# Quantized Gauge Field Theory of Chiral Symmetry*

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It will be shown that a consistent quantized gauge field theory of chiral symmetry can be formulated within the framework of the Gürsey models. The freedom of unitary transformations is emphasized, especially with respect to the problem of the decoupling of the $A_1$-meson field from the pion field. In the lowest order, our theory reproduces the same results as those calculated from other approaches. However, the higher-order terms are different.

## I. INTRODUCTION

AFTER the many successes of current-algebra calculations,[1] various Lagrangian models with the correct commutation relations and the PCAC (partially conserved axial-vector current) condition have been formulated.[2–4] The major aims are several: (1) It is easier to read off the matrix elements from an interacting Lagrangian. (2) They can be used to understand the off-mass-shell corrections, about which current-algebra technique makes no statement. (3) The different exchange mechanisms involved in the processes become transparent in a Lagrangian language. (4) The singular behaviors of the commutators can be partially answered.

In all these approaches, it has been emphasized repeatedly that only the "tree" graphs need be taken into account. In fact, exact current algebra and PCAC are not necessary. For a particular process we need to have these algebraic constraints only to the order of the "tree" graph.

In view of the success of the universal coupling of the $\rho$ meson,[5] it has been very tempting to speculate on formulating a gauge theory[6] of $SU(2)\times SU(2)$. This program received even more impetus after the celebrated Weinberg spectral sum rules[7] and the $\pi^+$-$\pi^0$ electromagnetic mass calculation[8] appeared. As has been explained by Lee, Weinberg, and Zumino[9] (LWZ), if the algebra of currents is replaced by the algebra of fields, the gauge theory provides us with more information about the Schwinger terms.[10] Besides, it makes the Weinberg sum rules come out as a result of the canonical commutation relations (i.e., kinematics).

In the work of LWZ, it was assumed that the pion field would cause no complication in the formal commutation relations. Thus it becomes necessary to demonstrate that the incorporation of the pion field can indeed be carried out,[11] and what further consequences may follow.

In this paper, we shall show that we can formulate a consistent quantized gauge theory within the framework of the Gürsey models.[12] In Sec. II, we shall discuss the general transformation properties of the nucleon field and the pion field which characterize the different Gürsey models.[3,12] Lagrangians are also constructed here.

Section III is devoted to the quantization of such models. We shall show that a consistent canonical scheme can be devised if proper fields are chosen as the independent ones. The commutation relations of LWZ follow.

In all these Lagrangians, bilinear coupling appears between the $A_1$ field and the pion field. In Sec. IV, we therefore choose as an example a particular model to demonstrate how a unitary gauge transformation[6] can be carried out to decouple them. In the lowest order, our transformed Lagrangian is the same as the ones discussed by other authors.[4] However, the higher-order terms according to our approach are different.

In this section and in Sec. V, we emphasize the importance of identification of fields with particles. Equivalently, we argue that unless further dynamical principles are assumed, the gauge transformations to decouple the $A_1$-$\pi$ fields are not unique. Because of different choices of the symmetry-breaking Lagrangian to maintain strong PCAC conditions, they give rise to different higher-order processes. The $\sigma$ terms[13] are also discussed here.

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## II. GÜRSEY MODELS

As is well known,[14] the mass term in the Lagrangian of a free isodoublet fermion field is not invariant under the chiral phase transformation

$$\Psi \to e^{g\gamma_5 \frac{1}{2}\tau\cdot\delta\theta}\Psi\,; \tag{1}$$

to make it so, we write the Lagrangian as[3,12]

$$\mathfrak{L}_1 = -\bar{\Psi}\gamma_\mu \frac{1}{i}\partial^\mu \Psi - M\bar{\Psi}U(\varphi)\Psi\,,$$

where $U(\varphi)$ is a unitary matrix function of the pion field $\varphi$ and satisfies the transformation property

$$U(\varphi) \to e^{-g\gamma_5\frac{1}{2}\tau\cdot\delta\theta}U(\varphi)e^{-g\gamma_5\frac{1}{2}\tau\cdot\delta\theta}. \tag{2}$$

If we now make the transformation parameters $\delta\theta$ space-time–dependent, we must introduce additional fields $V_\mu$ and $A_\mu$,[6] which transform as

$$V_\mu \to V_\mu + g\delta\theta\times A_\mu$$

and

$$A_\mu \to A_\mu + g\delta\theta\times V_\mu - \partial_\mu\delta\theta. \tag{3}$$

