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Phonon-Quasiparticle Interactions in Dilute Solutions of He^3 in Superfluid He^4. II. Phonon Boltzmann Equation and First Viscosity*

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We construct the phonon Boltzmann equation taking into account previously calculated phonon-He^3 scattering and absorption processes; when phonon-phonon interactions can be neglected this equation is exactly soluble. The first viscosity of solutions containing 1.3 and 5% molar concentrations of He^3 is calculated from this equation for $T \lesssim 0.6$°K.

I. INTRODUCTION

THIS paper is a continuation of recent work[1] on the effects of phonon-He^3 quasiparticle interactions on the transport properties of dilute solutions of He^3 in superfluid He^4 at low temperatures. Introducing He^3 impurities into pure He^4 changes the transport properties of the superfluid in two ways. First, the He^3 quasiparticles, carrying energy, momentum, and spin, provide a new mechanism for transport. At extremely low temperatures, where the thermal excitations of the He^4 are negligible, the contributions of the He^3 dominate the transport coefficients. Second, transport by the elementary excitations of the He^4, phonons and rotons, is limited by their interactions with the He^3, either by the scattering or by the absorption or emission of excitations by the He^3. In the temperature range up to about 0.6°K for a few percent He^3 concentration, where rotons can be neglected, the mean free paths of the phonons are determined entirely by their interactions with the He^3 (except at very low temperatures where boundary scattering is important). One of the purposes of this paper is to construct the Boltzmann equation for the phonons in this temperature region, taking into account essentially exactly the absorption and scattering of the phonons by the impurities.

The scattering of phonons from He^3 leads to a completely tractable collision term in the phonon Boltzmann equation. There are several reasons for this. First, the He^3 quasiparticle excitations have a particle-like energy-momentum relation

$$\epsilon_p = \epsilon_0 + p^2/2m, \tag{1}$$

where $m \approx 2.34 m_3$ is the He^3 effective mass; and they have typical velocities much smaller than s, the velocity of first sound in pure He^4 at zero temperature; the scattering of phonons from He^3 is basically elastic. Furthermore, the average scattering rate has the particularly simple form

$$\Gamma(q,\theta) = \frac{xsq^4}{8\pi n_4}(-1.4+0.69\cos\theta)^2 \tag{2}$$

involving only s, p and d waves; θ is the phonon scattering angle and q is the initial and final wave number; n_4 is the number density of pure He^4 at $T=0$, and $x = n_3/n_4$ is the He^3 molar concentration, where n_3 is the He^3 density. The scattering rate (2) has the q^4 dependence characteristic of Rayleigh scattering. Lastly, since in the temperature region (above roughly a tenth of a degree for a few percent He^3 concentration) where phonons become numerous enough to play a role in transport processes, the He^3-He^3 scattering rates are sufficiently rapid that one may, in calculating the phonon transport properties, make the simplifying assumption that the He^3 quasiparticles are in local thermal equilibrium. At extremely low temperatures the phonon contributions to transport can generally be neglected.

The absorption of phonons by the He^3, which becomes kinematically possible because of He^3-He^3 interactions, is the mechanism responsible for ultrasonic attenuation at very low temperatures. The absorption rate, calculated in I in terms of the He^3-He^3 scattering times, is proportional to q^2 at long wavelengths, and there it dominates the scattering rate of Eq. (2). Consequently, at low temperatures the phonon contributions to the transport coefficients are quite sensitive to the size of the absorption rate.

The one application of the phonon Boltzmann equation we make in this paper is to the calculation of the first viscosity of dilute solutions. In the temperature region below 0.6°K, to which we restrict ourselves, rotons and phonon-phonon interactions can be ignored. To lowest order in x and ρ_{ph}, the phonon normal mass density, the first viscosity is a sum of a He^3 contribution η_3 and a phonon contribution η_{ph}. We emphasize that this latter contribution is very unlike that in pure He^4, because here the phonon mean free paths are determined by the He^3-phonon interactions. The viscosity in the higher temperature region has been calculated by Zharkov,[2] including roton effects as well as phonon-

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[1] G. Baym and C. Ebner, Phys. Rev. 164, 235 (1967); referred to hereafter as I.

[2] V. N. Zharkov, Zh. Eksperim. i Teor. Fiz. 33, 929 (1957) [English transl.: Soviet Phys.—JETP 6, 714 (1958)].

phonon interactions; however, that theory fails to include the important phonon absorption mechanism, and is not applicable, for a few percent He^3, to the temperature regimes where our calculation is valid.

II. PHONON BOLTZMANN EQUATION

In this section we derive the phonon Boltzmann equation for dilute solutions, starting from the expressions for phonon scattering and absorption rates calculated in I. The phonon Boltzmann equation has the general form

$$\partial n_{\mathbf{q}}/\partial t+\nabla_{\mathbf{q}}\omega_{\mathbf{q}}\cdot\nabla_{\mathbf{r}}n_{\mathbf{q}}-\nabla_{\mathbf{r}}\omega_{\mathbf{q}}\cdot\nabla_{\mathbf{q}}n_{\mathbf{q}}=I_{3\text{-ph}}+I_{\text{ph-ph}}\,, \tag{3}$$

where $n_{\mathbf{q}}$ is the phonon distribution function and $\omega_{\mathbf{q}}$ is the local phonon energy; in the presence of a superfluid flow with velocity $\mathbf{v}_s$, the phonon energy is given by

$$\omega_{\mathbf{q}}=sq+\mathbf{q}\cdot\mathbf{v}_s\,, \tag{4}$$

where the sound velocity s may depend on position and time. $I_{3\text{-ph}}$ and $I_{\text{ph-ph}}$ are the impurity-phonon and phonon-phonon collision operators, respectively; $I_{3\text{-ph}}$ includes both the scattering and absorption or emission of phonons by the He^3. We shall neglect boundary scattering throughout.

