Superconvergence and Regge Poles. I. Odd-Signature Exchanges*

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A method is used in which the usual analytic properties of the πN scattering amplitude are exploited in a more complete way to obtain information about ρ-type Regge poles. It is found that the low-energy data determine the forward Reggeized ρ-meson-exchange parameters quite accurately. These are found to be in complete agreement with high-energy analyses. We are also able to see that in the forward direction, only ρ-meson exchange is allowed to a very good approximation. The evaluation of low-energy parameters and the prediction of forward pion charge-exchange scattering are also discussed. A criterion is given relating the validity of the "interference" model to the behavior of the general sum rule.

I. INTRODUCTION

THE importance of analytic constraints relating Regge-pole contributions to low-energy behavior has been long appreciated.[1,2] Only relatively recently, however, have these requirements been cast into a convenient form. The step required was the introduction of sum rules of the superconvergent type, the advantage being that the Regge parameters then appear explicitly. The papers of Igi and Matsuda[3] and of Logunov *et al.*[3] introduced a sum rule relating an integral over scattering cross sections to a Regge amplitude, under the assumption that there were no secondary trajectories with intercepts above $\alpha=-1$. It soon became clear that the point $\alpha=-1$ did not really have any deep significance[4] and that it was possible to write the sum rule in terms of a complete set of Regge trajectories. At this point the use of this type of sum rule was of value as a consistency condition to supplement the high-energy data, a single sum rule by itself being unable to define the parameters of even a single Regge pole (which, of course, involves two parameters, the residue γ and the intercept α). The existence of a second superconvergent sum rule was shown[5,6] to follow from the "inverse" dispersion relation of Gilbert.[7] Olsson[5] was able to demonstrate that this new sum rule combined with the old one implied Regge parameters for the pion charge-exchange amplitude consistent with those obtained from the high-energy data alone.

Liu and Okubo[8] have shown how the π-N analytic constraint can be generalized to include the ordinary dispersion relation and the Gilbert relation as special cases. Using this technique, a generalized superconvergence relation for the Regge parameters has been derived by Olsson[9] and used to investigate in detail the Regge ρ-exchange parameters in the forward direction.

In this paper, we discuss at greater length this generalized sum rule (GSR) and its application to the change-exchange (c.e.) process. Section II contains a derivation of the GSR and a description of its properties. To evaluate the GSR both the real and imaginary parts of the c.e. amplitude are required from threshold up to an energy in the asymptotic region which we take to be 5 GeV. The calculation of the real part ReT from the imaginary part ImT and related topics are the subject of Sec. III. The actual evaluation of the GSR is covered in Sec. IV and the results are discussed in Sec. V.

The main conclusions of this analysis using the GSR are twofold. First, a single Regge pole satisfies the GSR, and the parameters derived using only low-energy data are quite precisely defined and agree well with analyses using high-energy data. Secondly, if the GSR results are combined with the high-energy data, an upper limit can be placed on the contribution of a secondary ρ trajectory, the ρ', in the forward direction. It is found that the strength of ρ' must be considerably weaker than that of the ρ and makes more attractive the suggestion[10] that the ρ' residue vanishes in the forward direction.

II. GENERAL SUM RULE

We shall be considering the amplitude combination $T(\omega)=T_{\pi^-p}(\omega)-T_{\pi^+p}(\omega)$.[11] This amplitude (charge exchange) is odd under crossing and is normalized by the optical theorem to the total cross-section difference

$$\Delta_{\pi p}=\sigma_t(\pi^-p)-\sigma_t(\pi^+p)=(4\pi\hbar^2/k)\,\mathrm{Im}T(\omega). \qquad (1)$$

At sufficiently high energies the c.e. angular distribution is well represented by the exchange of a single ρ

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[1] K. Igi, Phys. Rev. 130, 820 (1963).

[2] G. Höhler, J. Baacke, and R. Strauss, Phys. Letters 21, 223 (1966).

[3] K. Igi and S. Matsuda, Phys. Rev. Letters 18, 625 (1967); A. Logunov, L. Soloviev, and A. Tavkhelidze, Phys. Letters 24B, 181 (1967).

[4] D. Horn and C. Schmid, California Institute of Technology Report No. CALT-68-127 (unpublished).

[5] M. G. Olsson, Phys. Rev. Letters 19, 550 (1967).

[6] V. Meshcheriakov, K. Rerikh, A. Tavkhelidze, and V. Zhuravlev, Phys. Letters 25B, 341 (1967).

[7] W. Gilbert, Phys. Rev. 108, 1078 (1957).

[8] Y. Liu and S. Okubo, Phys. Rev. Letters 19, 190 (1967).

[9] M. G. Olsson, University of Wisconsin Report No. C00-881-119 (unpublished); Phys. Letters 26B, 310 (1968).

[10] L. Sertorio and M. Toller, Phys. Rev. Letters 19, 1146 (1967).

[11] The incident pion energy is ω, and its associated momentum is k. The pion mass is μ and the nucleon mass is M; energy units, unless otherwise specified, are GeV. $f^2\approx0.081$ and $\hbar^2=0.389$ GeV mb.

Regge pole.[2,12] In the forward direction the Regge ρ-exchange amplitude can be written as

$$T^R(\omega)=(2\gamma/M)(\omega/\omega_0)^\alpha(i+\tan\tfrac{1}{2}\pi\alpha),$$

where ω_0 is a scale factor arbitrarily[12] chosen to be $\omega_0=s_0/2M$, with $s_0=1$ GeV^2. The constants γ, the residue, and the intercept α have values as determined from $\Delta_{\pi p}$ and forward π c.e. data[13] of roughly

$$\gamma_\rho\cong0.33\,,\quad \alpha_\rho\cong0.57\,.$$

The existence of polarization in the π c.e. scattering at high energy[14] requires an additional small contribution to the asymptotic amplitude besides the ρ. The simplest assumption[15] is that there is a lower-lying trajectory, the ρ', with the same quantum numbers as the ρ. So, in general, we shall adopt the following expression for the high-energy amplitude:

$$T^R(\omega)=\frac{2}{M}\sum_{(\gamma,\alpha)}\gamma\left(\frac{\omega}{\omega_0}\right)^\alpha(i+\tan\tfrac{1}{2}\pi\alpha)$$
$$=\frac{2i}{M}\sum_{(\gamma,\alpha)}\frac{\gamma}{\cos\frac{1}{2}\pi\alpha}\left(\frac{\omega}{\omega_0}\right)^\alpha e^{-i\pi\alpha/2}.\quad(2)$$

It is not possible, however, with present accuracy in the high-energy data to learn much about such lower-lying trajectories. At this point we shall use some general properties of $T(\omega)$ to obtain additional information about the Regge parameters.

