## APPENDIX II

We consider operators $\mathcal{O}^{\pm}=\{V_1,A_2\}\pm\{V_2,A_1\}$ which are

$$\begin{pmatrix}\text{even}\\ \text{odd}\end{pmatrix}$$

under interchanges of $V$ and $A$. It is shown here that

(1) $[F_a{}^5,[F_b{}^5,\mathcal{O}^+]]=[F_a,[F_b,\mathcal{O}^+]]$;

(2) double commutation with $F_a{}^5$, $F_b{}^5$ does not change the symmetry under $V$, $A$ interchange.

Both results follow from the identity

$$\begin{aligned}&[F_a{}^5,[F_b{}^5,\mathcal{O}^{\pm}]]\\ &=(\{[F_a{}^5,[F_b{}^5,V_1]],A_2\}\pm\{V_2,[F_a{}^5,[F_b{}^5,A_1]]\})\\ &+(\{V_1,[F_a{}^5,[F_b{}^5,A_2]]\}\pm\{[F_a{}^5,[F_b{}^5,V_2]],A_1\})\\ &+(\{[F_a{}^5,V_1],[F_b{}^5,A_2]\}\pm\{[F_a{}^5,A_1],[F_b{}^5,V_2]\})\\ &+(\{[F_b{}^5,V_1],[F_a{}^5,A_2]\}\pm\{[F_b{}^5,A_1],[F_a{}^5,V_2]\}).\end{aligned}\quad(24)$$

That the symmetry property is retained is clear from inspection. To see property (1), note that we may replace $F^5$ by $F$ in the double-commutation terms on the right-hand side, since

$$[F_a{}^5,[F_b{}^5,\mathcal{O}]]=[F_a,[F_b,\mathcal{O}]],\quad \mathcal{O}=A \text{ or } V.$$

For the single-commutation terms, since

$$[F_a{}^5,V]=[F_a,A],\quad [F_a{}^5,A]=[F_a,V],$$

replacement of $F^5$ by $F$ must be accompanied by interchange of $V$ and $A$, under which the symmetric case (+) will just transform into the same terms with $F^5$ replaced by $F$, but the antisymmetric case (−) transforms into minus the terms with $F^5$ replaced by $F$. Thus property (1) holds for $\mathcal{O}^+$ but not for $\mathcal{O}^-$.

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# Radiative Corrections to $K_{e3}{}^0$ Decays and the $\Delta I=\frac{1}{2}$ Rule

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The radiative corrections to the Dalitz plot in $K_{e3}{}^0$ decays are calculated assuming a phenomenological weak $K$-$\pi$ vertex and using perturbation theory. The answer depends logarithmically on a cutoff, as is the case for nuclear $\beta$ decay. An interesting feature of these decays is that they offer a means of measuring the $q^2$ dependence of the form factors $f_\pm(q^2)$. The radiative corrections contribute an additional energy dependence which cannot be separated experimentally. It is found that the radiative corrections are considerable, i.e., greater than 3% in absolute magnitude, over a large portion of the Dalitz plot, and are not particularly sensitive to a reasonable choice of cutoff. The corrections to the lepton and pion spectra, and the decay rate are also given. A comparison with previous results for $K_{e3}{}^0$ reveals that the $K_{e3}{}^+$ correction is of the same order of magnitude but everywhere more positive. In particular, the ratio of the decay rates $\Gamma(K_{e3}{}^0)/\Gamma(K_{e3}{}^+)$, which is equal to 2 according to the $\Delta I=\frac{1}{2}$ rule, must be modified by a factor $(1+\delta)$ due to the radiative corrections. It is found that $\delta\simeq 1\frac{1}{4}\%$ and is independent of the cutoff.

## I. INTRODUCTION

THE subject of this paper is the estimate of the radiative corrections to the three-body leptonic decays of neutral kaons, $K_{l3}{}^0$ for short. The same approach is adopted as that used in previous papers[1,2] concerned with the radiative corrections to $K_{l3}{}^\pm$. These estimates are relevant for several reasons. Recent experimental interest in these decays will result in measurements of sufficient precision to be sensitive to radiative corrections.[3] Thus, in a measurement of the energy dependence of the phenomenological form factors $f_\pm(q^2)$, one must allow for an additional, unavoidable $q^2$ dependence due to radiative effects. Moreover, electromagnetic interactions do not conserve isospin; therefore, the radiative corrections will modify predictions based upon isospin selection rules, such as the $\Delta I=\frac{1}{2}$ rule, which relate the experimentally measured decay rates of charged and neutral kaons. Finally, the presence of an electromagnetic final-state interaction [Fig. 1(b)] in the $K_{l3}{}^0$ (but not $K_{l3}{}^+$) radiative correction gives rise to an apparent violation of time-reversal invariance in the measurement of the transverse lepton polarization.[4] The numerical results presented in this paper depend upon the overwhelming simplification which results from neglecting the lepton mass (in particular, this excludes lepton polarization). Thus, the numerical results apply only to $K_{e3}{}^0$.