It is simplest to construct the collision integrals in the frame in which $\mathbf{v}_s=0$; we denote quantities measured in this frame by a bar. Then the rate at which phonon-impurity scattering changes $n_{\mathbf{q}}$ is

$$\begin{aligned}\bar{I}_{\text{scatt}}=-2\sum_{\mathbf{pp}'\mathbf{q}}2\pi\delta(\bar{\epsilon}_{\mathbf{p}}+\bar{\omega}_{\mathbf{q}}-\bar{\epsilon}_{\mathbf{p}'}-\bar{\omega}_{\mathbf{q}'})|\langle\mathbf{p}'\mathbf{q}'|T|\mathbf{pq}\rangle|^2\\ \times[\bar{n}_{\mathbf{q}}(1+\bar{n}_{\mathbf{q}'})\bar{f}_{\mathbf{p}}(1-\bar{f}_{\mathbf{p}'})\\ -(1+\bar{n}_{\mathbf{q}})\bar{n}_{\mathbf{q}'}(1-\bar{f}_{\mathbf{p}})\bar{f}_{\mathbf{p}'}]\,,\end{aligned} \tag{5}$$

where $\langle\mathbf{p}'\mathbf{q}'|T|\mathbf{pq}\rangle$ is the matrix element for scattering of a phonon of momentum $\mathbf{q}$ by a quasiparticle of momentum $\mathbf{p}$ to final states $\mathbf{q}'$, $\mathbf{p}'$; the factor of 2 is from the sum over spin states. (The generalization to independent spin up and spin down fermion distributions is trivial and not necessary here.) $f_{\mathbf{p}}$ is the He^3 distribution function and $\epsilon_{\mathbf{p}}$ the quasiparticle energy in the laboratory frame, and we let the volume of the system be unity.

We shall need only the form of (5) linearized about global equilibrium. Since $\mathbf{v}_s$ is a small quantity, the transformation back to the laboratory produces only first-order deviations from equilibrium. In equilibrium the square bracket in (5) times the δ function vanishes by detailed balancing. Thus the deviations of the matrix elements from equilibrium do not contribute, and we may take for $\langle\mathbf{p}'\mathbf{q}'|T|\mathbf{pq}\rangle$ its equilibrium value (1.24). Furthermore, the deviations of the phonon and quasiparticle energies from their equilibrium values lead to terms of relative order v/s in the collision integral; v is a typical quasiparticle thermal velocity. For example, $\delta\bar{\omega}_q-\delta\bar{\omega}_{q'}=\delta s(q-q')$, while to lowest order $q=q'$ in the scattering. Hence, we need take into account only the deviations of the distribution functions from their equilibrium values. Using the fact that the scattering is elastic in lowest order, we may set $q=q'$ and $p=p'$ in the equilibrium parts of (5); after linearizing, (5) thus becomes

$$\begin{aligned}\bar{I}_{\text{scatt}}=-2\sum_{\mathbf{pp}'\mathbf{q}}2\pi\delta(\epsilon_p+\omega_q-\epsilon_{p'}-\omega_{q'})|\langle\mathbf{p}'\mathbf{q}'|T|\mathbf{pq}\rangle|^2\\ \times\{f_p{}^0(1-f_p{}^0)[\delta\bar{n}_{\mathbf{q}}-\delta\bar{n}_{\mathbf{q}'}]\\ +n_q{}^0(1+n_q{}^0)[\delta\bar{f}_{\mathbf{p}}-\delta\bar{f}_{\mathbf{p}'}]\}\,,\end{aligned} \tag{6}$$

where we have written $f_{\mathbf{p}}=f_p{}^0+\delta f_{\mathbf{p}}$ and $n_{\mathbf{q}}=n_q{}^0+\delta n_{\mathbf{q}}$. The transformation back to the laboratory frame is trivial. Since the phonon momentum $\mathbf{q}$ is independent of the reference frame in which it is measured, we have $n_{\mathbf{q}}=\bar{n}_{\mathbf{q}}$; furthermore, the rate of change of this distribution function is the same in both frames so that

$$I_{\text{scatt}}=\bar{I}_{\text{scatt}}\,. \tag{7}$$

In addition, $\bar{f}_{\mathbf{p}}=f_{\mathbf{p}+m_3\mathbf{v}_s}$, where m_3 is the bare He^3 mass. Thus in the laboratory frame

$$\begin{aligned}I_{\text{scatt}}=-2\sum_{\mathbf{pp}'\mathbf{q}'}2\pi\delta(\epsilon_p+\omega_q-\epsilon_{p'}-\omega_{q'})|\langle\mathbf{p}'\mathbf{q}'|T|\mathbf{pq}\rangle|^2\\ \times\{f_p{}^0(1-f_p{}^0)[\delta n_{\mathbf{q}}-\delta n_{\mathbf{q}'}]+n_q{}^0(1+n_q{}^0)\\ \times[\delta f_{\mathbf{p}+m_3\mathbf{v}_s}-\delta f_{\mathbf{p}'+m_3\mathbf{v}_s}]\}\,.\end{aligned} \tag{8}$$

In the temperature regions where scattering of phonons by He^3 plays a role, the He^3-He^3 scattering times are sufficiently short that we may assume the He^3 to be in local thermal equilibrium[3]; thus we have

$$\begin{aligned}f_{\mathbf{p}+m_3\mathbf{v}_s}=\{\exp\beta_3'[\epsilon_0'+p^2/2m\\ -\mathbf{p}\cdot(\mathbf{v}_3-\mathbf{v}_s)-\mu_3']+1\}^{-1}\,,\end{aligned} \tag{9}$$

where the primes denote local equilibrium values and $\mathbf{v}_3$ is the local mean velocity of the He^3 distribution. Linearizing (9) implies that

$$\begin{aligned}\delta f_{\mathbf{p}+m_3\mathbf{v}_s}-\delta f_{\mathbf{p}'+m_3\mathbf{v}_s}\\ =-\beta(\mathbf{p}-\mathbf{p}')\cdot(\mathbf{v}_s-\mathbf{v}_3)f_p{}^0(1-f_p{}^0)\,.\end{aligned} \tag{10}$$