The scattering amplitude $T(\omega)$ has rather simple analytic and reflection properties in the complex ω plane.[16] These properties imply a relation[16] between ReT and ImT which can be expressed as[17]

$$\mathrm{Re}T(\omega)=\frac{4f^2}{\omega}+\frac{2\omega}{\pi}\mathrm{P}\int_\mu^\infty\frac{d\omega'}{\omega'^2-\omega^2}\mathrm{Im}T(\omega').\quad(3)$$

If $T(\omega)$ decreases rapidly enough at high energy, namely, the leading Regge intercept is below -1, the sum rule (3) can be converted into the superconvergent form by multiplying by ω and letting ω become large, giving

$$f^2=\frac{1}{2\pi}\int_\mu^\infty d\omega\,\mathrm{Im}T(\omega)\,.$$

Following the work of Liu and Okubo,[8] we define the quantity

$$F(\omega)=(-ik)^{-\epsilon-1}T(\omega)\,.\quad(4)$$

The factor $(-ik)^{-\epsilon-1}$ has branch points at the elastic thresholds $\omega=\pm\mu$, and is real between them. Thus $F(\omega)$ has the same analytic properties as the amplitude $T(\omega)$ and hence a dispersion relation similar to (3) is valid for the range $\alpha-2<\epsilon<1$, where α is the leading trajectory intercept. For a sufficiently rapidly decreasing $T(\omega)$ at high energy (i.e., if $\omega\,\mathrm{Re}F\rightarrow0$) a superconvergence relation is valid. This condition is satisfied when the leading intercept is less than ϵ. The resulting superconvergence relation is[18]

$$\frac{f^2}{\mu^{\epsilon+1}}\left(1-\frac{\mu^2}{4M^2}\right)^{-(1+\epsilon)/2}=\frac{1}{2\pi}\int_\mu^\infty d\omega\,\mathrm{Im}F(\omega)\,.$$

We now consider an energy $\omega=\bar\omega$ high enough that the scattering amplitude can be replaced by a Regge expansion. Multiplying the above expression by $M\pi\bar\omega$, and dividing up the integration range into two parts, gives

$$(M\pi/\mu)(\bar\omega/\mu)^\epsilon f^2(1-\mu^2/4M^2)^{-(1+\epsilon)/2}=I_1+I_2\,,$$
$$I_1=\tfrac{1}{2}M\bar\omega^\epsilon\int_\mu^{\bar\omega}d\omega\,\mathrm{Im}F(\omega)\,,\quad(5)$$
$$I_2=\tfrac{1}{2}M\bar\omega^\epsilon\int_{\bar\omega}^\infty d\omega\,\mathrm{Im}F^R(\omega)\,.$$

From (2) and (4) we find that

$$\mathrm{Im}F^R(\omega)=\frac{2}{M}\frac{1}{k^{\epsilon+1}}\sum_{(\gamma,\alpha)}\frac{\gamma}{\cos\frac{1}{2}\pi\alpha}\left(\frac{\omega}{\omega_0}\right)^\alpha\sin\tfrac{1}{2}\pi(\alpha-\epsilon)\,.$$

The second integral in (5) can now be explicitly carried out:

$$I_2=\sum_{(\gamma,\alpha)}\frac{\gamma}{\cos\frac{1}{2}\pi\alpha}\frac{\bar\omega^\epsilon}{\omega_0{}^\alpha}\sin\tfrac{1}{2}\pi(\alpha-\epsilon)\int_{\bar\omega}^\infty\frac{d\omega\,\omega^\alpha}{k^{\epsilon+1}}\,.$$

The integral is convergent by the assumption that $\alpha<\epsilon$, but cannot be carried out in closed form. However, if we approximate[19] the momentum by the energy, which is certainly accurate at energies above $\bar\omega$ (which we will take as 5 GeV in our numerical calculation), the integral

[12] R. Logan, Phys. Rev. Letters 14, 414 (1965).

[13] The normalization is the same as in V. Barger and M. Olsson, Phys. Rev. Letters 18, 294 (1967).

[14] P. Bonamy *et al.*, Phys. Letters 23, 501 (1966). A recent measurement by D. Drobnis *et al.*, Phys. Rev. Letters 20, 274 (1968), is consistent with zero polarization at 5 GeV/c.

[15] R. Logan, J. Beaupre, and L. Sertorio, Phys. Rev. Letters 18, 259 (1967).

[16] M. L. Goldberger, Phys. Rev. 99, 986 (1955); R. Karplus and M. Ruderman, *ibid.* 98, 771 (1955); M. Goldberger, H. Miyazawa, and R. Oehme, *ibid.* 99, 986 (1955). $T(\omega)$ is real for $-\mu\leq\omega\leq+\mu$ except for simple poles at $\omega=\pm\mu^2/2M$. At $\omega=\pm\mu$ there are branch points. Crossing symmetry for $T(\omega)$ is $T(\omega)=-T(-\omega)$.

[17] We have made the approximation $\omega_B\ll\mu$ in the nucleon-pole term. Note that this well-known relation is convergent without subtractions for $\alpha<1$.

[18] The factors on the left side arise from the relation $k_B{}^2=-\mu^2(1-\mu^2/4M^2)$.

[19] A series expansion of the integrand allows an exact calculation of the left side. The correction to $I(\epsilon)$ is given by

$$\delta I(\epsilon)=\sum_{(\gamma,\alpha)}\frac{\gamma}{\cos\frac{1}{2}\pi\alpha}\left(\frac{\bar\omega}{\omega_0}\right)^\alpha\sum_{m=1}^\infty\frac{\sin\frac{1}{2}\pi(\alpha-\epsilon)}{\alpha-\epsilon-2m}\left(\frac{\mu}{\bar\omega}\right)^{2m}B(\epsilon,m),$$

where

$$B(\epsilon,m)=\frac{1}{2^m}\prod_{j=1}^m(\epsilon+2j-1).$$

It is easily verified that this correction is quite negligible in all cases of interest to us.

is elementary with the result

$$I_2=-\sum_{(\gamma,\alpha)}\frac{\gamma}{\cos\frac{1}{2}\pi\alpha}\left(\frac{\bar{\omega}}{\omega_0}\right)^{\alpha}\frac{\sin\frac{1}{2}\pi(\alpha-\epsilon)}{\alpha-\epsilon}.$$

The first integral can be expressed in terms of $T(\omega)$ using

$$\mathrm{Im}F(\omega)=(1/k^{\epsilon+1})[\cos\tfrac{1}{2}\pi\epsilon\,\mathrm{Re}T(\omega)-\sin\tfrac{1}{2}\pi\epsilon\,\mathrm{Im}T(\omega)].$$

Combining the three terms, our GSR is obtained:

$$I(\epsilon)=\sum_{(\gamma,\alpha)}\frac{\gamma}{\cos\frac{1}{2}\pi\alpha}\left(\frac{\bar{\omega}}{\omega_0}\right)^{\alpha}\frac{\sin\frac{1}{2}\pi(\alpha-\epsilon)}{\alpha-\epsilon}$$
$$=-\frac{M\pi}{\mu}\left(\frac{\bar{\omega}}{\mu}\right)^{\epsilon}f^2\left(1-\frac{\mu^2}{4M^2}\right)^{-1/2(1+\epsilon)}+\tfrac{1}{2}M\int_{\mu}^{\bar{\omega}}\frac{d\omega}{k}\left(\frac{\bar{\omega}}{k}\right)^{\epsilon}$$
$$\times[\cos\tfrac{1}{2}\pi\epsilon\,\mathrm{Re}T(\omega)-\sin\tfrac{1}{2}\pi\epsilon\,\mathrm{Im}T(\omega)].\quad(6)$$

This sum rule is only valid in the region $\alpha<\epsilon<1$, the upper limit arising from the bad behavior of the integrand[20] at threshold for $\epsilon>1$.

We shall now argue that this sum rule actually holds for all ϵ less than unity.[21] The following points should be noted:

(a) The integral converges and is of course finite for all $\epsilon<1$; also, the nucleon-pole term is finite and well defined for all ϵ.

(b) This being true, we expect that there is a formula in terms of α and ϵ for the left side of (6) also when $\alpha>\epsilon$.

(c) $I(\epsilon)$ is analytic in α and ϵ, which can be continued from $\alpha<\epsilon$ to $\alpha>\epsilon$.

(d) Hence $I(\epsilon)$ as given in (6) is a valid formula for all $\alpha>\epsilon$.