* Supported by the National Science Foundation.
[1] E. S. Ginsberg, Phys. Rev. **142**, 1035 (1966).
[2] E. S. Ginsberg, Phys. Rev. **162**, 1570 (1967).
[3] Princeton Conference on $K$ mesons, Princeton, N. J., 1967 (unpublished).
[4] N. Byers, S. W. MacDowell, and C. N. Yang, in *Proceedings of the Seminar in High-Energy Physics and Elementary Particles, Trieste, 1965* (International Atomic Energy Agency, Vienna, 1965), p. 953.

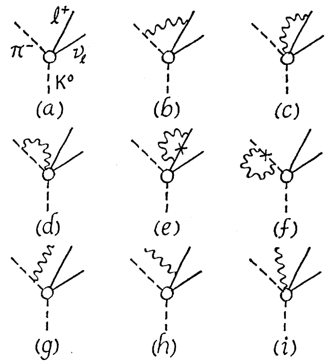


FIG. 1. Feynman diagrams for the radiative corrections to $K_{e3}^0$ to first order in $\alpha$; (a) zero-order process, (b)–(f) virtual diagrams, (g)–(i) inner bremsstrahlung.

The assumptions which underlie the present calculation may be summarized as follows[1,2]: (i) Assume the usual phenomenological hadronic contribution to the weak $K$-$\pi$ vertex [this depends on the form factors $f_\pm(q^2)$]. (ii) Calculate radiative effects using first-order perturbation theory in $\alpha$. (iii) Neglect the modification of the form factors themselves due to radiative effects (electromagnetic corrections to strong-interaction renormalization graphs). And (iv) neglect the small $q^2$ dependence of the form factors in calculating the radiative corrections[5] (i.e., if the form factors are expanded in terms of a small parameter $\lambda$, neglect terms of order $\lambda\alpha$ and higher). The simple model outlined above fails in one respect, namely, that the radiative corrections depend logarithmically on a cutoff.[6] It is customary to regard an estimate of radiative corrections as useful provided the numerical result is not sensitive to reasonable choices for the cutoff (i.e., $\Lambda \sim m_p$). For $K_{e3}^0$, the radiative corrections to the Dalitz plot and to the positron spectrum are not especially sensitive to $\Lambda$. (Because of cancellation between various terms, the correction to the pion spectrum and the lifetime is much less certain.) However, the radiative correction to the ratio of the decay rates of charged and neutral kaons does not depend on the cutoff (to first order in $\alpha$). The calculation is described in Sec. II and the results for $K_{e3}^0$ modes are given in Sec. III.

## II. CALCULATION

The zero-order transition matrix element for the process $K_{l3}^0$ [Fig. 1(a)] may be written as

$$\mathfrak{M}_0 = -i(2\pi)^{-2} m_l^{1/2} (4E_K E_\pi E_l)^{-1/2} \delta^{(4)}(p_K - p_\pi - p_l - p_\nu) \times f_+ \bar{u}_\nu M_0 \tfrac{1}{2}(1 - i\gamma_5) v_l , \quad (1)$$

where

$$M_0 = 2p_K \cdot \gamma - (1-\xi) p_l \cdot \gamma . \quad (2)$$

The notation is the same as in Ref. 2. Note that $p_i$ denotes *either* the four momentum *or* the magnitude of the three momentum of the $i$th particle, depending on the context. Also $m_K$ or $m_\pi$ denote the mass of the meson charge state appropriate to the process under consideration, be it $K_{l3}^0$ or $K_{l3}^+$. The same remark applies to other symbols used in this paper and in Ref. 2, e.g., the form factors $f_+(q^2)$ and $\xi = f_-(q^2)/f_+(q^2)$, which *a priori* need not be the same for $K_{l3}^0$ and $K_{l3}^+$. In view of (iii) and (iv) above, the momentum-transfer dependence of the form factors will be suppressed. Throughout this paper, any expression which is not explicitly Lorentz invariant is meant to apply to the barycentric system, i.e., the rest system of the kaon.