Furthermore, it is sufficiently accurate in the temperature region in which (8) is important to neglect the exclusion principle for the He^3 both in the matrix elements and in the final states. Thus, if we write

$$\langle\mathbf{p}'\mathbf{q}'|T|\mathbf{pq}\rangle=\delta_{\mathbf{p}+\mathbf{q},\mathbf{p}'+\mathbf{q}'}T(q,\theta)\,, \tag{11}$$

where θ is the angle between $\mathbf{q}$ and $\mathbf{q}'$, (8) simplifies

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[3] The temperature characterizing the local equilibrium for the He^3 quasiparticles is not the same as the temperature characterizing local equilibrium for the fermion-phonon system as a whole. This point is discussed further in Sec. III.

to

$$I_{\text{scatt}}=-n_3\sum_{q'}2\pi\delta(sq-sq')\,|T(q,\theta)|^2\times\left[\delta n_{\mathbf{q}}-\delta n_{\mathbf{q}'}-\frac{\partial n_q^{\,0}}{\partial\omega_q}(\mathbf{q}-\mathbf{q}')\cdot(\mathbf{v}_s-\mathbf{v}_3)\right]$$

$$=-\int\frac{d\Omega_{q'}}{2\pi}\Gamma(q,\theta)\times\left[\delta n_{\mathbf{q}}-\delta n_{\mathbf{q}'}-\frac{\partial n_q^{\,0}}{\partial\omega_q}(\mathbf{q}-\mathbf{q}')\cdot(\mathbf{v}_s-\mathbf{v}_3)\right],\quad(12)$$

where $\Gamma(q,\theta)$ is given by (2).

Since $q=q'$ in (12), the eigenfunctions of the scattering operator are simply spherical harmonics. We write

$$\delta n_{\mathbf{q}}=(\partial n_q^{\,0}/\partial\omega_q)q\Phi(\mathbf{q})\quad(13)$$

and expand $\Phi(\mathbf{q})$ in spherical harmonics

$$\Phi(\mathbf{q})=\sum_{l=0}^{\infty}\Phi_l(\mathbf{q}),\quad(14)$$

where

$$\Phi_l(q)=\sum_{m=-l}^{l}Y_{lm}(\Omega_q)\Phi_{lm}(q).\quad(15)$$

Then, making use of the addition theorem for spherical harmonics we find

$$I_{\text{scatt}}=-q\frac{\partial n_q^{\,0}}{\partial\omega_q}\left[\frac{\hat{q}\cdot(\mathbf{v}_3-\mathbf{v}_s)}{\tau_1(q)}+\sum_{l=1}^{\infty}\frac{\Phi_l(\mathbf{q})}{\tau_l(q)}\right],\quad(16)$$

where

$$\tau_l(q)^{-1}=\int_{-1}^{1}d\cos\theta[1-P_l(\cos\theta)]\Gamma(q,\theta),\quad l\geq1.\quad(17)$$

The net effect of the τ_1 terms in (16) is to urge the phonon distribution to relax about $\mathbf{v}_3$, the local He^3 velocity. Because $\Gamma(q,\theta)$ contains terms only up to $P_2(\cos\theta)$, all of the $\tau_l(q)^{-1}$ for $l\geq3$ are equal to

$$\tau_0^{-1}=\int_{-1}^{1}d(\cos\theta)\,\Gamma(q,\theta).\quad(18)$$

Numerically we have

$$\begin{aligned}\tau_0&=5.48n_4/xsq^4,\\ \tau_1&=4.35n_4/xsq^4,\\ \tau_2&=5.96n_4/xsq^4.\end{aligned}\quad(19)$$

We turn now to the contribution I_{abs} to $I_{3\text{-ph}}$ due to absorption or emission of phonons by the He^3 quasiparticles; this is the mechanism that attenuates ultrasound at low temperatures. The net absorption rate, as calculated in I, Eq. (42), is

$$\Gamma_{\text{abs}}(q)\equiv\frac{1}{\tau_a(q)}=\frac{4}{3}\frac{m_4P_f(T)}{m^2n_4}\times\frac{(1+\alpha+\delta m/m_4)^2q^2\tau_\eta[1+(\hbar sq/2\pi KT)^2]}{[1+(\hbar sq/2\pi KT)^2]^2+s^2q^2\tau_\eta^{\,2}},\quad(20)$$

where $\delta m=m-m_3$, m_4 is the bare He^4 mass and $P_f(T)$ is the pressure of an ideal Fermi gas of effective mass m and density n_3, α is the fractional excess molar volume of He^3 in He^4, and τ_η, estimated in I, is the He^3-He^3 scattering time appropriate to viscosity. When the He^3 is in local equilibrium at temperature T_3' and velocity $\mathbf{v}_3$, the absorption process causes the phonons to relax to local equilibrium at temperature T_3' and velocity $\mathbf{v}_3$. We write the net rate of absorption $I_{\text{abs}}(q)$ as the difference of an emission and absorption rate,

$$I_{\text{abs}}=-n_{\mathbf{q}}\Sigma^{>}(\mathbf{q},\omega_{\mathbf{q}}')+(1+n_{\mathbf{q}})\Sigma^{<}(\mathbf{q},\omega_{\mathbf{q}}'),\quad(21)$$

where $\Sigma^{>}$ and $\Sigma^{<}$ are the separate absorption and emission rates, and ω_q' is the local phonon energy. Using the detailed balancing relation that holds when the He^3 is in local equilibrium[4]

$$\Sigma^{<}(\mathbf{q},\omega)=e^{-\beta_3'(\omega-\mathbf{q}\cdot\mathbf{v}_3)}\Sigma^{>}(\mathbf{q},\omega)\quad(22)$$