The Lagrangian

$$\mathfrak{L}_2 = -\bar{\Psi}\gamma_\mu\frac{1}{i}\partial^\mu\Psi - M\bar{\Psi}U(\varphi)\Psi + g\bar{\Psi}\gamma_\mu\tfrac{1}{2}\tau\Psi\cdot V^\mu + ig\bar{\Psi}\gamma_\mu\gamma_5\tfrac{1}{2}\tau\Psi\cdot A^\mu \tag{4}$$

is invariant under these combined transformations.

We digress to discuss the transformation property of the unitary matrix $U(\varphi)$. In general, it can be written as

$$U(\varphi) = \lambda + \gamma_5\rho\varphi\cdot\tau\,, \tag{5}$$

where $\lambda$ and $\rho$ are functions of $\varphi^2$. The unitarity condition is

$$\lambda^2 + \rho^2\varphi^2 = 1\,. \tag{6}$$

The transformation property in Eq. (2) gives

$$\begin{aligned}\delta U &= \{-g\gamma_5\tfrac{1}{2}\tau\cdot\delta\theta,\, \lambda+\gamma_5\rho\varphi\cdot\tau\}\\ &= -g\lambda\gamma_5\tau\cdot\delta\theta + g\rho\varphi\cdot\delta\theta.\end{aligned}$$

On the other hand, direct variation of Eq. (5) gives

$$\delta U = 2\frac{\delta\lambda}{\delta\varphi^2}\varphi\cdot\delta\varphi + \gamma_5\left(2\frac{\delta\rho}{\delta\varphi^2}\varphi\cdot\delta\varphi\varphi\cdot\tau + \rho\delta\varphi\cdot\tau\right),$$

which implies the consistency conditions

$$g\rho\delta\theta\cdot\varphi = 2\frac{\delta\lambda}{\delta\varphi^2}\delta\varphi\cdot\varphi \tag{7}$$

and

$$-g\lambda\delta\theta = 2\frac{\delta\rho}{\delta\varphi^2}\varphi\cdot\delta\varphi\varphi + \rho\delta\varphi. \tag{8}$$

It can easily be shown that these two conditions combine to yield

$$\frac{\delta}{\delta\varphi^2}(\lambda^2+\rho^2\varphi^2) = 0\,,$$

which is equivalent to the unitarity condition (6).

In general, $\delta\theta$ is not along the direction of $\varphi$. Therefore we shall parametrize a general transformation on the pion field as[15]

$$\delta\varphi = p\delta\theta + q\varphi\cdot\delta\theta\varphi\,,$$

where $p$ and $q$ are functions of $\varphi^2$. Let us consider the case when $p\neq 0$ and $\delta\theta\cdot\varphi\neq 0$. Equations (7) and (8) lead to

$$\frac{\delta(\ln\rho)}{\delta\varphi^2} = \frac{-q}{2(p+q\varphi^2)}$$

and

$$-g\lambda = \rho p\,.$$

That is, for a given set of $p\neq 0$ and $q$, we can determine $\lambda$ and $\rho$ by these equations together with the unitarity condition.[16] A particularly simple case is when $q=0$. Then, with $\rho=$ constant, or with a proper choice of scale, we may write $\rho = g/m$ and

$$U = [1-(g^2/m^2)\varphi^2]^{1/2} + \gamma_5(g/m)\varphi\cdot\tau\,. \tag{9}$$

When $p=0$ but $\delta\theta\cdot\varphi\neq 0$, we have from Eq. (8)

$$-g\lambda\delta\theta = \left(2\frac{\delta\rho}{\delta\varphi^2}q\varphi^2 + q\rho\right)\varphi\cdot\delta\theta\varphi\,.$$