The lepton-pion energy correlation (Dalitz plot) of the zero-order process is easily obtained from Eqs. (1) and (2):

$$\Gamma_0(E_l, E_\pi) = (2\pi)^{-3} |f_+|^2 (2E_K)^{-1} \times \{[m_K^2 + m_l^2 - H^2 - m_l^2 \operatorname{Re}(1-\xi)] \times (H^2 - m_\pi^2 + q^2 - m_l^2) - (m_K^2 - \tfrac{1}{4} m_l^2 |1-\xi|^2)(q^2 - m_l^2)\} , \quad (3)$$

where

$$q^2 = (p_K - p_\pi)^2 \quad \text{and} \quad H^2 = (p_K - p_l)^2 . \quad (4)$$

Equation (3) simplifies in the barycentric system. Recalling the notation of Ref. 2, we have

$$\Gamma_0(E_l, E_\pi) = (2\pi)^{-3} |f_+|^2 \{[2m_K E_l - m_l^2 \operatorname{Re}(1-\xi)] E_\nu - (m_K^2 - \tfrac{1}{4} m_l^2 |1-\xi|^2)(W_\pi - E_\pi)\} . \quad (5)$$

The contribution to the transition matrix element from the first-order virtual diagrams [Fig. 1(b)–1(f)] can be evaluated, and with a definition similar to Eq. (1), is found to be

$$M_{\text{virtual}} = 2p_K \cdot \gamma \tfrac{1}{2} A - p_l \cdot \gamma (1-\xi) \tfrac{1}{2} B , \quad (6)$$

where

$$A = (\alpha/\pi)[\tfrac{3}{2} \ln(\Lambda/m_l) - 1 + 2\ln(m_l/\lambda) + t_1 - \tfrac{1}{2} t_2 m_l^2 (1-\xi)/h^2] , \quad (7)$$

$$B = (\alpha/\pi)[-\tfrac{3}{2} \ln(\Lambda/m_l) - (7/4) + 2\ln(m_l/\lambda) + t_1 - 2t_2/(1-\xi)] , \quad (8)$$

and

$$h^2 = (p_l + p_\pi)^2 = m_K(m_K - 2E_\nu) . \quad (9)$$

In Eqs. (7) and (8), $\Lambda$ is the ultraviolet cutoff and $\lambda$ is the "fictitious" photon mass (infrared cutoff). It follows from Eq. (6) that in the interference term of the matrix element, $A$ and $B$ occur only in the combination $\tfrac{1}{2}(A^* + B)(1-\xi)$. Therefore, to first order in $\alpha$, only the

[5] Present experiments are consistent with constant form factors; S. H. Aronson and K. Wendell Chen, Phys. Rev. Letters **20**, 287 (1968).

[6] Some recent remarks on the logarithmic divergence in $\beta$ decay are to be found in Julian Schwinger, Phys. Rev. Letters **19**, 1501 (1967).

real part of $t_1$ contributes. It is found that

$$\mathrm{Re}(t_1)=\frac{a}{\Delta}\left[\tfrac{1}{3}\pi^2+2\,\mathrm{Li}_2\left(\frac{a+2m_\pi^2-\Delta}{a+2m_\pi^2+\Delta}\right)+2\,\mathrm{Li}_2\left(\frac{a+2m_l^2-\Delta}{a+2m_l^2+\Delta}\right)+2\left(1-\ln\frac{\Delta}{\lambda^2}+\ln\frac{a+m_\pi^2+m_l^2}{\Delta}\right)\ln\frac{a+\Delta}{2m_lm_\pi}\right.$$
$$\left.+\tfrac{1}{2}\left(\ln\frac{a+2m_\pi^2-\Delta}{a+2m_\pi^2+\Delta}\right)^2+\tfrac{1}{2}\left(\ln\frac{a+2m_l^2-\Delta}{a+2m_l^2+\Delta}\right)^2\right],\quad(10)$$

$$t_2=\ln\frac{m_l}{m_\pi}+\left(\frac{a+2m_\pi^2}{\Delta}\right)\left(i\pi-\ln\frac{a+\Delta}{2m_lm_\pi}\right),\quad(11)$$

where

$$a=m_K^2-H^2-q^2\quad\text{and}\quad\Delta=(a^2-4m_l^2m_\pi^2)^{1/2},\quad(12)$$

and $\mathrm{Li}_2(Z)$ is the dilogarithm function.

As mentioned above, the imaginary part of $t_2$ would contribute to the transverse lepton polarization (i.e., normal to the decay plane) even if the weak interaction were invariant under time reversal (i.e., $\xi$ real).[4]

The contribution of the virtual diagrams to the Dalitz plot is

$$\Gamma_v(E_l,E_\pi)=(2\pi)^{-3}|f_+|^2(2E_K)^{-1}\{[(m_K^2+m_l^2-H^2)(H^2-m_\pi^2+q^2-m_l^2)-m_K^2(q^2-m_l^2)]\,\mathrm{Re}A$$
$$-\tfrac{1}{2}m_l^2\,\mathrm{Re}[(1-\xi)(A^*+B)](H^2-m_\pi^2+q^2-m_l^2)+\tfrac{1}{4}m_l^2\,\mathrm{Re}B|1-\xi|^2(q^2-m_l^2)\}.\quad(13)$$