and writing

$$\Gamma_{\text{abs}}(\mathbf{q})=(1-\exp[-\beta_3'(\omega_{\mathbf{q}}'-\mathbf{q}\cdot\mathbf{v}_3)])\Sigma^{>}(\mathbf{q},\omega_{\mathbf{q}}')\quad(23)$$

we find that the net absorption rate is

$$I_{\text{abs}}=-(n_{\mathbf{q}}-n_{\mathbf{q}}^{(3)})/\tau_a(q),\quad(24)$$

where

$$n_{\mathbf{q}}^{(3)}=\{\exp[\beta_3'(\omega_{\mathbf{q}}'-\mathbf{q}\cdot\mathbf{v}_3)]-1\}^{-1}\quad(25)$$

is the local phonon equilibrium distribution that is characterized by the local He^3 temperature and velocity. Using Eqs. (11) and (2) and linearizing the local equilibrium distribution, viz.,

$$n_{\mathbf{q}}^{(3)}=n_q^{\,0}+q(\partial n_q^{\,0}/\partial\omega_q)[\delta(\beta_3's)+\hat{q}\cdot(\mathbf{v}_s-\mathbf{v}_3)],$$

we find that the absorption term in the Boltzmann equation becomes

$$I_{\text{abs}}=-q\frac{\partial n_q^{\,0}}{\partial\omega_q}\frac{\Phi(\mathbf{q})-\beta^{-1}\delta(\beta_3's)-\hat{q}\cdot(\mathbf{v}_s-\mathbf{v}_3)}{\tau_a}.\quad(26)$$

On the left-hand side of the Boltzmann equation we may write $\nabla_q n_q=sq\partial n_q^{\,0}/\partial\omega_q$ and $\nabla_r\omega_{\mathbf{q}}=q\nabla_r s+\nabla_r(\mathbf{q}\cdot\mathbf{v}_s)$. Assembling all the pieces, the phonon

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[4] See, for example, L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962), Eq. (10.5).

Boltzmann equation becomes finally

$$\frac{\partial n_{\mathbf{q}}}{\partial t}+\nabla_q\omega_q\cdot\nabla_r n_{\mathbf{q}}-s\hat{q}\cdot\nabla_r(sq+\mathbf{q}\cdot\mathbf{v}_s)\frac{\partial n_q^0}{\partial\omega_q}$$
$$=-q\frac{\partial n_q^0}{\partial\omega_q}\left[\frac{\Phi(\mathbf{q})-\beta^{-1}\delta(\beta_3's)+\hat{q}\cdot(\mathbf{v}_3-\mathbf{v}_s)}{\tau_a(q)}+\frac{\hat{q}\cdot(\mathbf{v}_3-\mathbf{v}_s)}{\tau_1(q)}+\sum_{l=1}^{\infty}\frac{\Phi_l(\mathbf{q})}{\tau_l(q)}\right]+I_{\text{ph-ph}}. \qquad (27)$$

At present we shall not specify a form for $I_{\text{ph-ph}}$. At temperatures below about 0.6°K, where $I_{\text{ph-ph}}$ may be neglected, the Boltzmann equation (27) is exactly soluble. The essential reason for this is that $\tau_l(q)=\tau_0(q)$ for $l\geq 3$.

III. STRESS TENSOR AND FIRST VISCOSITY

In order to calculate the viscosity coefficients of the dilute solutions we must first construct the form of the stress tensor for the system. The stress tensor is determined from the equation of motion for the total momentum, which is given by

$$\mathbf{g}=\sum_{\mathbf{p}}\mathbf{p}f_{\mathbf{p}}+\sum_{\mathbf{q}}\mathbf{q}n_{\mathbf{q}}+m_4n_4\mathbf{v}_s. \qquad (28)$$

The phonon momentum can be written in linear order as

$$\sum_{\mathbf{q}}\mathbf{q}n_{\mathbf{q}}=\rho_{\text{ph}}(\mathbf{v}_n-\mathbf{v}_s), \qquad (29)$$

thus defining the phonon normal velocity and normal mass denslty; the phonon normal mass density is given by the Landau result

$$\rho_{\text{ph}}=-\sum_{\mathbf{q}}\frac{q^2}{3}\frac{\partial n_q^0}{\partial\omega_q}=\frac{2\pi^2}{45}\frac{(KT)^4}{s^5\hbar^3}. \qquad (30)$$

$\mathbf{g}$ can alternatively be written as

$$\mathbf{g}=\sum_{\mathbf{p}}(\mathbf{p}+\delta m\mathbf{v}_s)f_p+\rho_{\text{ph}}\mathbf{v}_n+\rho_s\mathbf{v}_s, \qquad (31)$$

where the superfluid mass density is given by

$$\rho_s=m_4n_4-\rho_{\text{ph}}-\delta mn_3. \qquad (32)$$

When the He^3 are in local equilibrium $\sum_p(\mathbf{p}+\delta m\mathbf{v}_s)f_p=mn_3\mathbf{v}_3$. From the phonon Boltzmann equation (27) we find the linearized equation

$$\frac{\partial}{\partial t}\sum_{\mathbf{q}}\mathbf{q}n_{\mathbf{q}}+\nabla_{\mathbf{r}}\cdot\sum_{\mathbf{q}}s\hat{q}\mathbf{q}n_{\mathbf{q}}+\rho_{\text{ph}}s\nabla s=\sum_q\mathbf{q}(I_{\text{scatt}}+I_{\text{abs}}). \qquad (33)$$

From the He^3 Boltzmann equation in I(A2) we have

$$\frac{\partial}{\partial t}\sum_{\mathbf{p}}\mathbf{p}f_{\mathbf{p}}+\nabla_r\cdot\sum_p\frac{\mathbf{pp}}{m}f_{\mathbf{p}}+n_3\nabla\epsilon_0=\sum_{\mathbf{p}}\mathbf{p}\left(\frac{\partial f_{\mathbf{p}}}{\partial t}\right)_{3\text{-ph}}. \qquad (34)$$