Hence, if $\delta\theta$ is not along the direction of $\varphi$, it follows that $g\lambda=0$. Upon imposing a physical requirement that

$$\lambda = 1 + O(g) + \cdots\,,$$

which means that we should recover the free massive fermion Lagrangian as $g\to 0$, we infer that $g=0$. This implies that there is no chiral symmetry in our system, which is not a desirable result. We therefore assert that when $p=0$ and $\delta\theta\cdot\varphi\neq 0$, $\delta\theta$ must be along $\varphi$. For a given $q$ the quantities $\lambda$ and $\rho$ are determined by the unitarity condition and

$$g\rho = 2\frac{\delta\lambda}{\delta\varphi^2}q\varphi^2\,.$$

An example in this case is

$$U = e^{-2(g/m)\gamma_5\tau\cdot\varphi}\,, \tag{10}$$

which gives

$$\delta\varphi = \tfrac{1}{2}m\delta\theta = m\delta\theta\cdot\varphi\varphi/2\varphi^2\,.$$

Finally, when $\delta\theta\cdot\varphi = 0$, we have for $p>0$

$$\rho = -g/(p^2+g^2\varphi^2)^{1/2}$$

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[14] We use the metric $(-1,1,1,1)\cdot\mu,\nu=0,1,2,3$, $k,l=1,2,3$, $\{\gamma_\mu,\gamma_\nu\}=-2g_{\mu\nu}$, $\gamma_k=-\gamma_k^\dagger$, $\gamma_5=\gamma^0\gamma^1\gamma^2\gamma^3=-\gamma_5^\dagger$, and $\gamma_0=\gamma_0^\dagger$. The symbols $\alpha,\beta,\gamma=1,2,3$ are isospin indices. Dot and vector products refer to the isospin space.

[15] This is also independently observed by L. S. Brown (Ref. 3).

[16] J. Wess and B. Zumino (Ref. 4) considered a particular case in this class.

and

$$\lambda = p/(p^2+g^2\varphi^2)^{1/2}.$$

We shall abbreviate

$$U^{1/2}=\sigma-\gamma_5\tau\cdot\eta. \tag{11}$$

It can be shown that we have the condition

$$\sigma^2+\eta^2=1. \tag{12}$$

For example, if $U$ is given as Eq. (9), we have

$$\sigma=(1+\kappa)^{1/2}/\sqrt{2},$$

$$\eta=-\frac{1}{\sqrt{2}}\frac{g}{m}\left(\frac{1}{1+\kappa}\right)^{1/2}\varphi,$$

where

$$\kappa=\left(1-\frac{g^2}{m^2}\varphi^2\right)^{1/2}.$$

From the transformation property

$$\delta\varphi=-m\kappa\delta\theta$$

we have

$$\delta\sigma=-\tfrac{1}{2}g\eta\cdot\delta\theta,$$

$$\begin{aligned}\delta\eta&=\tfrac{1}{2}g\sigma^{-1}[\eta\cdot\delta\theta\eta+(\sigma^2-\eta^2)\delta\theta]\\&=\tfrac{1}{2}g\sigma^{-1}[(\delta\theta\times\eta)\times\eta+\sigma^2\delta\theta].\end{aligned} \tag{13}$$

On the other hand, when $U$ is given by Eq. (10), we have

$$\delta\eta=\tfrac{1}{2}g\sigma\delta\theta$$

and

$$\delta\sigma=-\tfrac{1}{2}g\eta\cdot\delta\theta. \tag{14}$$

It is seen that we can obtain this from Eq. (13) once we are reminded that now $\delta\theta$ is along the direction of $\varphi$. [Although the transformation relations (13) are obtained from a particular unitary matrix (9), we suspect that they have general validity.]

From this point on, we shall limit ourselves to these two models specified by matrix (9) and matrix (10). As was mentioned before, Eq. (13) covers the transformation properties in both cases. We define a new fermion field $\psi$ through the relation

$$\psi=U^{1/2}\Psi. \tag{15}$$

It shall be taken as the field operator that describes the nucleons. Then, the Lagrangian (4) becomes

$$\mathfrak{L}_2=-\bar\psi[\gamma_\mu 1/i\partial^\mu+M]\psi+g\bar\psi\gamma_\mu\tfrac{1}{2}\tau\psi\cdot V'^\mu+ig\bar\psi\gamma_\mu\gamma_5\tfrac{1}{2}\tau\psi\cdot A'^\mu, \tag{16}$$

with

$$V'^\mu=V^\mu+(2/g)d^\mu\eta\times\eta+2\sigma A^\mu\times\eta,$$

$$A'^\mu=A^\mu+(2/g)\sigma^2d^\mu(\sigma^{-1}\eta)+2(A^\mu\times\eta)\times\eta,$$