The first-order matrix element for inner bremsstrahlung [Fig. 1(g)–1(i)] can be written

$$\mathfrak{M}_{\mathrm{IB}}=-i(2\pi)^{-3}(m_l\alpha)^{1/2}(4E_KE_\pi E_lE_\gamma)^{-1/2}\times\delta^{(4)}(p_K-p_\pi-p_l-p_\nu-k)\times f_+\bar{u}_\nu\tfrac{1}{2}(1+i\gamma_5)M_{\mathrm{IB}}v_l,\quad(14)$$

where

$$M_{\mathrm{IB}}=[2p_k\cdot\gamma+m_l(1-\xi)]\left(\frac{p_l\cdot\epsilon}{p_l\cdot k}-\frac{p_\pi\cdot\epsilon}{p_\pi\cdot k}+\frac{k\cdot\gamma\epsilon\cdot\gamma}{2p_l\cdot k}\right)\quad(15)$$

and $k$, $E_\gamma$, and $\epsilon$ are the photon four momentum, energy, and polarization, respectively.

It is assumed that the experimental apparatus will detect the lepton and pion in the final state and that their observed momenta are fit to three-body kinematics with zero missing mass. The kinematics of the three-body ($\pi l\nu$) and four-body ($\pi l\nu\gamma$) final states is discussed in Ref. 2. Under these conditions the contribution of the inner bremsstrahlung to the Dalitz plot is

$$\Gamma_{\mathrm{IB}}(E_l,E_\pi)=(\alpha/\pi)\left\{\Gamma_0(E_l,E_\pi)I_0(E_l,E_\pi)+(2\pi)^{-3}|f_+|^2m_K^{-1}\tfrac{1}{4}\int_0^{x_{\max}}dx\times\sum c_{m,n}(p_i,p_j)I_{m,n}(p_i,p_j)\right\},\quad(16)$$

where

$$x=(p_K-p_l-p_\pi)^2=(p_\nu+k)^2\quad(17)$$

is the invariant mass squared of the undetected particles, and

$$x_{\max}=(E_\nu-p_\pi+p_l)(E_\nu+p_\pi-p_l).\quad(18)$$

As in Ref. 2, the first term contains the infrared divergence; it is evaluated with $k^2=\lambda^2$ and then the limit $\lambda\to0$ is taken. The remainder of the terms in Eq. (16), the so called "real" inner bremsstrahlung, may be evaluated with $k^2=0$. The invariant integrals are defined[2] by

$$I_{m,n}(p_i,p_j)=\frac{1}{2\pi}\int\frac{d^3p_\nu d^3k}{E_\nu E_\gamma}\frac{\delta^{(4)}(p_K-p_\pi-p_l-p_\nu-k)}{(p_i\cdot k)^m(p_j\cdot k)^n},\quad(19)$$

where $i$ and $j$ run over $K$, $l$, and $\pi$. The infrared-divergent integral is

$$I_0(E_l,E_\pi)=\lim_{\lambda\to0}\tfrac{1}{4}\int_{\lambda^2}^{x_{\max}}dx[2p_l\cdot p_\pi I_{1,1}(p_l,p_\pi)-m_l^2I_{2,0}(p_l,p_\pi)-m_\pi^2I_{0,2}(p_l,p_\pi)].\quad(20)$$

In Eq. (16) the coefficients $c_{m,n}(p_i,p_j)$, which are different from zero, are

$$\begin{aligned}
c_{1,0}(p_l,p_\pi)&=p_K\cdot p_l(H^2-m_l^2+q^2-m_\pi^2)+(m_K^2-H^2-q^2+x)c'-\tfrac{1}{2}(q^2+m_l^2)c'',\\
c_{0,1}(p_l,p_\pi)&=-p_K\cdot p_\pi(H^2-m_l^2+q^2-m_\pi^2)+(m_K^2-H^2-q^2+x)c'+\tfrac{1}{2}(2m_K^2-H^2-2q^2+m_\pi^2+x)c'',\\
c_{1,1}(p_l,p_\pi)&=\tfrac{1}{2}x(m_K^2-H^2-q^2+x)(c'+\tfrac{1}{2}c''),\\
c_{1,-1}(p_l,p_K)&=3H^2+2q^2-2m_K^2-2m_l^2-m_\pi^2-x,\\
c_{1,-1}(p_\pi,p_K)&=H^2+m_\pi^2-x,\\
c_{2,-1}(p_l,p_K)&=-\tfrac{1}{2}m_l^2(H^2-m_l^2+q^2-m_\pi^2+x+2c'),\\
c_{2,-1}(p_\pi,p_K)&=m_\pi^2(m_K^2-H^2+m_l^2\,\mathrm{Re}\xi-c''),\\
c_{1,-2}(p_l,p_K)&=-2,\\
c_{2,-2}(p_l,p_K)&=2m_l^2,
\end{aligned}\quad(21)$$

where

$$\begin{aligned}
c'&=m_K^2-3p_K\cdot p_l-p_K\cdot p_\pi+m_l^2\,\mathrm{Re}(1-\xi)-\tfrac{1}{2}x,\\
c''&=m_K^2-\tfrac{1}{4}m_l^2|1-\xi|^2.
\end{aligned}\quad(22)$$