Adding (33) and (34) to the equation of motion for the superfluid

$$m_4\frac{\partial\mathbf{v}_s}{\partial t}+\nabla\mu_4=0, \qquad (35)$$

we then derive

$$\frac{\partial\mathbf{g}}{\partial t}+n_3\nabla\epsilon_0+n_4\nabla\mu_4+\rho_{\text{ph}}s\nabla s+\nabla\cdot\left(\sum_q s\hat{q}\mathbf{q}n_{\mathbf{q}}+\sum_p\frac{\mathbf{pp}}{m}f_{\mathbf{p}}\right)=0. \qquad (36)$$

The right-hand sides of (33) and (34) are equal and opposite in sign due to over-all momentum conservation in collisions.

To put (36) in the form of a Navier-Stokes hydrodynamic equation we must isolate the terms in the equation representing the deviations from local thermal equilibrium for the combined fermion-phonon system. The temperature T characterizing this local equilibrium is determined from the requirement that the variations in the total energy as determined from the exact distribution functions be the same as those determined from the local equilibrium functions. Explicitly the variation in total energy E, for fixed $\mathbf{v}_s$, is

$$\delta E=\mu_4\delta n_4+\sum_{\mathbf{p}}\epsilon_p\delta f_{\mathbf{p}}+\sum_{\mathbf{q}}\omega_q\delta n_{\mathbf{q}}; \qquad (37)$$

thus to first order the local equilibrium distributions obey

$$\sum_{\mathbf{p}}\epsilon_p(\delta f_{\mathbf{p}}-\delta f_{\mathbf{p}}^{le})+\sum_{\mathbf{q}}\omega_q(\delta n_{\mathbf{q}}-\delta n_{\mathbf{q}}^{le})=0, \qquad (38)$$

where $\delta f_{\mathbf{p}}^{le}$ and $\delta n_{\mathbf{q}}^{le}$ are the local equilibrium deviations for the impurities and phonons, respectively. As we mentioned earlier, at the temperatures and concentrations of interest, the He^3 tend to come into local equilibrium themselves, at temperature T_3', much before common He^3-phonon equilibrium is established. Because of He^3-phonon energy exchange via phonon absorption, the common local equilibrium temperature T differs from the initial T_3'.

Proceeding, we note that the local equilibrium distribution, $n_{\mathbf{q}}^{le}$, obeys

$$\nabla\cdot\sum_{\mathbf{q}}s\hat{q}\mathbf{q}n_{\mathbf{q}}^{le}=\sum_q\frac{sq}{3}\frac{\partial n_q^0}{\partial\omega_q}\frac{1}{\beta}\nabla(\beta sq)=-\frac{s\rho_{\text{ph}}}{\beta}\nabla(\beta s)$$
$$=-s\rho_{\text{ph}}\nabla s+S_{\text{ph}}\nabla T, \qquad (39)$$

where $S_{\text{ph}}=s^2\rho_{\text{ph}}/T$ is the equilibrium phonon entropy per unit volume. Also,

$$\nabla\cdot\sum_{\mathbf{p}}\frac{\mathbf{pp}}{m}f_p^{le}=\sum_{\mathbf{p}}\frac{p^2}{3m}\frac{\partial f_p^0}{\partial\epsilon_p}\frac{1}{\beta}\nabla[\beta(\epsilon_p-\mu_3^{le})]$$
$$=S_3\nabla T-n_3\nabla(\epsilon_0-\mu_3^{le}), \qquad (40)$$

where $S_3=[\frac{5}{2}P_f(T)+(\epsilon_0-\mu_3)n_3]/KT$ is the He^3 entropy.

In local thermodynamic equilibrium the Gibbs-Duhem relation is

$$\delta P(n_3,n_4,T)=n_3\delta\mu_3{}^{le}+n_4\delta\mu_4{}^{le}+S\delta T\,, \tag{41}$$

where P is the local thermodynamic pressure, $\mu_4{}^{le}$ is the He^4 chemical potential for local equilibrium, and $S=S_3+S_{\rm ph}$ is the total entropy density. Combining (36) and (39)–(41) we obtain

$$(\partial \mathbf{g}/\partial t)+\nabla\cdot\boldsymbol{\pi}=0\,, \tag{42}$$

where the stress tensor is

$$\pi_{ij}=\sum_{\mathbf{p}}\frac{p_ip_j}{m}(f_{\mathbf{p}}-f_{\mathbf{p}}{}^{le})+\sum_{\mathbf{q}}\frac{sq_iq_j}{q}(n_{\mathbf{q}}-n_{\mathbf{q}}{}^{le})$$
$$+\delta_{ij}[P+n_4(\mu_4-\mu_4{}^{le})]\,. \tag{43}$$

The difference of μ_4 from its local equilibrium value can be found by first noting from (37) that

$$\delta\mu_4=\left(\frac{\partial^2E}{\partial n_4{}^2}\right)_{n_q,f_p}\delta n_4+\sum_{\mathbf{q}}\left(\frac{\partial\omega_q}{\partial n_4}\right)_{n_q,f_p}\delta n_{\mathbf{q}}$$
$$+\sum_{\mathbf{p}}\left(\frac{\partial\epsilon_p}{\partial n_4}\right)_{n_q,f_p}\delta f_{\mathbf{p}}\,. \tag{44}$$

Thus, to first order,

$$\mu_4-\mu_4{}^{le}=\sum_{\mathbf{q}}\left(\frac{\partial\omega_q}{\partial n_4}\right)_{n_q,f_p}(n_{\mathbf{q}}-n_{\mathbf{q}}{}^{le})$$
$$+\sum_{\mathbf{p}}\left(\frac{\partial\epsilon_p}{\partial n_4}\right)_{n_q}(f_{\mathbf{p}}-f_{\mathbf{p}}{}^{le})\,. \tag{45}$$

Equation (43) is the general form of the stress tensor from which the viscosity coefficients of the solutions can be inferred.