and

$$d^\mu\equiv\partial^\mu+gV^\mu\times. \tag{17}$$

This Lagrangian is invariant under the isospin transformation

$$V^\mu\to V^\mu-g\delta\omega\times V^\mu+\partial^\mu\delta\omega,$$

$$A^\mu\to A^\mu-g\delta\omega\times A^\mu,$$

and

$$\varphi\to\varphi-g\delta\omega\times\varphi.$$

Furthermore,

$$V'^\mu\to V'^\mu-\delta\omega\times V'^\mu+\partial^\mu\delta\omega$$

and

$$A'^\mu\to A'^\mu-g\delta\omega\times A'^\mu. \tag{18}$$

Under chiral isospin transformation, besides Eqs. (1), (3), and (13) [or (14)], we also have

$$V'^\mu\to V'^\mu+\delta\bar\theta\times V'^\mu-\partial^\mu\delta\bar\theta,$$

$$A'^\mu\to A'^\mu+\delta\bar\theta\times A'^\mu,$$

and

$$\psi\to\psi-\tfrac{1}{2}i\tau\cdot\delta\bar\theta\psi, \tag{19}$$

where

$$\delta\bar\theta=g\sigma^{-1}(\delta\theta\times\eta).$$

We must add to the Lagrangian other parts which contribute to the kinetic energies of the fields $V^\mu$, $A^\mu$, and $\varphi$. As usual, the combination

$$V_{\mu\nu}{}^2+A_{\mu\nu}{}^2,$$

where

$$V_{\mu\nu}=\partial_\mu V_\nu-\partial_\nu V_\mu+gV_\mu\times V_\nu+gA_\mu\times A_\nu$$

and

$$A_{\mu\nu}=d_\mu A_\nu-d_\nu A_\mu,$$

is invariant under $SU_2\times SU_2$, while the combination

$$-\tfrac{1}{2}m_\rho{}^2(A_\mu{}^2+V_\mu{}^2)$$

transforms as

$$m_\rho{}^2A_\mu\partial^\mu\delta\theta \tag{20}$$

under chiral $SU_2$ and as

$$-m_\rho{}^2V_\mu\partial^\mu\delta\omega \tag{21}$$

under isospin $SU_2$.

We still need an extra piece in the Lagrangian to correspond to the kinetic energy of the pion. Tentatively, let us assume that this term is $-\frac{1}{2}\bar m^2(A'^\mu)^2$, where $\bar m$ is some mass scale. The complete Lagrangian is then

$$\begin{aligned}\mathfrak{L}=&-\bar\psi\gamma_\mu\frac{1}{i}\partial^\mu\psi-M\bar\psi U(\varphi)\psi+g\bar\psi\gamma_\mu\tfrac{1}{2}\tau\psi\cdot V^\mu\\&+ig\bar\psi\gamma_\mu\gamma_5\tfrac{1}{2}\tau\psi\cdot A^\mu-\tfrac{1}{2}V_{\mu\nu}(\partial^\mu V^\nu-\partial^\nu V^\mu+gV^\mu\times V^\nu\\&+gA^\mu\times A^\nu)+\tfrac{1}{4}V_{\mu\nu}V^{\mu\nu}-\tfrac{1}{2}A_{\mu\nu}(d^\mu A^\nu-d^\nu A^\mu)\\&+\tfrac{1}{4}A_{\mu\nu}A^{\mu\nu}-\tfrac{1}{2}m_\rho{}^2(A_\mu A^\mu+V_\mu V^\mu)\\&-\tfrac{1}{2}\bar m^2A_\mu'A'^\mu+\mathfrak{L}',\end{aligned} \tag{22}$$

where $\mathfrak{L}'$ is so chosen that it provides us with the pion mass and that, after the unitary transformation which we shall introduce later on, we shall have the strong

PCAC condition. We shall return to this point in the subsequent sections. At this moment, it is only necessary to say that $\mathfrak{L}'$ is a function of $\varphi^2$ (with no derivatives).

