\pi)m_{K^*}^2=1.28\,m_{K^*}^2. \quad (25)$$

From (7), (16), and (22) we have

$$G_{K^{*0}K^+\pi^-}=((m_\rho^2+m_{K^*}^2)/2m_\rho^2)G_{\rho^0\pi^+\pi^-},$$

$$G_{K^*}/G_\rho=((m_\rho^2+m_{K^*}^2)/2m_\rho^2) \quad\text{and}\quad G_\rho G_{\rho^0\pi^+\pi^-}=m_\rho^2.$$

We therefore obtain

$$G_{K^*}G_{K^{*0}K^+\pi^-}=\left(\frac{m_\rho^2+m_{K^*}^2}{2m_\rho^2}\right)^2 m_\rho^2\simeq m_{K^*}^2. \quad (26)$$

Therefore, Eqs. (25) and (26) are not compatible. We can avoid this difficulty if we keep the contribution of the κ meson in Eq. (24). We then predict from Eqs. (24) and (26)

$$F_\kappa G_{\kappa^0K^-\pi^+}=\left[\left(\frac{F_K}{F_\pi}\right)-\left(\frac{m_\rho^2+m_{K^*}^2}{2m_\rho^2}\right)^2\left(\frac{m_\rho^2}{m_{K^*}^2}\right)\right] \times(m_\kappa^2-m_K^2)\simeq\left[\left(\frac{F_K}{F_\pi}\right)-1\right](m_\kappa^2-m_K^2). \quad (27)$$

This relation may be useful if the existence of the κ meson is confirmed. Since $\partial_\mu V_\mu^K=0$ in the $SU(3)$ limit, $F_\kappa\to 0$ in this limit. This *is explicitly borne out* in Eq. (27). [$F_K=F_\pi$, $m_{K^*}=m_\rho$ in the $SU(3)$ limit.] We have discussed in another place[17] an implication of this relation to the K_{l3} decay which seems to enable us to have a consistent description of the K_{l3}-decay form factors. In particular, the form factors $F_+(q^2)$ and $F_-(q^2)$ obtained by using Eq. (27) satisfy the Callan-Trieman[18] prediction (based on a soft-pion approach) made at the point $-q^2=m_K^2$.

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APPENDIX A

In deriving Eq. (9), we have dropped the κ contribution. We shall show below that this is justified in our approximation, which is good, effectively, to the order $O(\epsilon)$. From the commutator $[V_{\bar{K}^0},V_\mu^{\pi^-}(x)]=V_\mu^{K^-}(x)$, the commutator $[V_0^{\pi^-}(x),\dot{V}_{\bar{K}^0}]=\partial_\mu V_\mu^{K^-}(x)$ is derived. We insert this commutator between the vacuum state and the $K^{*+}(\mathbf{q})$ state with $|\mathbf{q}|=\infty$:

$$\langle 0|V_0^{\pi^-}(x)|\rho^+\rangle\langle\rho^+|\dot{V}_{\bar{K}^0}|K^{*+}(\mathbf{q})\rangle -\langle 0|\dot{V}_{\bar{K}^0}|\kappa^0\rangle\langle\kappa^0|V_0^{\pi^-}(x)|K^{*+}(\mathbf{q})\rangle =\langle 0|\partial_\mu V_\mu^{K^-}(x)|K^*(\mathbf{q})\rangle,\quad |\mathbf{q}|=\infty. \quad (A1)$$

[16] In fact, we have derived in a rather similar way (using PCAC) the sum rule

$$G_\rho^2/m_\rho^2-G_{A_1}^2/m_{A_1}^2=F_\pi^2,$$

which is the same as the one obtained by Weinberg from the consideration of spectral functions in the chiral $SU(2)\times SU(2)$ symmetry [S. Weinberg, Phys. Rev. Letters **18**, 507 (1967)]. However, we cannot get the second sum rule of Weinberg $G_{A_1}=G_\rho$ unless we assume that the A_π is a strict generator of the chiral $SU(2)\times SU(2)$ group. This does not mean that we rule out the possibility of having $G_{A_1}\simeq G_\rho$ when the A_π is not a strict generator, which is actually the case. Details will be discussed elsewhere.

[17] S. Matsuda and S. Oneda, Phys. Rev. **169**, 1172 (1968).

[18] C. G. Callan and S. B. Treiman, Phys. Rev. Letters **16**, 153 (1966); M. Suzuki, *ibid.* **16**, 212 (1966); V. S. Mathur, S. Okubo, and L. K. Pandit, *ibid.* **16**, 371 (1966).