In the hydrodynamic limit[1]

$$\sum_{\mathbf{p}}\frac{\mathbf{p}(\mathbf{p}\cdot\nabla)}{m}(f_{\mathbf{p}}-f_{\mathbf{p}}{}^{le})=-\eta_3[\nabla^2\mathbf{v}_3+\tfrac{1}{3}\nabla(\nabla\cdot\mathbf{v}_3)]\,, \tag{46}$$

where $\eta_3=P_f(T)\tau_\eta$ and τ_η is the He^3-He^3 quasiparticle collision time appropriate to viscosity. More generally, whenever $vk\ll|\omega+i/\tau_\eta|$ where v is a typical He^3 velocity, η_3 in (41) is replaced by $\eta_3(1-i\tau_\eta\omega)^{-1}$. The detailed form of the phonon contribution to the stress tensor is determined by solving the phonon Boltzmann equation [27] for $n_{\mathbf{q}}-n_{\mathbf{q}}{}^{le}$.

In this paper we confine our attention to a transverse driving force on the system corresponding to a measurement of the first or shear viscosity. Because of the relatively long phonon mean free paths in very low-temperature solutions the results of a shear-viscosity measurement depend sensitively on the frequency of the driving force, and we must consider the AC and DC cases separately. For zero, or very low-frequency conditions both the impurities and phonons may be treated hydrodynamically. This limit is realized, for example, in viscosity measurements utilizing capillary flow,[5] slowly oscillating disks,[6] low-frequency second sound attenuation,[7] or a rotating cylinder viscometer.[8] On the other hand, in an AC viscosity measurement,[9] where one observes the damping of a crystal immersed in the solution and oscillating in a transverse vibrational mode at frequencies on the order of 10 Kc/sec, a new situation appears: the He^3 impurities can be treated hydrodynamically but the phonons must be treated in a low frequency but intermediate to high wave number regime.

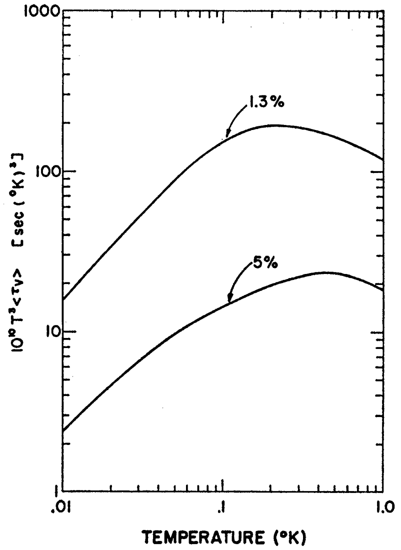

FIG. 1. Phonon viscosity lifetimes in 1.3 and 5% solutions of He^3 in He^4.

We assume that all spatial variations occur in the z direction only and that all velocities are in the x direction. Because a transverse probe cannot excite longitudinal variations in linear order, all longitudinal variations such as $\mathbf{v}_s$, δP, δs, and $\delta\beta$ are zero. Neglecting phonon-phonon collisions at low temperatures, the Fourier-transformed Boltzmann equation becomes

$$\left[\omega-\nabla_q\omega_q\cdot\mathbf{k}+i\left(\frac{1}{\tau_0}+\frac{1}{\tau_a}\right)\right]\Phi(\mathbf{q})$$
$$=-i\left[\hat{q}\cdot\mathbf{v}_3\left(\frac{1}{\tau_a}+\frac{1}{\tau_1}\right)\frac{\Phi_0}{\tau_0}\right.$$
$$\left.+\Phi_1\left(\frac{1}{\tau_1}-\frac{1}{\tau_0}\right)+\Phi_2\left(\frac{1}{\tau_2}-\frac{1}{\tau_0}\right)\right]. \tag{47}$$

This equation is readily solved in general for $\Phi(q)$ by

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[9] See, for example, R. W. H. Webeler and D. C. Hammer, Phys. Letters **19**, 533 (1965).

dividing both sides by the square bracket on the left and taking angular moments to solve for the explicit Φ_{lm} on the right. Since

$$\hat{q}\cdot\mathbf{v}_3=-(2\pi/3)^{1/2}(Y_{11}-Y_{1,-1})v_3,$$

the driving term in (47), and hence Φ, contains only $m=\pm 1$ components and $\Phi_{l,-1}=-\Phi_{l1}$.

We first construct the solution of (47) in the hydrodynamic, i.e., ω and $k\rightarrow 0$ limit. For ω and $k=0$ we have

$$\Phi(\mathbf{q})=-\hat{q}\cdot\mathbf{v}_3;\tag{48}$$

the phonons equilibrate at the local He^3 velocity. To determine the next order in ω and k, we substitute (48) for the coefficient of $\omega-s\hat{q}\cdot\mathbf{k}$ in (47). Thus

$$(\omega-s\hat{q}\cdot\mathbf{k})(-\hat{q}\cdot\mathbf{v}_3)$$
$$=-i\left[(\hat{q}\cdot v_3+\Phi_1)\left(\frac{1}{\tau_1}+\frac{1}{\tau_a}\right)+\Phi_2\left(\frac{1}{\tau_2}+\frac{1}{\tau_a}\right)\right.$$
$$\left.+\sum_{l=3}\Phi_l\left(\frac{1}{\tau_0}+\frac{1}{\tau_a}\right)\right],\tag{49}$$

and by comparing coefficients of the various spherical harmonics we see that only $\Phi_{1,\pm1}$ and $\Phi_{2,\pm1}$ are nonzero and that