Because of Eqs. (20) and (21), respectively, we have the PCAC condition[17]

$$-\partial_\mu A^\mu \cong \text{const}\times\varphi \tag{23}$$

and the conserved-vector-current (CVC) condition[18]

$$\partial_\mu V^\mu = 0. \tag{24}$$

### III. QUANTIZATION

We shall assume that $V_k$, $A_k$, and $\varphi$ are independent variables. It follows from the Lagrangian (22) that we have the equal-time canonical commutation relations

$$[V^{0k}(x),V_l(x')]=i\delta_{kl}\delta(\mathbf{x}-\mathbf{x}'),$$
$$[A^{0k}(x),A_l(x')]=i\delta_{kl}\delta(\mathbf{x}-\mathbf{x}'),$$

and

$$[p^0(x),\eta(x')]=i\delta(\mathbf{x}-\mathbf{x}'). \tag{25}$$

In the last expression, we have defined

$$p^0=(2\bar{m}^2/g)(\sigma A'^0+\sigma^{-1}\eta\cdot A'^0\eta), \tag{26}$$

which is seen to be the conjugate momentum to the field $\eta$.

The Euler's equations, which follow from variations with respect to $A_0$ and $V_0$, successively, give

$$m_\rho^2A^0=-d_kA^{0k}+gV^{0k}\times A_k+ig\bar{\psi}\gamma^0\gamma_5\tfrac{1}{2}\tau\psi-\tfrac{1}{2}g\sigma^{-1}[(\sigma^2-\eta^2)p^0+\eta\cdot p^0\eta]$$

and

$$m_\rho^2V^0=-\partial_kV^{0k}+gV^{0k}\times V_k+gA^{0k}\times A_k+g\bar{\psi}\gamma^0\tfrac{1}{2}\tau\psi+gp^0\times\eta. \tag{27}$$

We have here expressed the dependent variables $A^0$ and $V^0$ in terms of the independent ones. It takes little effort to show that

$$i\int m_\rho^2A^0\cdot\delta\theta d^3x \quad\text{and}\quad \frac{1}{i}\int m_\rho^2\rho^0\cdot\delta\omega d^3x$$

are the generators for the chiral $SU_2$ and the isospin $SU_2$, respectively. That is,

$$i\left[\int m_\rho^2A^0\cdot\delta\theta d^3x,\,\eta\right]=\tfrac{1}{2}g\sigma^{-1}[(\sigma^2-\eta^2)\delta\theta+\eta\cdot\delta\theta\eta],\quad\text{etc.,}$$

and

$$\frac{1}{i}\left[\int m_\rho^2V^0\cdot\delta\omega d^3x,\,\eta\right]=-g\delta\omega\times\eta,\quad\text{etc.} \tag{28}$$

After some algebra, it is also verified that

$$[m_\rho^2A_\alpha{}^0(x),m_\rho^2A_\beta{}^0(x')]=ig\epsilon_{\alpha\beta\gamma}m_\rho^2V_\gamma{}^0(x)\delta(\mathbf{x}-\mathbf{x}'), \tag{29}$$

and the other commutation relations listed in Ref. 9. Another interesting commutator is

$$[m_\rho^2A_\alpha{}^0(x),\eta_\beta(x')]=-\tfrac{1}{2}ig\sigma^{-1}[(\sigma^2-\eta^2)\delta_{\alpha\beta}+\eta_\alpha\eta_\beta]\delta(\mathbf{x}-\mathbf{x}'), \tag{30}$$

which differs from that of the $\sigma$ model.[13]

Before we go on to Sec. IV, let us stress the importance of choosing $V_k$, $A_k$, and $\eta$ as independent variables. If we first redefine new variables[4]

$$A_\mu\to a_\mu-(2/g)d_\mu\eta$$

and treat $V_k$, $a_k$ (as the $A_1$-meson field), and $\eta$ independently, we can show that the quantization procedure cannot be carried through. Rather than presenting the actual algebra, let us argue by noticing that transformations which depend on derivatives in general change the commutation relations of the fields. Therefore there is no guarantee that the commutator algebra will be preserved after such transformations. On the other hand, if one is interested only in the $S$-matrix elements, then such transformations can be justifiably performed.[19]