The right-hand side of Eq. (20) can be evaluated by noting that terms such as $\lambda^2 x$, $\lambda^4$, etc., do not contribute in the limit $\lambda \to 0$. After several judicious changes in variables and a certain amount of brute force, the result is

$$I_0(E_l,E_\pi)=\frac{a}{\Delta}\left\{\left(2\ln\frac{w_{\max}}{w_{\min}}+\ln\frac{\Delta}{\lambda^2}\right)\ln\frac{a+\Delta}{2m_l m_\pi}-\ln\frac{w_{\max}+2\Delta}{2\Delta}\ln\frac{(w_{\max}+2\Delta)\Delta}{2m_l^2 m_\pi^2}-\ln\frac{a+\Delta}{2m_l m_\pi}\ln\frac{w_0}{2m_l m_\pi}\right.$$
$$\left.+2\,\mathrm{Li}_2\left(-\frac{a-\Delta}{\Delta}\right)-2\,\mathrm{Li}_2\left(-\frac{w_{\max}}{a+\Delta}\right)-2\,\mathrm{Li}_2\left(-\frac{a-\Delta}{w_{\max}+2\Delta}\right)+\tfrac{1}{2}\,\mathrm{Li}_2\left(-\frac{a+\Delta}{w_0}\right)-\tfrac{1}{2}\,\mathrm{Li}_2\left(-\frac{a-\Delta}{w_0}\right)\right\}$$
$$+\ln\frac{(H^2-m_\pi^2)(q^2-m_l^2)}{x_{\max}^2}+\left(\ln\frac{a+\Delta+w_{\max}}{2m_l m_\pi}\right)^2-\left(\ln\frac{a+\Delta}{2m_l m_\pi}\right)^2-\ln\frac{m_l m_\pi}{\lambda}, \quad (23)$$

where

$$w_{\max}=x_{\max}-\Delta+[(a+x_{\max})^2-4m_l^2 m_\pi^2]^{1/2},$$
$$w_{\min}=\Delta^{-3/2}(a+\Delta)[a(H^2-m_\pi^2)(q^2-m_l^2)-m_\pi^2(q^2-m_l^2)^2-m_l^2(H^2-m_\pi^2)^2]^{1/2}, \quad (24)$$
$$w_0=\frac{am_\pi^2(q^2-m_l^2)^2+am_l^2(H^2-m_\pi^2)^2-4m_l^2m_\pi^2(H^2-m_\pi^2)(q^2-m_l^2)}{a(H^2-m_\pi^2)(q^2-m_l^2)-m_\pi^2(q^2-m_l^2)^2-m_l^2(H^2-m_\pi^2)^2},$$

and $a$ and $\Delta$ are given by Eq. (12). It is easily verified that the infrared divergent terms in $\Gamma_{\mathrm{IB}}(E_l,E_\pi)$ exactly cancel the corresponding terms in $\Gamma_v(E_l,E_\pi)$, as required. The remaining invariant integrals $I_{m,n}$ are given in the Appendix of Ref. 2 for the case $k^2=\lambda^2=0$.

The evaluation of the "real" inner-bremsstrahlung contribution to $\Gamma_{\mathrm{IB}}$ by analytic means does not appear to be feasible. A numerical integration is always possible, however. In the limit of vanishing lepton mass, useful analytic results can be obtained. In the remainder of this paper the discussion is restricted to $K_{e3}^0$ modes, for which $m_l \to 0$ is a good approximation, and the subscript $l$ will be omitted from the lepton variables.

The total radiative correction to the $K_{e3}^0$ Dalitz plot is obtained by combining Eqs. (13) and (16):

$$\Gamma_{\mathrm{RC}}(E,E_\pi)=\Gamma_v(E,E_\pi)+\Gamma_{\mathrm{IB}}(E,E_\pi)$$
$$=(\alpha/\pi)\left\{\Gamma_0(E,E_\pi)T_0+\frac{|f_+|^2}{4(2\pi)^3}\sum_{i=1}^{8}T_i\right\}, \quad (25)$$