$$\Phi(\mathbf{q})=-\hat{q}\cdot\mathbf{v}_3(1+i\omega\tau_\kappa)+is\hat{q}\cdot\mathbf{k}\hat{q}\cdot\mathbf{v}_3\tau_v,\tag{50}$$

where $\tau_\kappa(q)=(\tau_1^{-1}+\tau_a^{-1})^{-1}$ is the lifetime for phonon thermal conductivity and $\tau_v(q)=(\tau_2^{-1}+\tau_a^{-1})^{-1}$ is the lifetime for phonon viscosity. Then the phonon contribution to the stress tensor, the sum over q in Eq. (43), is given by

$$\pi_{\mathrm{ph},xz}=s\sum_{\mathbf{q}}q_xq_z\frac{\partial n_q^0}{\partial\omega_q}\Phi=-\tfrac{1}{5}is^2kv_3\rho_{\mathrm{ph}}\langle\tau_v\rangle$$
$$=-i\eta_{\mathrm{ph}}kv_3,\tag{51}$$

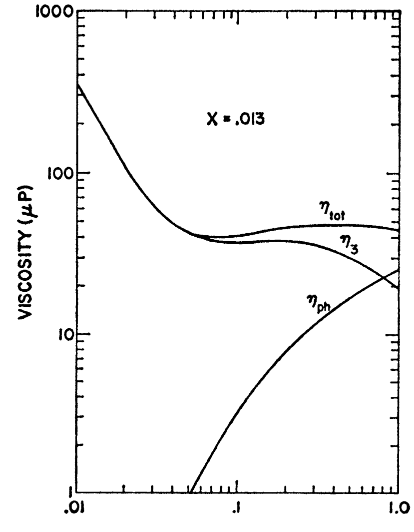

FIG. 2. Phonon, He^3, and total viscosities in a 1.3% solution of He^3 in He^4.

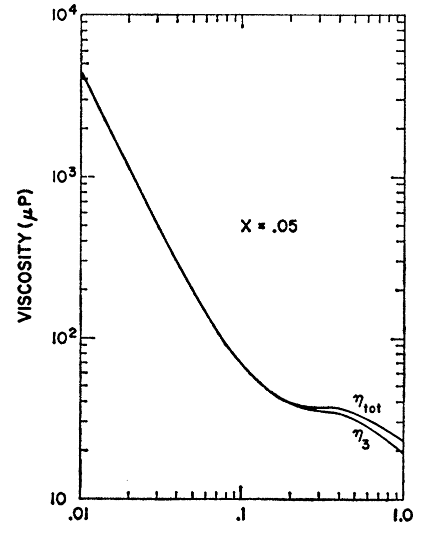

FIG. 3. He^3 and total viscosities in a 5% solution of He^3 in He^4.

where, for any function A of q,

$$\langle A\rangle\equiv-\frac{1}{\rho_{\mathrm{ph}}}\sum_{\mathbf{q}}\frac{q^2}{3}\frac{\partial n_q^0}{\partial\omega_q}A(q).\tag{52}$$

Thus

$$\eta_{\mathrm{ph}}=\tfrac{1}{5}s^2\rho_{\mathrm{ph}}\langle\tau_v\rangle.\tag{53}$$

The average lifetime for viscosity $\langle\tau_v\rangle$, evaluated numerically, is shown as a function of temperature for 1.3 and 5% He^3 concentrations in Fig. 1. Note that as in the calculation of thermal conductivity,[1] $\langle\tau_v\rangle$ diverges if τ_a is neglected.

We see from (48) and (29) that in the hydrodynamic limit the phonon normal velocity v_n equals the He^3 normal velocity v_3; then (51), (46), and (43) imply that

$$\pi_{xy}=-i(\eta_3+\eta_{\mathrm{ph}})kv_3.\tag{54}$$

The first viscosity of the solution is then

$$\eta_{\mathrm{tot}}=\eta_3+\eta_{\mathrm{ph}}=P_f(T)\tau_\eta+\tfrac{1}{5}s^2\rho_{\mathrm{ph}}\langle\tau_v\rangle.\tag{55}$$

η_{tot} is plotted in Fig. 2 for a 1.3% solution and in Fig. 3 for a 5% solution. Although η is plotted up to 1°K, these calculations are valid only below 0.6°K since we have neglected rotons and phonon-phonon interactions.

We should point out that discrepancies between the high-temperature phonon thermal conductivity calculated in I and measurements of the thermal conductivity[10] above 0.65°K do not appear to be attributable entirely to the neglected rotons and phonon-phonon interactions. This suggests that τ_η for high temperatures was considerably overestimated in I. A reduction in τ_η

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[10] T. P. Ptukha, Zh. Eksperim. i Teor. Fiz. **40**, 1583 (1961) [English transl.: Soviet Phys.—JETP **13**, 1112 (1961)].

would lead to a decrease in both η_3 and η_{ph}. Measurements of the viscosity below 0.6°K would serve to clarify this situation. At present the viscosity has been measured[5] in the hydrodynamic regime down to 1°K; the results are of the same order of magnitude as our calculated values but also suggest that the theoretical τ_η is too large.