### IV. A MODEL

In this section, we shall assume a certain form for the unitary matrix. The reason for doing so is only to fix the scale $m$. We must remark that our general approach is independent of such a specific choice. We write

$$U=e^{-2(g/m)\gamma_5\tau\cdot\varphi}. \tag{31}$$

It is clear that

$$\varphi\to\varphi+\tfrac{1}{2}m\delta\theta \tag{32}$$

under an infinitesimal chiral transformation. The explicit forms for the parameters in Eq. (11) are

$$\eta=\frac{(g/m)\varphi\sinh((-g^2\varphi^2/m^2)^{1/2})}{(-g^2\varphi^2/m^2)^{1/2}}$$

and

$$\sigma=\cosh((-g^2\varphi^2/m^2)^{1/2}). \tag{33}$$

The symmetry-breaking part of the Lagrangian can be taken as

$$\mathfrak{L}'\cong-\tfrac{1}{8}\mu^2\varphi^2, \tag{34}$$

where $\mu$ will become the mass of the pion. The PCAC condition (23) is

$$-\partial_\mu A^\mu\cong(m\mu^2/8m_\rho^2)\varphi. \tag{35}$$

The approximate signs here mean that $\mathfrak{L}'$ should be so chosen that after the unitary transformation, which we shall introduce below, Eq. (35) becomes Eq. (44).

The term $-\frac{1}{2}\bar{m}^2A'^\mu A_\mu'$ in the Lagrangian (22)

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[19] S. Kamefuchi, L. O'Raifeartaigh, and A. Salam, Nucl. Phys. **28**, 529 (1961); H. J. Borchers, Nuovo Cimento **15**, 784 (1960).

contains a cross term

$$-(2\bar{m}^2/g)\sigma^2\partial^\mu(\sigma^{-1}\eta)A_\mu. \qquad (36)$$

Such a term implies that the kinetic-energy matrix of the $A$ field and the $\varphi$ field is not diagonal. Therefore it is necessary to define new field variables.[6] From the structure of Eq. (36) we deduce that a unitary transformation with the generator

$$iG=i\int\frac{2m_\rho^2}{g}\frac{\bar{m}^2}{m_\rho^2}\sigma\eta\cdot A^0 \qquad (37)$$

can accomplish this purpose. We shall elaborate more on unitary transformations in Sec. V. It is now evident that we are essentially performing a chiral transformation, with

$$\delta\theta=(2/g)\sigma\eta\bar{m}^2/m_\rho^2. \qquad (38)$$

In general, of course, the various fields after this transformation will become very complicated. We shall use as examples the problems of $\rho\rightarrow 2\pi$, $A_1\rightarrow\rho+\pi$, and $\pi^+$–$\pi^0$ electromagnetic mass difference to show what the program is. Then, the relevant parts are

$$\bar{A}_k\equiv e^{i\epsilon G}A_k e^{-i\epsilon G}\cong A_k-(2/g)ad_k\eta\,,$$

$$e^{i\epsilon G}\varphi e^{-i\epsilon G}\cong\varphi+a\varphi\,,$$

$$e^{i\epsilon G}\eta e^{-i\epsilon G}\cong\eta+a\eta\,,$$

and

$$\rho_k\equiv e^{i\epsilon G}V_k e^{-i\epsilon G}\cong V_k+2a\eta\times A_k-(2/g)a^2\eta\times d_k\eta\,, \qquad (39)$$

in which

$$a=e^{\epsilon(\bar{m}/m_\rho)^2}-1\,.$$

In addition, the combinations $V_{kl}^2+A_{kl}^2$ and $A_k'$ are both invariant. Substituting all these into the Lagrangian (22), we have

$$\begin{aligned}\bar{\mathfrak{L}}&\equiv e^{i\epsilon G}\mathfrak{L}e^{-i\epsilon G}\\&\cong-\tfrac{1}{4}(\partial_\mu V_\nu-\partial_\nu V_\mu+gV_\mu\times V_\nu+gA_\mu\times A_\nu)^2\\&\quad-\tfrac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu+gV_\mu\times A_\nu-gV_\nu\times A_\mu)^2\\&\quad-\tfrac{1}{2}m_\rho^2[(A_\mu-(2a/g)d_\mu\eta)^2+\rho_\mu^2]\\&\quad-\tfrac{1}{2}\bar{m}^2[A_\mu+(2/g)d_\mu\eta+2(A_\mu\times\eta)\times\eta]^2\\&\quad-\tfrac{1}{8}\mu^2(\varphi+a\varphi)^2. \qquad (40)\end{aligned}$$

In order to retrieve Weinberg's mass relation,[7] $2m_\rho^2=m_A^2$, we put

$$\bar{m}^2=m_\rho^2.$$

However, note that this is not a necessary condition for consistency in the theory.