There are several values of ϵ which have particular interest. The original sum rule studied by Igi and Matsuda,[3] Logunov *et al.*,[3] and Horn and Schmid[4] corresponds to $\epsilon=-1$. By (4) we see that, for this value of ϵ, $F(\omega)=T(\omega)$. The sum rule in this case (assuming one Regge pole) reduces to

$$I(-1)=\frac{\gamma}{\alpha+1}\left(\frac{\bar{\omega}}{\omega_0}\right)^{\alpha}=-\frac{M\pi}{\bar{\omega}}f^2+\frac{M}{2\bar{\omega}}\int_{\mu}^{\bar{\omega}}d\omega\,\mathrm{Im}T(\omega).\quad(7)$$

The evaluation of this sum rule is straightforward since by the optical theorem the integrand is proportional to $\Delta_{\pi p}$. The "higher moment" sum rules[4] are given by $\epsilon=-3, -5, \cdots$; however, in practice they are of little use.

For the value $\epsilon=0$ the GSR reduces to the sum rule of Olsson[5] and of Meshcheriakov *et al.*[6]:

$$I(0)=\frac{\gamma}{\alpha}\left(\frac{\bar{\omega}}{\omega_0}\right)^{\alpha}\tan\tfrac{1}{2}\pi\alpha=-\frac{M\pi f^2}{\mu}\left(1-\frac{\mu^2}{4M^2}\right)^{-1/2}$$
$$+\tfrac{1}{2}M\int_{\mu}^{\bar{\omega}}\frac{d\omega}{k}\,\mathrm{Re}T(\omega).\quad(8)$$

This sum rule was evaluated by Olsson[5] with the aid of forward π c.e. cross-section data $(d\sigma/d\Omega)$ which gives the following[22] $\mathrm{Re}T(\omega)$:

$$|\mathrm{Re}T(\omega)|=\frac{\sqrt{s}}{M\hbar}\left[2\frac{d\sigma}{d\Omega}-\left(\frac{q\Delta_{\pi p}}{4\pi\hbar}\right)^2\right]^{1/2}.\quad(9)$$

The sign is defined by low-energy phase-shift analyses. Evaluation of (8) by use of (9) combined with the evaluation of the first sum rule (7) provides two pieces of information which then define the parameters of a single Regge pole. It is interesting to note that the Regge-pole parameters thus obtained were consistent with those resulting from an analysis of high-energy data.[5]

There is one further special value of ϵ which should be noted. If $\epsilon=\alpha_\rho-2$, where α_ρ is the "leading" intercept, then the corresponding Regge term on the left side of (6) vanishes because of the zero of $\sin\frac{1}{2}\pi(\epsilon-\alpha_\rho)$. Thus, if there were only one Regge exchange in the forward c.e. amplitude and, further, if we knew from another source (high-energy data) what the value of α_ρ is, then the proper evaluation of the sum rule $I(\alpha_\rho-2)$ should give a zero result.[23] If the result is nonzero, this is proof of the existence of secondary Regge exchanges. Our main result will be that indeed $I(\alpha_\rho-2)$ is consistent with zero.

Some general properties of the GSR should be mentioned. As ϵ becomes positive, the sum rule becomes more convergent and the Regge term becomes correspondingly less well determined. We find that in practice the content of the GSR becomes seriously reduced for ϵ greater than 0.5 in the case of the c.e. amplitude. Note that it is just this range ($\epsilon>0.57$) that $F(\omega)$ is superconvergent. At $\epsilon=1$ the sum rule as derived breaks down since a new pole appears.[20] The GSR at $\epsilon=1$ then becomes indentical to the ordinary dispersion relation (3) evaluated at threshold.[24]

On the other hand, as ϵ becomes negative, the integrand becomes strongly influenced by the $k^{-\epsilon}$ term

[20] At $\epsilon=+1$, $F(\omega)$ acquires a simple pole at $\omega=\pm\mu$ which introduces a subtraction constant. Sum rules near $\epsilon=+1$, however, are of no value for the c.e. amplitude because of the rapid convergence of the integrand and hence small Regge contribution (see Ref. 24).

[21] The author would like to thank Professor C. Goebel for suggesting this argument. Of course, the GSR can also be derived using the standard methods of Refs. 3 and 4.

[22] The quantity $\sqrt{s}$ is the usual c.m. energy; q is the c.m. momentum. See Ref. 11 for previously defined kinematic quantities.

[23] Because of the small correction terms described in Ref. 19, $I(\alpha_\rho-2)$ does not completely vanish. The ρ-pole correction contributes about 0.0015, which is much smaller than the experimental error in the evaluation of $I(\alpha_\rho-2)$.

[24] It has been noticed for some time that the dispersion relation (3) evaluated at threshold has an extremely small high-energy contribution [J. Hamilton and W. Woolcock, Rev. Mod. Phys. 35, 737 (1963)]. This again illustrates the lack of content for the GSR when $\epsilon>\alpha_\rho$ (i.e., in the superconvergent region).

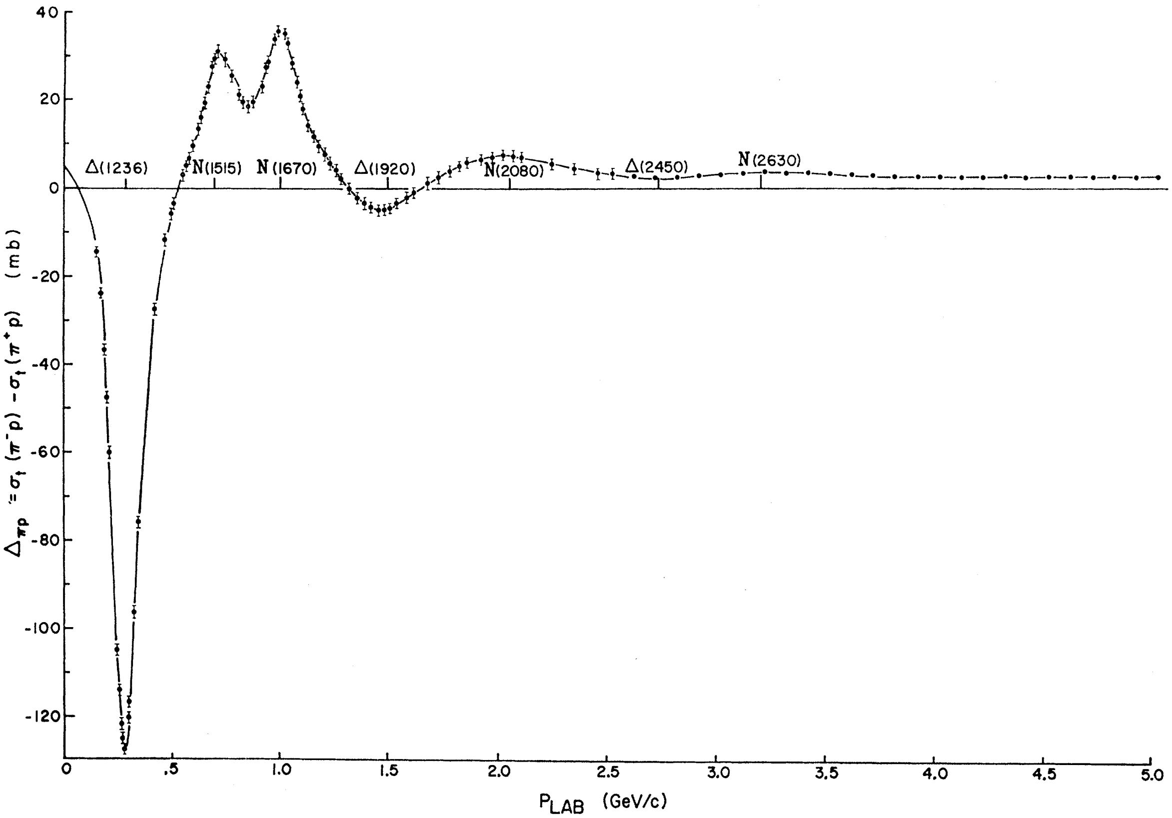

FIG. 1. Available data (Ref. 27) for the total cross-section difference $\Delta_{\pi p} = \sigma_t(\pi^- p) - \sigma_t(\pi^+ p)$. The curve drawn represents a statistical fit to the data involving 25 parameters.

which enhances the energy region near $\omega = \bar{\omega}$. Now, since $T(\omega)$ is well represented by the Regge expansion in this region and since the nucleon-pole term becomes rapidly unimportant, the GSR reduces to an identity in the Regge parameters for sufficiently negative ϵ. In our case, we find that for $\epsilon < -2$ little that is new is obtained.[25]

To sum up this section, we have derived a sum rule which is valid for all values of a parameter ϵ which are less than unity. In practice only values of ϵ not too close to unity or not too negative will provide useful information. The question of how much information is contained in the evaluated GSR will be dealt with later in detail; however, it is physically clear why more information can be provided by the GSR than by the original sum rule at $\epsilon = -1$. To see this, remember that as ϵ varies, the integrand is changed by the over-all factor $k^{-\epsilon}$ which takes various "moments" of the experimental data and hence extracts new knowledge about the Regge expansion.