where

$$T_0=\tfrac{3}{2}\ln(\Lambda/m)+\tfrac{1}{2}\pi^2-1-\mathrm{Li}_2(\eta)+\ln(\eta-1)+\ln[(H^2-m_\pi^2)/q^2]+\tfrac{1}{2}\,\mathrm{Li}_2[m_K^2m_\pi^{-2}(1-4EE_\nu/q^2)]$$
$$-2\,\mathrm{Li}_2[m_\pi^{-2}x_{\max}(1-\eta)^{-1}]+2\{-1+\ln[m_\pi m^{-1}(\eta-1)]\}\ln(x_{\max}/q^2),$$
$$T_1=4E(E_\pi+p_\pi)(m_K-4E+2W-m_K^{-1}E(E_\pi+p_\pi))(L_1+L_3),$$
$$T_2=2(m_Km_\pi^2+2H^2E_\pi)\{\mathrm{Li}_2[m_K(E_\pi+p_\pi)^{-1}]-\pi^2/6+\mathrm{Li}_2[m_\pi^{-2}(m_K-2E)(E_\pi+p_\pi)]-\mathrm{Li}_2(H^2m_\pi^{-2})\},$$
$$T_3=[(H^2-m_\pi^2)(4E_\pi-3E+2E_\nu+W)-2m_\pi^2E_\nu]\ln(m_\pi^2x_{\max}(H^2-m_\pi^2)^{-2}), \quad (26)$$
$$T_4=2(2L_3-L_2)[m_K^{-1}\alpha_\pi\beta_\pi+\beta_\pi(4E_\nu+3E_\pi-2E)+\alpha_\pi\beta_\pi^{-1}(E_\pi(2x_{\max}+\alpha_\pi)-E_\nu(2\alpha_\pi+m_\pi^2))],$$
$$T_5=[4m_KE(E-E_\nu)+q^2(m_K-2W+2E)][\mathrm{Li}_2(\zeta)-\pi^2/6+(\ln\zeta)^2+\ln\zeta\,\ln(q^2/m^2)],$$
$$T_6=[2m_K(E^2-E_\nu^2+4EE_\nu)-q^2(m_K+8E-4W-q^2m_K^{-1})]\ln\zeta,$$
$$T_7=(2L_1+L_2)[\tfrac{1}{2}m_K(E_\nu-E+p_\pi)(3E_\nu-3E-p_\pi)+2x_{\max}(2E-E_\nu+\tfrac{1}{4}m_K^{-1}x_{\max})],$$
$$T_8=3m_K(E_\nu-E+p_\pi)E,$$

and

$$\zeta=2E(E_\nu+E+p_\pi)^{-1}, \quad \eta=m_\pi^{-2}(m_K^2-2m_KE_\nu), \quad (27)$$
$$\alpha_\pi=E_\pi E_\nu-p_\pi(E-p_\pi), \quad \beta_\pi=p_\pi E_\nu-E_\pi(E-p_\pi),$$
$$L_1=\ln(2E/m), \quad L_2=\ln\frac{E_\nu+E-p_\pi}{E_\nu-E+p_\pi}, \quad L_3=\ln\frac{E_\pi+p_\pi}{m_\pi}.$$

To the same approximation, namely $m_l \to 0$, the zero-order contribution to the Dalitz plot given by Eq. (5) can be written[2]

$$\Gamma_0(E,E_\pi)=(2\pi)^{-3}|f_+|^2 2m_KE(W-E)X_\pi, \quad (E\gg m) \quad (28)$$
$$=(2\pi)^{-3}|f_+|^2 2m_Kp_\pi^2X(1-X), \quad (q^2\gg m^2) \quad (29)$$

where

$$X=(E-E^{\min})/(E^{\max}-E^{\min}), \quad (30)$$
$$X_\pi=(E_\pi-E_\pi^{\min})/(E_\pi^{\max}-E_\pi^{\min}) \quad (31)$$

and the reader is referred to Ref. 2 for the relevant kinematics.

## III. RESULTS

The radiative corrections to the $K_{e3}^0$ Dalitz plot given by Eqs. (25)–(27) have been evaluated on a computer. The corrections are appreciable, averaging greater than 3% in absolute magnitude over a good portion of the Dalitz plot. A sampling of these results is shown in Figs. 2–4. The fractional radiative correction to the Dalitz plot is

$$\Delta_{\mathrm{RC}}(E,E_\pi)=\Gamma_{\mathrm{RC}}(E,E_\pi)/\Gamma_0(E,E_\pi). \qquad (32)$$

In terms of $\Delta_{\mathrm{RC}}$, the experimentally measured Dalitz plot and the zero-order Dalitz plot are related by

$$\Gamma_0(E,E_\pi)=\Gamma_{\mathrm{expt}}(E,E_\pi)[1+\Delta_{\mathrm{RC}}(E,E_\pi)]^{-1}. \qquad (33)$$

In Fig. 2 the fractional correction in percent has been plotted, for various lepton energies, as a function of $X_\pi$ (pion energy normalized to unit interval). Figure 3 shows a similar plot of $\Delta_{\mathrm{RC}}$ versus $X$ for various pion energies. A more comprehensive graphical view of the radiative corrections is obtained by marking discrete values of $\Delta_{\mathrm{RC}}$ directly within the boundaries of the Dalitz plot, as in Fig. 4.