To eliminate the term (36), we clearly want

$$a=1$$

or

$$\epsilon=\ln 2.$$

We must now identify the various fields for describing the different particles. We make the assumption that they are

$$\varphi\sim\pi\text{ meson},$$

$$A_\mu\sim A_1\text{ meson},$$

and

$$\rho_\mu\sim\rho\text{ meson}.$$

It is especially important to make the last identification if one wants to couple the $\rho$ meson directly to the photon (vector-dominance hypothesis),[20] For the CVC condition is now

$$\partial_\mu\rho^\mu=0\,, \qquad (41)$$

whereas

$$\partial_\mu V_\mu\neq 0$$

in this new description. Condition (41) is necessary to guarantee gauge invariance.

It is also noted that we should equate

$$m^2=8m_\rho^2$$

or

$$m=2m_A$$

in order to have the correct factor for the kinetic energy of the pion field in the Lagrangian (40).

We make use of Eq. (39) to express

$$V_\mu\cong\rho_\mu-(g/m_A)\varphi\times A_\mu+(g/2m_A^2)\varphi\times D_\mu\varphi\,, \qquad (42)$$

where now

$$D_\mu\equiv\partial_\mu+g\rho_\mu\times\,.$$

The relevant parts of the Lagrangian in this description are

$$\begin{aligned}\bar{\mathfrak{L}}\cong&-\tfrac{1}{4}(\partial_\mu\rho_\nu-\partial_\nu\rho_\mu)^2-\tfrac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)^2\\&-\tfrac{1}{2}(\partial_\mu\varphi)^2-\tfrac{1}{2}m_\rho^2\rho_\mu^2-\tfrac{1}{2}m_A^2A_\mu^2-\tfrac{1}{2}\mu^2\varphi^2\\&-g\rho_\mu\cdot(\varphi\times\partial^\mu\varphi)-\tfrac{1}{2}g^2(\rho_\mu\times\varphi)^2+(g/2m_A)\\&\times(\partial_\mu\rho_\nu-\partial_\nu\rho_\mu)\cdot[\varphi\times(\partial^\mu A^\nu-\partial^\nu A^\mu)]-(g^2/4m_A^2)\\&\times[(\partial_\mu\rho_\nu-\partial_\nu\rho_\mu)\times\varphi]^2+(g/2m_A)(\partial_\mu\rho_\nu-\partial_\nu\rho_\mu)\\&\times(\partial^\mu\varphi\times A^\nu-\partial^\nu\varphi\times A^\mu)-(g/2m_A^2)\\&\qquad\times(\partial_\mu\rho_\nu-\partial_\nu\rho_\mu)\cdot(D^\mu\varphi\times D^\nu\varphi). \qquad (43)\end{aligned}$$

This Lagrangian is the same as the ones proposed by several authors.[4,21] However, we must once again point out the difference in approach. The higher-order terms by this method are different. The PCAC condition (35) becomes

$$-\partial_\mu\left(\frac{m_\rho^2}{g}\bar{A}^\mu\right)=\frac{m_\rho^2}{g}\frac{\mu^2}{m_A}\varphi\,, \qquad (44)$$

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[20] M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 953 (1961); S. H. Patil and Y.-P. Yao, *ibid.* **153**, 1455 (1967); N. Kroll, T. D. Lee, and B. Zumino, *ibid.* **157**, 1376 (1967); T. D. Lee and B. Zumino, *ibid.* **163**, 1667 (1967); see also Ref. 3 for further references.

[21] G. C. Wick and B. Zumino, CERN Report No. 67/1082/5-TH.826, 1967 (unpublished); I. S. Gerstein, B. W. Lee, H. T. Nieh, and H. J. Schnitzer, Phys. Rev. Letters **19**, 1064 (1967).

where the axial-vector current is

$$\frac{m_\rho^2}{g}\bar{A}^\mu=\frac{m_\rho^2}{g}\left(A^\mu-\frac{1}{m_A}D^\mu\varphi+\text{additional terms}\right). \quad (45)$$

These additional terms are not present in the other approaches. In principle, they can be detected.