Of course, the actual evaluation of the GSR requires knowledge of both the real and imaginary parts of $T(\omega)$ from threshold up to $\omega = \bar{\omega} \equiv 5$ GeV. The most accurate data available are the $\Delta_{\pi p}$ which tell us only about $\mathrm{Im}T(\omega)$. We could use the $\mathrm{Re}T(\omega)$ as determined from the forward c.e. data, using (9) as was originally done to evaluate[5] $I(0)$; however, we shall proceed differently. The data for $\Delta_{\pi p}$ will be used to evaluate $\mathrm{Re}T(\omega)$ by use of the ordinary dispersion relation (3). As will be shown in Sec. III, it is possible to determine $\mathrm{Re}T(\omega)$ from threshold to 5 GeV without making important assumptions about the Regge expansion. The GSR can then be evaluated and the Regge parameters displayed.

III. REAL PART BY A DISPERSION-RELATION METHOD

In this section we discuss the parametrization of the experimental data, by the use of a least-squares minimization procedure, and how this fit to the data can generate the real part up to 5 GeV. Our method has the advantage that a realistic error can be assigned to not only the real part at each energy but to any quantity depending functionally upon $\mathrm{Re}T(\omega)$ and $\mathrm{Im}(\omega)$.[26] In the course of this process it is natural to reexamine certain dispersion-relation calculations of threshold parameters and of forward c.e. scattering in the light of the most recent experimental data.

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[25] This question will be considered quantitatively in Sec. IV.

[26] A more detailed treatment of the error calculation is contained in the Appendix.

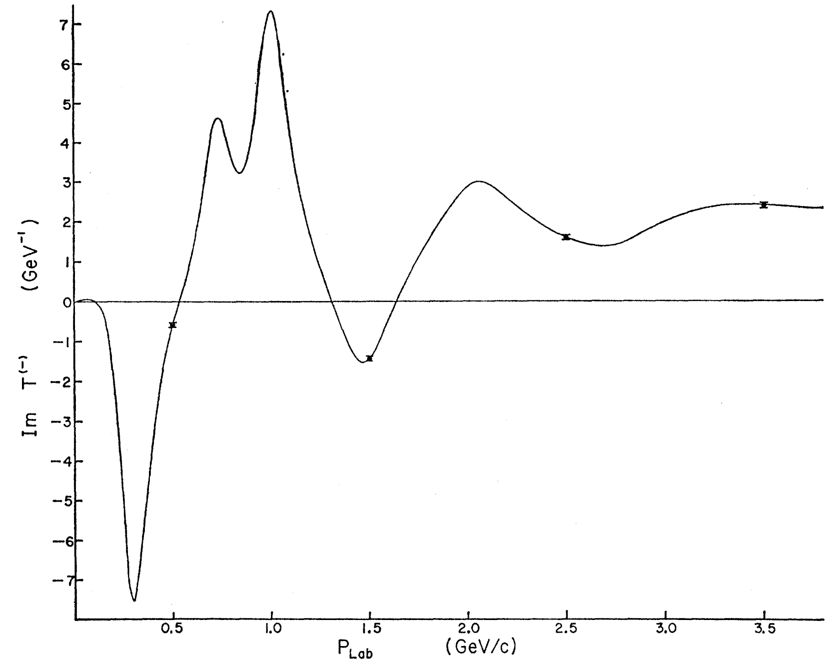

FIG. 2. Parametrized fit to the imaginary part of $T(\omega)$; the errors shown are representative of the error corridor calculated from the error matrix.

Since we wish to evaluate the GSR for a variety of values of the parameter ϵ and quote the error in each such evaluation, the best way to proceed is to fit a suitable analytic form to the experimental data. The values of the fitted parameters together with the "error matrix"[26] can then be used by numerical integration to rapidly evaluate the sum rules by machine. It will not be important that the fitted parameters have a straightforward interpretation in terms of resonance parameters, etc., but only that an adequate fit to the data is obtained. Nevertheless, our choice of interpolation formula closely follows the physics of the scattering process.

The data[27] to be fitted for $\Delta_{\pi p}$ are shown in Fig. 1. The energies at which measurements have been made range from threshold up to 22 GeV/c. There now exists high-accuracy experimental data in all energy regions with considerable overlap among experimental groups, indicating good agreement. The only exception to this statement is from threshold to slightly above the (3,3) resonance, where most experiments have been done at a single energy and then not even with both pion-charge states. These low-energy experiments are quite numerous, however, and their good agreement indicates an adequate knowledge of $\Delta_{\pi p}$ in the (3,3) resonance region.

Above 5 GeV/c, we use Regge parameters to fit the $\Delta_{\pi p}$. Between 2 and 5 GeV/c we use direct-channel resonances added[28] coherently with the Regge ρ-exchange amplitude. Below 2 GeV/c, a polynomial background is added to the direct-channel resonances to fit the data. The background parameters and all important resonance parameters are varied simultaneously to find the best least-squares fit. A χ^2 of 83 was obtained in the fit of 25 parameters to 180 pieces of cross-section data. The fit to the data is shown by the curve in Fig. 1.

The $\Delta_{\pi p}$ data have now been abstracted to a set of parameters and their associated error matrix. The result of this fit for $\mathrm{Im}T(\omega)$ is given in Fig. 2 and Table I. The errors shown are representative of the error corridor calculated from the error matrix,[26] and agree quite well with the actual experimental error at each point. We find that the smoothly varying "background" is accurately given by the Regge ρ-exchange amplitude even down to energies as low as the (3,3) resonance. Thus the "interference model,"[28] in which the whole amplitude is represented by the coherent sum of direct-channel resonances and cross-channel Regge exchanges, works surprisingly well, at least for the forward, no-spin-flip, c.e. π-N amplitude.

The usual forward-dispersion relation (3) can now be used to obtain $\mathrm{Re}T(\omega)$ from threshold to 5 GeV.[29] The

[27] Low-energy data were taken from the compilation of N. P. Klepikov, V. A. Meshcheryakov, and S. N. Sokolov, Dubna Report No. D-584. Other low-energy data used are those of O. T. Vik and H. Rugge, Phys. Rev. **129**, 2311 (1963). Medium-energy data used were from the recent experiment by K. F. Riley *et al.*, Phys. Rev. **168**, 1457 (1968). At higher energies the experiments used were those of A. Diddens *et al.*, Phys. Rev. Letters **10**, 262 (1963); A. Citron *et al.*, Phys. Rev. **144**, 1101 (1966); and K. J. Foley *et al.*, Phys. Rev. Letters **19**, 330 (1967).