In the numerical computation described above, the ultraviolet cutoff $\Lambda$ has been taken equal to the proton mass. The effect of varying $\Lambda$ is evident from Eq. (7)

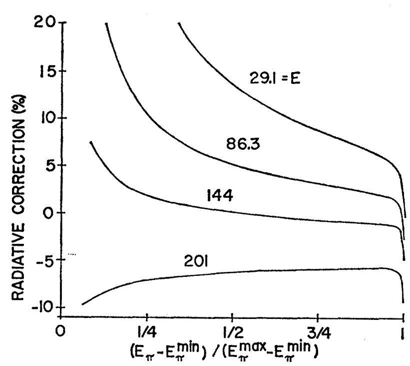


FIG. 2. Fractional radiative correction to the $K_{e3}^0$ Dalitz plot in percent as a function of $x_\pi$. Curves are labeled by positron energy in MeV.

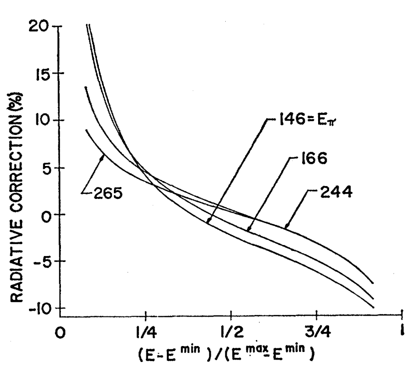


FIG. 3. Fractional radiative correction to the $K_{e3}^0$ Dalitz plot in percent as a function of $x$. Curves are labeled by pion energy in MeV.

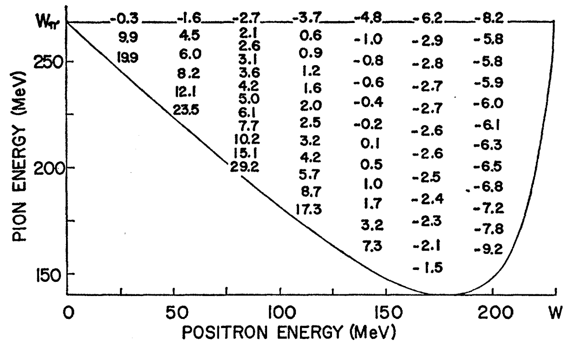


FIG. 4. Fractional radiative correction in percent at various points in the $K_{e3}^0$ Dalitz plot (indicated by the corresponding decimal points).

and has been noted previously.[1,2] Doubling the cutoff adds only 0.24% to the computed values of $\Delta_{\mathrm{RC}}$.

Equation (25) can be integrated numerically to give the radiative corrections to either the pion or positron spectrum.[7] Figures 5 and 6 show the results of this calculation. The corresponding spectra for $K_{e3}^+$ are also plotted for comparison. (The zero-order spectra differ slightly for $K_{e3}^0$ and $K_{e3}^+$ due to the difference in phase space, but this is not shown in the figures.) The radiative corrections to $K_{e3}^0$ are noticeably different from $K_{e3}^+$, being everywhere more positive than the $K_{e3}^+$ corrections. The radiative corrections to the positron spectrum are large enough to be relatively insensitive to reasonable variations in the cutoff, but the corrections to the pion spectrum are an order of magnitude smaller and are therefore less certain.

A final integration over the positron or pion spectrum gives the radiative corrections to the decay rate $\Gamma$. The present calculation results in a fractional change in lifetime[1] of

$$(\Delta\tau/\tau)_{K_{e3}^0}=-\Gamma_{\mathrm{RC}}/\Gamma_0 \simeq -0.80\%. \qquad (34)$$

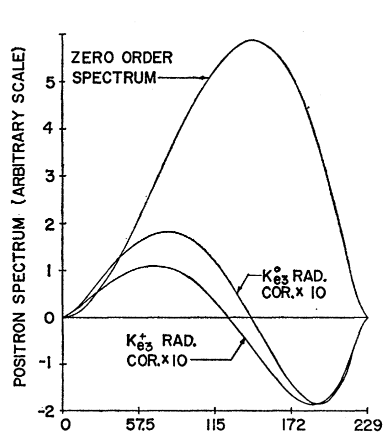


FIG. 5. Zero-order positron spectrum and radiative corrections for $\Lambda = m_p$; $K_{e3}^0$—present calculation; $K_{e3}^+$—Ref. 2.

[7] Expressions for the zero-order spectra are given in footnote 26 of Ref. 2.