The hydrodynamic expression (51) for the phonon stress tensor is valid for $\omega\langle\tau_v\rangle$ and $sk\langle\tau_v\rangle\ll 1$. A zero frequency experiment is in the hydrodynamic regime for small velocity gradients. For nonzero frequencies ω and k are related by combining the results (42) and (54) to yield[11]

$$k=(1+i)\left(\frac{\omega\rho_n}{2\eta_{\mathrm{tot}}}\right)^{1/2}, \tag{56}$$

where ρ_n is the normal mass density, given to lowest order in the concentration by

$$\rho_n=\rho-\rho_s=mn_3+\rho_{\mathrm{ph}}, \tag{57}$$

in the regime where rotons may be neglected. Thus, in order of magnitude we have

$$\begin{aligned} s|k|\langle\tau_v\rangle &\sim s[\omega mn_3/\eta_3]^{1/2}\langle\tau_v\rangle \\ &\sim[(s/v_f)((\langle\tau_v\rangle/\tau_\eta)^{1/2}](\omega\langle\tau_v\rangle)^{1/2}. \end{aligned} \tag{58}$$

Since $\tau_\eta\ll\langle\tau_v\rangle$ and $s/v_f\sim 10$, the inequality $s|k|\langle\tau_v\rangle\ll 1$ is extremely hard to satisfy even if $\omega\langle\tau_v\rangle\ll 1$. Using (56) and our numerical results for η_{tot} we find that the purely hydrodynamic result is valid to within 10% over the range 0.2 to 0.6°K, in which phonons are important, only for frequencies of the order of 0–10 cps. The frequency restrictions become slightly less strict as the temperature is increased. The hydrodynamic result for the He^3 viscosity, valid for $\omega\tau_\eta\ll 1$ and $v_f k\tau_\eta\ll 1$ only requires frequencies smaller than about 100 kc/sec for $T\gtrsim 0.01$°K.

As mentioned above, typical frequencies used in high frequency AC shear viscosity experiments are of the order of 10 kc/sec. For this frequency, over the temperature range where phonons are important, we still have $\omega\langle\tau_v\rangle\ll 1$ but $sk\langle\tau_v\rangle\gg 1$, except near 0.6°K where $sk\langle\tau_v\rangle\gtrsim 1$. To find π_{ph} in this regime we must solve (47) in the low frequency but high wave number limit. In this limit

$$\Phi(q)=\frac{[(2\pi/3)^{1/2}v_3\tau_\kappa^{-1}+\Phi_{11}t_1^{-1}](Y_{11}-Y_{1,-1})+t_2^{-1}\Phi_{21}(Y_{21}-Y_{2,-1})}{\tau_z^{-1}+is\mathbf{k}\cdot\hat{q}}; \tag{59}$$

for simplicity we have written $\tau_z^{-1}=\tau_0^{-1}+\tau_a^{-1}$, $t_1^{-1}=\tau_0^{-1}-\tau_1^{-1}$, and $t_2^{-1}=\tau_0^{-1}-\tau_2^{-1}$. The $l=1$, $m=1$ moment of (58) implies, for $sk\tau_z\approx sk\tau_v\gg 1$,

$$\Phi_{11}\approx\frac{3\pi}{4sk}\left[\left(\frac{2\pi}{3}\right)^{1/2}\frac{v_3}{\tau_\kappa}+\frac{\Phi_{11}}{t_1}\right]-\frac{i5^{1/2}\Phi_{21}}{skt_2}, \tag{60}$$

while the $l=1$, $m=1$ moment of the Boltzmann equation (47) yields

$$\Phi_{11}+i5^{-1/2}sk\tau_\kappa\Phi_{21}=(2\pi/3)^{1/2}v_3. \tag{61}$$

Thus from (60) and (61) we find

$$\Phi_{21}\approx\frac{(10\pi/3)^{1/2}v_3}{isk\tau_\eta}\approx\frac{5^{1/2}4}{3\pi i}\Phi_{11}, \tag{62}$$

for large $sk\tau_v$ and small $\omega\tau_v$. The phonon contribution to the stress tensor is then

$$\pi_{\mathrm{ph},xz}=(\tfrac{3}{10})^{1/2}s\rho_{\mathrm{ph}}\langle\Phi_{21}\rangle=-i\rho_{\mathrm{ph}}v_3\langle\tau_\kappa^{-1}\rangle k^{-1}; \tag{63}$$

in addition

$$v_n=(3/2\pi)^{1/2}\langle\Phi_{11}\rangle=3\pi v_3\langle\tau_\kappa^{-1}\rangle/4sk. \tag{64}$$

We now use (42) to relate ω and k; the v_n contribution to g can be neglected for $\rho_{\mathrm{ph}}\ll mn_3$. Thus

$$k^2=(imn_3\omega-\rho_{\mathrm{ph}}\langle\tau_\kappa^{-1}\rangle)/\eta_3. \tag{65}$$

To a very good approximation $\langle\tau_\kappa^{-1}\rangle=\langle\tau_1^{-1}\rangle$ which can be evaluated analytically as

$$\langle\tau_1^{-1}\rangle=3.9\times 10^{11}(xT^4)\ \mathrm{sec}^{-1}\ (°\mathrm{K})^{-4}. \tag{66}$$

This result is considerably larger than $\langle\tau_v\rangle^{-1}$ or $\langle\tau_\kappa\rangle^{-1}$ in the temperature range of interest. Thus below 0.3°K for $\omega\approx 10$ kc/sec we may approximate k^2 by $imn_3\omega/\eta_3$, and we find that the condition $sk\gg\langle\tau_\kappa\rangle^{-1}$ is satisfied only up to 0.2°K. The temperature range 0.2°–0.6°K is an intermediate wavenumber, low-frequency regime, in which the relation between k and ω can be found only by extensive numerical calculation; such a calculation has not yet been carried out.

The stress tensor, for $T\lesssim 0.2$°K, is, from (65), (63), and (46),

$$\pi_{xz}=-ik\eta_3[1-\rho_{\mathrm{ph}}\langle\tau_\kappa^{-1}\rangle/\omega mn_3]v_3\approx -ik\eta_3 v_3. \tag{67}$$

Thus in the low-frequency and high-wavenumber limit, the phonon contribution to the viscosity is completely negligible; this effect extends to higher temperature the range over which τ_η is measured directly. As can be seen from Fig. 2, the disappearance of the phonon viscosity in an AC experiment is most noticeable in the 1.3% solution at 0.2°K, where the reduction from the hydrodynamic result is about 20%.

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[11] Equation (56) is a standard hydrodynamic result. See, e.g., L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959), p. 89.