[28] V. Barger and M. Olsson, Phys. Rev. **151**, 1123 (1966); V. Barger and D. Cline, Phys. Rev. Letters **16**, 913 (1966); J. Baacke and M. Yvert, Nuovo Cimento **51**, 761 (1967).

[29] In the calculation the principal-value singularity was removed by subtraction of a quantity proportional to

$$\mathrm{P}\int_0^\infty \frac{dk'}{k'^2-k^2}=0.$$

TABLE I. The real and imaginary parts of the antisymmetric amplitude $T(\omega)$ [as defined in Eq. (1)] are tabulated as a function of the pion lab momentum k. The units of k are GeV/c, and $T(\omega)$ has the units $(\mathrm{GeV})^{-1}$. Re$T(\omega)$ was determined from Im$T(\omega)$ by the dispersion relation (3) and does not include uncertainty in the value of f^2 (here taken to be 0.081).

k	ImT	ReT	k	ImT	ReT
0	0	1.65±0.01	0.925	4.92±0.03	4.21±0.04
0.035	0.033±0.002	1.53±0.01	0.963	6.48±0.03	3.33±0.05
0.070	0.050±0.004	1.17±0.01	1.002	7.33±0.03	1.26±0.05
0.087	0.038±0.004	0.91±0.01	1.040	6.60±0.03	−0.97±0.045
0.105	0 ±0.005	0.60±0.01	1.078	5.06±0.03	−2.11±0.05
0.122	−0.082±0.006	0.24±0.01	1.117	3.65±0.03	−2.40±0.05
0.140	−0.227±0.007	−0.17±0.01	1.193	1.79±0.03	−2.08±0.05
0.175	−0.83 ±0.01	−1.11±0.015	1.270	0.59±0.03	−1.57±0.05
0.210	−2.21 ±0.015	−2.05±0.015	1.347	−0.47±0.03	−0.90±0.05
0.228	−3.34 ±0.015	−2.29±0.02	1.423	−1.35±0.04	0.28±0.06
0.245	−4.73 ±0.015	−2.12±0.02	1.50	−1.44±0.04	1.84±0.07
0.262	−6.16 ±0.015	−0.13±0.03	1.60	−0.44±0.05	3.21±0.07
0.280	−7.21 ±0.015	0.16±0.03	1.70	0.67±0.05	3.63±0.08
0.298	−7.57 ±0.02	1.91±0.03	1.80	1.55±0.07	3.64±0.08
0.315	−7.23 ±0.02	3.56±0.04	1.90	2.31±0.08	3.41±0.10
0.332	−6.43 ±0.04	4.77±0.05	2.00	2.89±0.10	2.84±0.12
0.350	−5.61 ±0.05	5.50±0.06	2.10	2.97±0.12	2.08±0.13
0.388	−3.8 ±0.1	6.52±0.06	2.20	2.61±0.13	1.59±0.11
0.427	−2.3 ±0.1	6.70±0.06	2.40	1.86±0.08	1.52±0.11
0.465	−1.30 ±0.07	6.62±0.06	2.60	1.43±0.05	1.93±0.10
0.503	−0.54 ±0.05	6.54±0.07	2.80	1.48±0.05	2.60±0.10
0.542	0.09 ±0.05	6.56±0.06	3.00	1.96±0.05	2.89±0.10
0.580	0.78 ±0.04	6.67±0.05	3.20	2.32±0.06	2.81±0.10
0.618	1.66 ±0.03	6.77±0.05	3.40	2.41±0.05	2.66±0.12
0.657	2.83 ±0.03	6.65±0.04	3.60	2.36±0.05	2.61±0.12
0.695	4.09 ±0.03	5.87±0.05	3.80	2.31±0.06	2.70±0.12
0.733	4.66 ±0.04	4.40±0.04	4.00	2.33±0.06	2.84±0.13
0.772	4.15 ±0.035	3.26±0.04	4.20	2.41±0.06	2.96±0.14
0.810	3.45 ±0.025	3.08±0.04	4.40	2.50±0.07	3.05±0.15
0.848	3.23 ±0.03	3.51±0.04	4.60	2.57±0.07	3.12±0.16
0.887	3.71 ±0.03	4.05±0.04	4.80	2.64±0.06	3.18±0.17
			5.00	2.70±0.05	3.25±0.18

result is shown in Fig. 3 and Table I along with representative errors.[26] The values of Re$T(\omega)$ near threshold are of course influenced by the magnitude of f^2, the direct-channel nucleon-pole term. Following the analysis of Hamilton and Woolcock,[24] we take f^2 to be 0.081 ±0.002. The GSR values, however, are quite independent of the value of f^2. To demonstrate this last statement it is only necessary to substitute Eq. (3) for Re$T(\omega)$ into the GSR and note that as $\bar{\omega}$ becomes very large, the dependence on f^2 vanishes; in practice, this is nearly the case.

We have at our disposal data for $\Delta_{\pi p}$ up to 22 GeV/c,

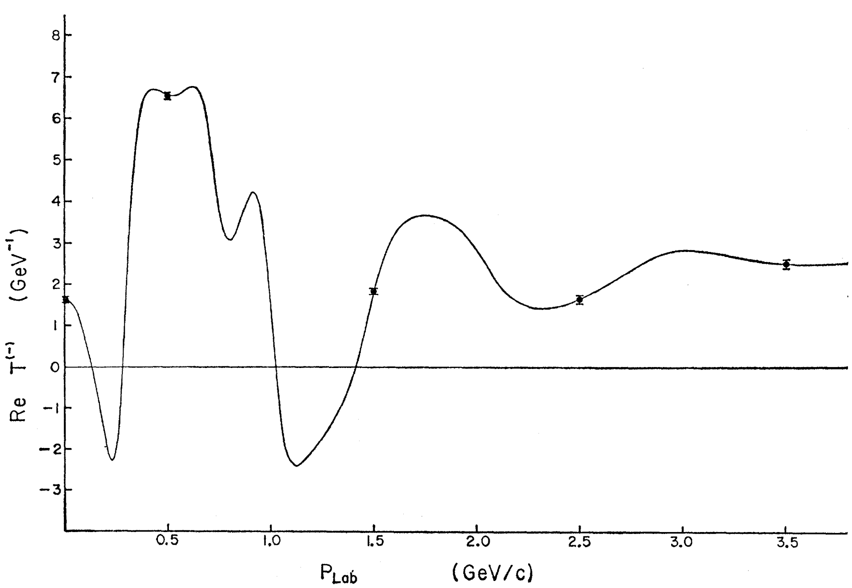

FIG. 3. Real part of $T(\omega)$ calculated from the imaginary part of $T(\omega)$ by use of the dispersion relation (3). The errors shown are representative of the error corridor obtained by taking account of parameter errors and correlations as described in the Appendix.

so that in order to evaluate the dispersion relation (3) we must make an extrapolation based upon the lower-energy information. This turns out to be no problem, however, since $\mathrm{Re}T(\omega)$ is needed only below 5 GeV and is not particularly sensitive to the high-energy extrapolation. To see this, we try the following variation: Repeat the least-squares fit to $\Delta_{\pi p}$, except using a two-pole Regge expression, the second pole being constrained to have the same residue and to have an intercept one unit below the ρ pole. The ρ residue and intercept now adjust to give a good fit to the data below 22 GeV. Using this new parametrization, the real part of $T(\omega)$ is computed. It is found that even at 5 GeV, $\mathrm{Re}T(\omega)$ has not changed appreciably, remaining well within its error. Thus we have obtained numerical values for $\mathrm{Re}T(\omega)$ below 5 GeV in a model-independent manner.