As in the derivation of our sum rules discussed in the text, the use of $SU(3)$ approximation for the charge operator V_K and the existence of a vacuum state in the states under consideration restrict the intermediate states (in the one-particle approximation, which is the same as in the spectral-function case) to the ρ and κ states. The right-hand side of (A1) vanishes, so we have

$$\lim_{|\mathbf{q}|\to\infty}[E_\rho(\mathbf{q})-E_{K^*}(\mathbf{q})] \times\langle 0|V_0^{\pi^-}(x)|\rho^+\rangle\langle\rho^+|V_{\bar{K}^0}|K^{*+}(\mathbf{q})\rangle =\lim_{|\mathbf{q}|\to\infty} m_\kappa\langle 0|V_{\bar{K}^0}|\kappa^0\rangle\langle\kappa^0|V_0^{\pi^-}(x)|K^{*+}(\mathbf{q})\rangle.$$

Apart from the factor $[E_\rho(\mathbf{q})-E_{K^*}(\mathbf{q})]$, which vanishes in the limit $|\mathbf{q}|\to\infty$, the terms on the left-hand side of the above equation are finite. Therefore, we have

$$\lim_{|\mathbf{q}|\to\infty}\langle 0|V_{\bar{K}^0}|\kappa^0\rangle\langle\kappa^0|V_0^{\pi^-}(x)|K^*(\mathbf{q})\rangle=0.$$

APPENDIX B

In a way similar to that of Appendix A, we can show that the pion electromagnetic form factor satisfies an unsubtracted dispersion relation. We take the commutator $[V_0^{\pi^0}(x),\dot{V}_{\pi^+}]=\partial_\mu A_\mu^{\pi^+}(x)$. As in (A1), we obtain

$$\lim_{|\mathbf{q}|\to\infty}\{\langle 0|V_0^{\pi^0}(x)|\rho^0\rangle\langle\rho^0|\dot{A}_{\pi^+}|\pi^-(\mathbf{q})\rangle -\langle 0|\dot{A}_{\pi^+}|\pi^-\rangle\langle\pi^-|V_0^{\pi^0}(x)|\pi^-(\mathbf{q})\rangle\} =\lim_{|\mathbf{q}|\to\infty}\langle 0|\partial_\mu A_\mu^{\pi^+}|\pi^-(\mathbf{q})\rangle. \quad \text{(B1)}$$

From the PCAC condition, the right-hand side of (B1) is zero. Then using the same argument as in Appendix A, we obtain

$$\lim_{|\mathbf{q}|\to\infty}\langle\pi^-(\mathbf{p})|V_0^{\pi^0}(x)|\pi(\mathbf{q})\rangle=0,$$

where $|\mathbf{p}|=0$.

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Subsidiary Condition in Quantum Electrodynamics

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The subsidiary condition $\partial A_\mu^{(+)}/\partial x_\mu|n\rangle=0$, usually known as the "Gupta-Bleuler" condition, is shown to be inadequate as a criterion for defining physical states in quantum electrodynamics in the Lorentz gauge. The condition is shown not to be covariant and to fail to define state vectors that remain in the physical subspace. An alternative subsidiary condition, which is satisfactory, is discussed and is shown to require an extensively different formulation of the collision problem in quantum electrodynamics. Some possible physical consequences of the inadequacy of $\partial A_\mu^{(+)}/\partial x_\mu|n\rangle=0$ are proposed; these include effects in the decays of short-lived particles, and the fact that in some types of strong interactions, acting simultaneously with electromagnetic ones, S-matrix elements may occur which predict transitions from the physical space into the part of space in which the subsidiary condition is violated. The solution to the collision problem for stable charged particles that have only electromagnetic interactions is shown to be identical to that obtainable from the present theory.

I. INTRODUCTION

THE correct formulation of quantum electrodynamics (QED) in the Lorentz gauge requires the imposition of a subsidiary condition [namely, $\chi^{(+)}(x)|n\rangle=0$, where $\chi^{(+)}(x)=\partial A_\mu^{(+)}(x)/\partial x_\mu$] on the "physical" state vectors and involves the use of a nondegenerate indefinite metric space instead of the usual Hilbert space in which quantum theories are ordinarily framed. The reasons for this have to do with the incompatibility of the subsidiary condition $\partial A_\mu(x)/\partial x_\mu=0$, as an operator identity, with the canonical quantization procedure and the commutation rules for the four-dimensional vector potential. This situation has been understood for a very long time and is discussed in detail in most standard texts.[1] For a set of noninteracting photons the resulting theory is clear and has the follow-

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[1] S. N. Gupta, Proc. Phys. Soc. (London) **63**, 681 (1950); W. Heitler, *The Quantum Theory of Radiation* (Clarendon Press, Oxford, England, 1954); G. Källén, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. 5; J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., London, 1955); A. J. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics* (Interscience Publishers, Inc., New York, 1965); P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, England, 1958), 4th ed.; S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1949); J. Hamilton, *The Theory of Elementary Particles* (Clarendon Press, Oxford, England, 1959); N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959); for discussions of the algebra of the indefinite metric space see R. Ascoli and E. Minardi, Nucl. Phys. **9**, 242 (1958); L. A. Maksimov, Zh. Eksperim. i Teor. Fiz. **36**, 465 (1959) [English transl.: Soviet Phys.—JETP **9**, 324 (1959)]; K. L. Nagy, Nuovo Cimento Suppl. **17**, 760 (1960); L. K. Pandit, *ibid.* **11**, 157 (1959); A. Uhlmann, Nucl. Phys. **12**, 103 (1959); H. J. Schnitzer and E. C. G. Sudarshan, Phys. Rev. **123**, 2193 (1961); E. C. G. Sudarshan, *ibid.* **123**, 2183 (1961).