## V. FURTHER COMMENTS ON UNITARY TRANSFORMATIONS

In order to eliminate the cross term (36), we made a unitary transformation (37) as expressed by the gauge parameters (38). However, we would like to emphasize that the choice of such a transformation is not unique. Different unitary transformations which get rid of the term (36) do give the same Lagrangian (43), but they give different higher-order terms. Besides, we have imposed the strong PCAC condition

$$\partial_\mu\bar{A}^\mu=\text{const}\times\varphi$$

in Sec. IV. This means that we want

$$e^{i\epsilon G}\mathcal{L}'e^{-i\epsilon G}=-\tfrac{1}{2}\mu^2\varphi^2. \quad (46)$$

We can instead assume that

$$\mathcal{L}'=-\tfrac{1}{8}\mu^2\varphi^2;$$

then we have

$$\partial_\mu A^\mu=\text{const}\times\varphi.$$

However, after a unitary transformation, we shall have some generalized PCAC condition

$$\partial_\mu\bar{A}^\mu=a\varphi+b\varphi^3+\cdots.$$

There is a distinction between these two approaches. In the former case, because $\mathcal{L}'$ must be different for various choices of the unitary transformations in order that Eq. (46) holds, we get different on-shell and off-shell matrix elements. In the latter case, different choices of unitary transformations give different off-shell matrix elements. However, they give the same on-shell amplitudes.[22] We do not know which is the correct approach; all we can conclude is that at this stage, the algebra of currents (or fields) and PCAC are not sufficient to determine many processes, e.g., pion-pion scattering lengths.[23] It is hoped that further dynamical principles can be devised to supplement and implement this program.

As an illustration that different unitary transformations give different physics, because we insist on a strong PCAC condition and hence the $\mathcal{L}'$'s are different, we now look at the commutator in Eq. (30). We shall calculate the right-hand side to order $\varphi^2$ only. Therefore we must calculate $e^{i\epsilon G}\eta e^{-i\epsilon G}$ to order $\varphi^3$. After some elementary but tedious computation, one finds that the transformation (37) and (38) leads to

$$[\bar{A}_\alpha{}^0(x),\varphi_\beta(x')]=\frac{-i}{m_A}\left[\left(1-\frac{7}{12}\frac{g^2}{m_A{}^2}\varphi^2\right)\delta_{\alpha\beta}+\frac{11}{6}\frac{g^2}{m_A{}^2}\varphi_\alpha\varphi_\beta\right]\delta(\mathbf{x}-\mathbf{x}'). \quad (47)$$

Both $I=0$ and $I=2$ $2\pi$ $s$-wave contributions appear on the right-hand side. This commutator has been used in describing the process $\eta\rightarrow3\pi$.[24] Here, in order to obtain a good fit for the energy dependence of the odd pion, it is noted that the commutator $[\bar{A}_\alpha{}^0,\varphi_\beta]$ should contain only an $I=0$ contribution. Equivalently, only $I=0$ $s$-wave enhancement should exist. Within our framework, we can accomplish this by choosing

$$\delta\theta'=(2/g)\sigma^{-37/9}\eta$$

and

$$iG=i\int m_\rho{}^2A^0\cdot\delta\theta'. \quad (48)$$

Then,

$$[\bar{A}_\alpha{}^0,\varphi_\beta]=(-i/m_A)(1-\tfrac{3}{2}\varphi^2)\delta_{\alpha\beta}. \quad (49)$$

Various low-order calculations based on this model [Eq. (43)] can be found in the existing literature.[4] We are looking further into higher-order detectable processes to differentiate our approach.

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[22] This point has been discussed by B. Zumino, Phys. Letters **25B**, 349 (1967); S. Weinberg, Phys. Rev. **166**, 1568 (1968).

[23] Y. Tomozawa, Nuovo Cimento **46**, 707 (1966); S. Weinberg, Phys. Rev. Letters **17**, 616 (1966); J. Schwinger, Phys. Letters **24B**, 473 (1967); N. Khuri, Phys. Rev. **153**, 1477 (1967); J. Sucher and C. H. Woo, Phys. Rev. Letters **18**, 723 (1967).

[24] A. D. Dolgov, A. I. Vainshtein, and V. I. Zakhorov, Phys. Letters **24B**, 425 (1967); W. A. Bardeen, L. S. Brown, B. W. Lee, and H. T. Nieh, Phys. Rev. Letters **18**, 1170 (1967).