Before using these expressions for $\mathrm{Re}T(\omega)$ and $\mathrm{Im}T(\omega)$ in a numerical evaluation of the GSR, let us examine their relation to earlier work and discuss the general situation regarding the low-energy π-N scattering parameters. The determination of the π-N scattering lengths and the coupling constant f^2 involves a rather elaborate analysis, using low-energy angular distributions, and was originally carried out by Hamilton and Woolcock.[24] In later[30] work, however, there has been considerable variation in these parameters. The most straightforward part of such analyses is the evaluation of the dispersion relation (3) at threshold, which assumes the following form:

$$\tfrac{1}{6}(a_1-a_3)\left(1+\frac{\mu}{M}\right)=f^2+\frac{\mu^2}{2\pi}\int_\mu^\infty \frac{d\omega}{k^2}\,\mathrm{Im}T(\omega)\,,$$

where a_1-a_3 is the πN s-wave scattering-length difference in inverse pion-mass units. The evaluation of the integral in terms of our parametrized fit to the most recent data[27] results in the relation

$$a_1-a_3=5.22f^2-(0.122\pm0.002)\,.$$

If Hamilton and Woolcock's[24] value for the π-N coupling constant, $f^2=0.081\pm0.002$, is introduced, we get $a_1-a_3=0.30\pm0.01$, which disagrees decidedly with their value $a_1-a_3=0.265\pm0.01$. This discrepancy has been previously noticed by Höhler *et al.*,[2] whose results (which do not quote an error) agree reasonably well with ours. Höhler *et al.*[2] were then led to remark that perhaps f^2 is not so well determined as it appeared, and we see that if $f^2=0.074$, Hamilton and Woolcock's value for a_1-a_3 is obtained.

Much of the confusion about low-energy parameters would be cleared up if there were some agreement among more direct analyses of experiment. A recent experiment and analysis of low-energy scattering by Donald *et al.*[31] finds that $a_1-a_3=0.29\pm0.015$. Other less direct analy-

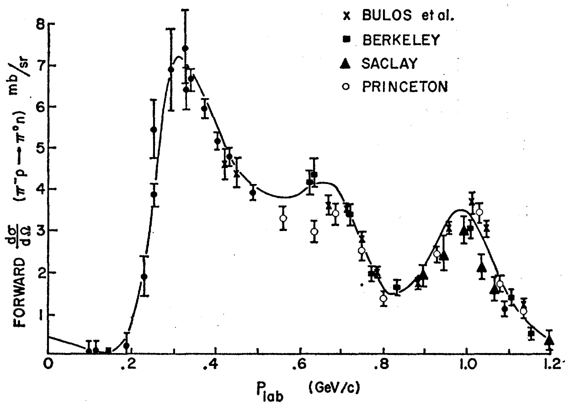

FIG. 4. Prediction of the forward π c.e. angular distribution using charge independence and the values of $T(\omega)$ from $\Delta_{\pi p}$ data [using the dispersion relation to obtain $\mathrm{Re}T(\omega)$]. The experimental data shown are those of Ref. 33.

ses or more difficult experiments tend to favor[31,32] a lower value of $a_1-a_3\cong0.25$. Apparently, the only resolution of this discrepency will be the accumulation of high-statistics data for elastic and c.e. scattering at very low energies.

Since we know both $\mathrm{Re}T(\omega)$ and $\mathrm{Im}T(\omega)$ up to 5 GeV/c, we can then predict the forward π-charge-exchange angular distribution $(d\sigma/d\Omega)$ by charge independence [Eq. (9)]. Figure 4 shows this prediction compared with experiment from threshold to above 1 GeV/c.[33] In general, there is good agreement with the data points and also with the curve of Höhler *et al.*[2] Between 0.9 and 2 GeV/c our prediction agrees quite well with that of Borgeaud *et al.*[34] We have assumed here a value of $f^2=0.081$. The question arises whether it is feasible to measure $d\sigma/d\Omega$ $(\theta=0)$ accurately enough to get a good determination of f^2. The best momentum to use is when $\mathrm{Re}T(\omega)$ is large and $\mathrm{Im}T(\omega)$ is small. This is the case for $0.35<k<0.45$ GeV/c. In this range we have

$$\delta(d\sigma/d\Omega)\approx12.5\delta(f^2)\,;$$

thus, if f^2 is to have an error of ±0.002, the forward differential cross section must be measured to ±0.025 mb/sr. There already exists a set of measurements[35] in this range whose errors are ±0.19 mb/sr, so at the present, f^2 is determined by the forward π-charge-exchange data to be $f^2=0.08\pm0.015$.

IV. EVALUATION OF THE GENERALIZED SUM RULE

In Sec. III we showed how the $\Delta_{\pi p}$ data from threshold up to 22 GeV/c allowed the calculation of the real

[30] V. K. Samaranayake and W. S. Woolcock, Phys. Rev. Letters **15**, 936 (1965); J. Hamilton, Phys. Letters **20**, 687 (1966).

[31] R. A. Donald, W. H. Evans, W. Hart, P. Mason, D. E. Plane, and E. J. C. Read, Proc. Phys. Soc. (London) **87**, 445 (1966). This paper contains references to earlier work.

[32] A. Donnachie and G. Shaw, Nucl. Phys. **87**, 556 (1967).

[33] The experimental data for $d\sigma/d\Omega$ below 1 GeV/c are compiled in L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. **138**, B190 (1965). Other data shown: Bulos *et al.*, Phys. Rev. Letters **13**, 558 (1964); C. Chiu *et al.*, Phys. Rev. **156**, 1415 (1967); P. Borgeaud *et al.*, Phys. Letters **10**, 134 (1964); W. S. Risk, Phys. Rev. **167**, 1249 (1968).

[34] P. Borgeaud *et al.*, Phys. Letters **10**, 134 (1964).

[35] J. Caris *et al.*, Phys. Rev. **121**, 893 (1961).

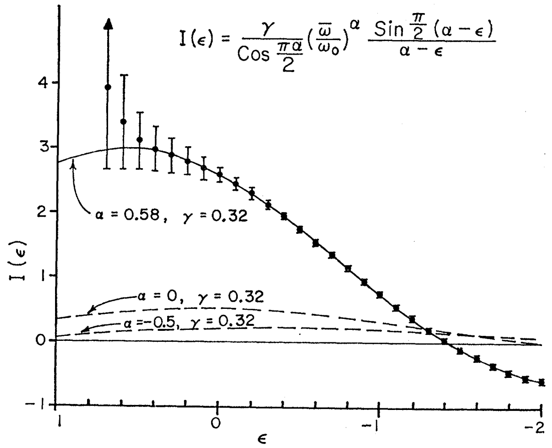

FIG. 5. Sum rules $I(\epsilon)$ for $-2\leq\epsilon<+1$. The curves drawn correspond to single Regge-pole predictions.

part in a model-independent manner below 5 GeV/c. We shall now make use of this information to evaluate the GSR formula (6) directly. The asymptotic energy $\bar{\omega}$ is chosen to be 5 GeV. If this is indeed high enough,[36] we do not gain additional information by varying $\bar{\omega}$. Although the GSR explicitly depends strongly on the value of the πN coupling constant f^2 (at least for large ϵ), we have seen in Sec. III that it is not necessary to know f^2 well since it is nearly cancelled by the pole term appearing in $\mathrm{Re}T(\omega)$ from (3). We shall want to evaluate the GSR for $\epsilon<1$ and down to a value (of ϵ) below which the integral is dominated by contributions near $\bar{\omega}$. The resulting values of the GSR must also be generated by the Regge-pole expansion, thus determining Regge parameters.

Using our expressions for $\mathrm{Re}T(\omega)$ and $\mathrm{Im}T(\omega)$, the GSR can be numerically integrated, giving a resulting set of values $I(\epsilon)$ shown in Fig. 5 (for $\frac{1}{10}$ unit intervals of ϵ). We shall now attempt to answer the question of how negative we can go in ϵ. In Sec. III it was shown that changing to a two-pole extrapolation above 22 GeV/c has a negligible effect on $\mathrm{Re}T(\omega)$. If this two-pole fit is substituted into the GSR, the resulting $I(\epsilon)$ can be compared with that derived from a single-pole extrapolation. It is seen that the $I(\epsilon)$'s are quite similar until ϵ approaches -3. At $\epsilon=-3$ the two differ by a standard deviation.[37] Thus we have evidence that by $\epsilon=-3$ the GSR content is becoming model-dependent.

A realistic error has been assigned to $I(\epsilon)$ by keeping track of all parameter correlations by means of the error matrix.[26] The indicated errors in Fig. 5 are obtained in this manner. Table II contains a tabulation of $I(\epsilon)$.

[36] 5 GeV is certainly asymptotic by the usual criteria: (i) $\omega\gg$masses in problem; (ii) above direct-channel resonances; (iii) many absorbative channels open; (iv) cross section changing smoothly.

[37] It might be noted that $\epsilon=-3$ corresponds to the first "higher moment" sum rule of Horn and Schmid (Ref. 4), which thus, at least for the forward nonflip amplitude, is not useful.

TABLE II. Values of $I(\epsilon)$ obtained from the sum rule (6) for ϵ in the interval $-2<\epsilon<+1$. These results have been plotted in Fig. 5.

ϵ	$I(\epsilon)$	ϵ	$I(\epsilon)$
0.6	3.53±0.58	−0.8	1.17±0.03
0.5	3.25±0.44	−0.9	0.96±0.03
0.4	3.09±0.34	−1.0	0.76±0.03
0.3	2.97±0.27	−1.1	0.56±0.02
0.2	2.86±0.22	−1.2	0.37±0.03
0.1	2.75±0.18	−1.3	0.19±0.03
0	2.65±0.11	−1.4	0.02±0.03
−0.1	2.51±0.10	−1.5	−0.13±0.03
−0.2	2.35±0.09	−1.6	−0.26±0.03
−0.3	2.18±0.07	−1.7	−0.38±0.03
−0.4	1.99±0.06	−1.8	−0.48±0.03
−0.5	1.79±0.06	−1.9	−0.56±0.03
−0.6	1.59±0.05	−2.0	−0.62±0.03
−0.7	1.38±0.04		

Because of the finite errors, the smooth variation of $I(\epsilon)$, and the limited range of ϵ, the experimentally determined $I(\epsilon)$ contains only a finite amount of information. A good measure of this content has been suggested by Kreps.[38] A polynomial series[39] is fitted to the $I(\epsilon)$ and the number of nonvanishing coefficients (larger than their error) measures the amount of information contained in $I(\epsilon)$. The result of such a fit is that there are four nontrivial numbers which can be extracted from the $I(\epsilon)$ of Fig. 5 or Table II.

The $I(\epsilon)$ curve must also be expressible by a set of Regge parameters.[40] In fact, by (6) we must have

$$I(\epsilon)=\sum_{(\gamma,\alpha)}\frac{\gamma}{\cos\frac{1}{2}\pi\alpha}\left(\frac{\bar{\omega}}{\omega_0}\right)^{\alpha}\frac{\sin\frac{1}{2}\pi(\alpha-\epsilon)}{\alpha-\epsilon}.$$

First, we might note that the integral used in the evaluation of $I(\epsilon)$ is always finite, and hence at right-signature points (α odd) the residue must have at least a simple zero. This is just equivalent to the usual requirement that the Regge amplitude is always finite.[41]

In Fig. 5 we illustrate the $I(\epsilon)$ expected for various single-pole terms. It is seen that for $\gamma\approx0.33$ and $\alpha\approx0.58$ an optimum fit to the data results. If a single pole is fitted to the experimentally determined $I(\epsilon)$ by varying its residue and intercept, the following set of parameters results:

$$\alpha_\rho=0.58\pm0.01\,,\quad \gamma_\rho=0.32\pm0.01\,. \qquad (10)$$

This is in excellent agreement with the results of single-pole analyses of high-energy data.[13,42] The agreement of

[38] R. Kreps (private communication).

[39] Tschebyscheff polynomials were chosen because of their rapid convergence.

[40] Of course, we have only considered the leading term of the asymptotic expansion of $Q_{-\alpha-1}(z)$. The next term lies two units below and its residue is much too small to be detected. Real evidence for the Regge picture is obtained if the set of power-law coefficients is rapidly convergent.

[41] In particular, if $T^R=(2\gamma/M)(\omega/\omega_0)^\alpha(i+\tan\frac{1}{2}\pi\alpha)$, then γ must have a zero at odd-signature points if T^R is to be finite.

[42] G. Höhler, J. Baacke, H. Schlaile, and P. Sonderegger, Phys. Letters 20, 79 (1966); other more model-dependent analyses are given in Refs. 43 and 44.

the various methods for calculating the ρ Regge parameters makes these parameters among the best-determined numbers in high-energy physics.

Clearly, the addition of a second Regge pole will not improve agreement with experiment and hence will not determine any new parameters (see Fig. 5). However, we can place an upper limit on the residue for any given ρ' intercept. A particularly strong statement can be made if we use some information from high-energy π c.e. scattering. The angular distributions at high energy are not sensitive to the existence of the ρ'. This can be seen in a practical way by examining the parameters determined from essentially the same π c.e. data by a number of different investigators, some using a single-pole fit[42,43] and others assuming both a ρ and ρ'.[44] In particular, there is complete agreement concerning the forward intercept, which we take as[42]

$$\alpha_\rho = 0.575 \pm 0.01 .$$

When this is used in conjunction with the GSR result, and if the ρ' intercept is chosen to be one unit below the ρ, the result is

$$\gamma_{\rho'} < \tfrac{1}{10}\gamma_\rho . \tag{11}$$

A similar limit could be obtained for $\alpha_{\rho'} = 0$.

V. DISCUSSION AND CONCLUSIONS

The existence of a ρ' trajectory has been suggested among other things[45] to provide a nonvanishing π c.e. polarization[15] which is apparently observed at high energy.[14] The forward helicity-nonflip ρ' residue in these early models violates badly the inequality (11). Indications of this were noticed using a single sum rule,[46] leading Sertorio and Toller[10] to propose that the ρ' is a conspiring trajectory of the type (Gribov-Volkov-β-type)[47] whose residue vanishes as $\sqrt{t}$ at the forward direction. Our result strongly supports this hypothesis. Recently, an analysis by Gajdicar *et al.*[44] has demonstrated that it is possible to explain the polarization with an arbitrarily small helicity-nonflip ρ' amplitude. It seems reasonable to assume that since the nonflip amplitude plays a minor role[44] in explaining polarization and since it must be quite small in the forward direction due to the GSR, that it probably vanishes at $t=0$.

[43] R. J. N. Phillips and W. Rarita, Phys. Rev. **139**, B1336 (1965); C. Chiu, R. Phillips, and W. Rarita, *ibid.* **153**, 1485 (1967); F. Arbab and C. Chiu, *ibid.* **141**, 1045 (1966).

[44] W. Rarita and B. Schwarzchild, Phys. Rev. **162**, 1378 (1967); T. J. Gajdicar, R. K. Logan, and J. W. Moffat, University of Toronto Report, 1967 (unpublished).

[45] Rarita and Schwarzchild (Ref. 44) also propose an explanation of the K^+ c.e. angular distribution using a ρ'.

[46] See Igi *et al.* (Ref. 3) and Sertorio *et al.* (Ref. 10). One must be careful in evaluating the $\epsilon=-1$ sum rule. The data of Citron *et al.* (Ref. 27), which are used from 2.5–6.0 GeV/c, have a *systematic* error far in excess of the statistical errors. Thus the neglect of the large systematic error results in an unrealistically low error in the sum-rule evaluation.

[47] V. N. Gribov and D. V. Volkov, Zh. Eksperim. i Teor. Fiz. **44**, 1068 (1963) [English transl.: Soviet Phys.—JETP **17**, 720 (1963)].

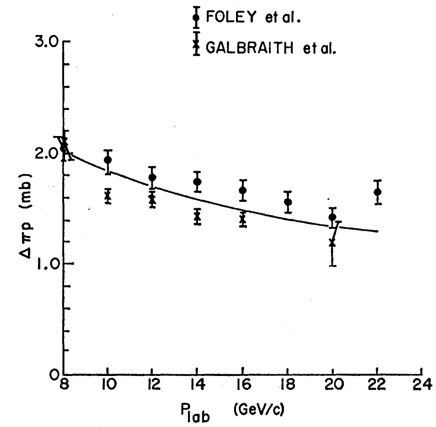

FIG. 6. Prediction of $\Delta_{\pi p}$ at high energy, using the Regge parameters [Eq. (10)] obtained from low-energy data. The high-energy measurements are from Refs. 27 and 52.

As complicated as the situation may be in the non-forward directions, there appears to be a certain simplification at $t=0$ in that only ρ exchange contributes. Another interesting aspect of the forward c.e. amplitude is seen if the GSR is evaluated using only resonances. To do this we set all the nonresonant (smoothly varying background) parameters to be zero in the expression for $\mathrm{Im}T(\omega)$. Inserting this resonance part of $\mathrm{Im}T(\omega)$ into the dispersion relation (3) gives the corresponding resonance part of $\mathrm{Re}T(\omega)$. The modified amplitude is then used to evaluate the GSR. The result is that $I(\epsilon)$ is consistent with zero for all ϵ. This unusual result is equivalent to the fact that the interference model[28] works so well for the forward π c.e. amplitude. To see that this is a reasonably fair statement we remember that the interference model claims that down to low energies

$$T(\omega) = T_{\mathrm{res}}(\omega) + T^R(\omega) ,$$

where $T^R(\omega)$ is the asymptotic Regge expression. Using for $T^R(\omega)$ an expression which differs from (2) only at very low energies,[48] we substitute into the GSR. $I(\epsilon)$ then cancels with the integral over $T^R(\omega)$ and implies that the resonance contribution to the integral must cancel with the nucleon pole, as is observed. We can thus tell under what circumstances the interference model will fail at low energies. Whenever resonances enter into an amplitude in such a way that cancellation with the nucleon-pole term is impossible, then the background amplitude is badly approximated by the Regge amplitude. This will be the case in the helicity-flip c.e. amplitude or the nonflip, non-c.e. amplitude where all resonant contributions enter with the same sign. Now, since the interference model is presumably always valid above *some* energy, our criterion indicates that the two above-mentioned amplitudes violate the interference model at a higher energy than in the more favorable

[48] Assume $\mathrm{Im}T^R(\omega) = (2\gamma/M)(\omega/k)(k/\omega_0)^\alpha$.

case of forward c.e. scattering. This can be seen directly from the fact that for $\pi^{\pm}p$ total cross sections the Regge amplitude does not represent the nonresonant background below 2.5 GeV/c.[49]

As for the basic reasons why the resonances and nucleon-pole term exactly cancel in the expression for the Regge amplitude $I(\epsilon)$ in the case considered (forward π c.e.), we must rely on speculation. The explanation might be as follows: ρ exchange dominates both high and low c.e. scattering[50]; thus the "interference" model is likely to work and hence the resonance cancellation follows. In π non-c.e. scattering the resonances cannot cancel and in fact add to give, through the GSR, a large Regge background. Because there can be no cancellation of resonances, the interference model breaks down, and we know that at threshold the Adler condition[51] makes the non-c.e. amplitude small. Similar conclusions follow in the helicity-flip c.e. amplitude, since it is nucleon-pole terms and not ρ exchange which dominate this amplitude at threshold.

Finally, it is of interest to plot our ρ-exchange prediction for high-energy $\Delta_{\pi p}$ as measured in two recent experiments. The parameters of (10) when substituted into (1) result in the curve in Fig. 6. Also plotted are the experimental data of Foley *et al.*[27] and Galbraith *et al.*[52]

In conclusion, we have evaluated a set of sum rules which constrain the parameters describing ρ-like Regge exchanges. These constraints are given in Table I and should be used in future analyses involving ρ exchange. We have explicitly written the $I(\epsilon)$ expected for a discrete set of Regge poles, but the generalization to a continuous superposition of poles (or a Regge cut) is straightforward.

[49] V. Barger and M. Olsson, Phys. Rev. **148**, 1428 (1966).

[50] At low energies it is well known that ρ exchange predicts the correct s-wave scattering lengths [see J. Sakurai, Phys. Rev. Letters **17**, 1021 (1966)]; J. Hamilton, in *Strong Interactions and High Energy Physics, Scottish Universities' Summer School, 1963*, edited by R. G. Moorhouse (Oliver and Boyd, London, 1964), shows how the nucleon-pole contributions vanish for π c.e. near threshold.

[51] S. Adler, Phys. Rev. **137**, B1022 (1965).

[52] Galbraith *et al.*, in *Proceedings of the Athens Conference on Resonant Particles, Ohio University, Athens, Ohio, 1965*, edited by B. A. Munir (Ohio University Press, Athens, Ohio, 1965), p. 522.

APPENDIX

The assignment of errors and their propagation through dispersion relations, sum rules, and other derived quantities will be discussed here in greater detail.

The available data for $\Delta_{\pi p}$ are fitted by standard least-squares techniques by an expression containing N parameters $\{p_i\}$. The result of this process is a set of "best" values of these parameters and their correlation or error matrix[53] $T_{ij}=\langle\Delta p_i\Delta p_j\rangle_{\rm av}$. Any function $H(p_j)$ these parameters can be evaluated and the expected error in $H(p_j)$ is determined by

$$\Delta H(p_j)=\sum_{i,j}^{N}\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}T_{ij}.$$

We have in the course of the analysis calculated all of the derivatives $\partial\,\mathrm{Im}T(\omega)/\partial p_j$; thus by means of the dispersion relation we can find

$$\frac{\partial\,\mathrm{Re}T(\omega)}{\partial p_j}=\frac{4f^2}{\omega}+\frac{2\omega}{\pi}\mathrm{P}\int_\mu^\infty\frac{d\omega'}{\omega'^2-\omega^2}\frac{\partial\,\mathrm{Im}T(\omega')}{\partial p_j}.$$

Thus any function of $\mathrm{Re}T(\omega)$ and $\mathrm{Im}T(\omega)$ can be assigned an error:

$$\Delta G=\sum_{i,j}^{N}\frac{\partial G}{\partial p_i}\frac{\partial G}{\partial p_j}T_{ij},$$

where

$$\frac{\partial G}{\partial p_i}=\frac{\partial G}{\partial(\mathrm{Re}T)}\frac{\partial(\mathrm{Re}T)}{\partial p_i}+\frac{\partial G}{\partial(\mathrm{Im}T)}\frac{\partial(\mathrm{Im}T)}{\partial p_i}.$$

The function G is evaluated at a given energy and the error applies to this energy. G may, for example, be the forward π c.e. cross section or it may be a value of the interband in the GSR [Eq. (6)]. In the latter case, the error in $I(\epsilon)$ is calculated by integration of ΔG from threshold to $\bar{\omega}$.

[53] The theory of the least-squares method is available from numerous sources. Two references which might be mentioned because of their high-energy physics orientation are J. Orear, University of California Report No. UCRL-8417 (unpublished); F. T. Solmitz, Ann. Rev. Nucl. Sci. **14**, 375 (1964).