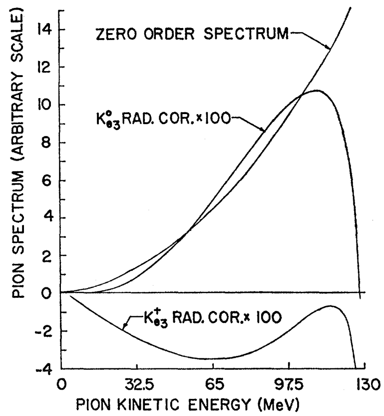


FIG. 6. Zero-order pion spectrum and radiative corrections for $\Lambda = m_p$; $K_{e3}{}^0$—present calculation; $K_{e3}{}^+$—Ref. 2.

The corresponding number for $K_{e3}{}^+$ (recalculated[8] with slightly greater numerical precision than in Ref. 2) is

$$(\Delta\tau/\tau)_{K_{e3}{}^+} \simeq +0.45\%. \tag{35}$$

These numbers are sensitive to variations in the cutoff, but the divergent term is of the same form for both; namely, $-(3\alpha/2\pi)\ln(\Lambda/m_p)$. This implies that, to first order in $\alpha$, the radiative correction to the ratio of the decay rates for $K_{e3}{}^0$ and $K_{e3}{}^+$ is independent of the cutoff, provided that the same value of $\Lambda$ is used in both corrections. Using a notation analogous to that in Eq. (33),

$$\Gamma_{\text{expt}} = \Gamma_0 + \Gamma_{\text{RC}} = \Gamma_0(1 - \Delta\tau/\tau), \tag{36}$$

it is found that

$$\frac{\Gamma_{\text{expt}}(K_{e3}{}^0)}{\Gamma_{\text{expt}}(K_{e3}{}^+)} = \frac{\Gamma_0(K_{e3}{}^0)}{\Gamma_0(K_{e3}{}^+)}(1+\delta), \tag{37}$$

where

$$\delta = (\Delta\tau/\tau)_{K_{e3}{}^+} - (\Delta\tau/\tau)_{K_{e3}{}^0} \simeq 1\tfrac{1}{4}\%. \tag{38}$$

It is plausible to assume that the $\Delta S = \Delta Q$ and $\Delta S = -\Delta Q$ amplitudes have the same general Lorentz covariant form as in (i) above. Then the *fractional* radiative corrections to the decays $K^0 \to \pi^- e^+ \nu$ and $\bar{K}^0 \to \pi^- e^+ \nu$ are identical. (Of course, the values of $\Gamma_{\text{RC}}$ may be quite different since there is no reason to suppose that the form factors for $\Delta S = \Delta Q$ and $\Delta S = -\Delta Q$ are identical.) Using the *TCP* theorem[9] and the standard notation[10] for the long- and short-lived kaon states, the decay rate for $K_L$ into both charged modes $\pi^\mp e^\pm \nu$ is easily related to the $K_{e3}{}^0$ decay rate:

$$\frac{\Gamma_{\text{expt}}(K_L \to \pi^\mp e^\pm \nu)}{\Gamma_{\text{expt}}(K^0 \to \pi^- e^+ \nu)} = 1 - 4\,\text{Re}\left(\frac{pq^*\chi^*}{|p|^2 + |q|^2}\right) + |\chi|^2, \tag{39}$$

where $\chi$ is the ratio of the $\Delta S = -\Delta Q$ and $\Delta S = \Delta Q$ amplitudes. To lowest order in the *CP*-violating parameter $\epsilon$, the $\Delta S = -\Delta Q$ factor in Eq. (39) reduces to $|1-\chi|^2$. Using Eq. (37), one finds

$$\frac{\Gamma_{\text{expt}}(K_L \to \pi^\mp e^\pm \nu)}{\Gamma_{\text{expt}}(K^+ \to \pi^0 e^+ \nu)} = |1-\chi|^2(1+\delta)\frac{\Gamma_0(K^0 \to \pi^- e^+ \nu)}{\Gamma_0(K^+ \to \pi^0 e^+ \nu)}. \tag{40}$$

Equation (40) shows that the combination of $K_{e3}$ decay rates which is experimentally measurable contains a factor of $(1+\delta)$ due to radiative corrections, where, on the basis of the model assumed here, $\delta$ is given by Eq. (38).

It is not the purpose of the present paper to discuss the application of Eq. (40) to existing experimental data, but a few brief remarks are in order. According to the compilation of Rosenfeld *et al.*[11] the left-hand side of Eq. (40) is $1.73 \pm 0.10$. The factor on the right-hand side involving the zero-order rates can be expressed in terms of the $\Delta I = \frac{1}{2}$ and $\frac{3}{2}$ amplitudes:

$$\frac{\Gamma_0(K^0 \to \pi^- e^+ \nu)}{\Gamma_0(K^+ \to \pi^0 e^+ \nu)} = \left|\frac{2^{1/2}A(\frac{1}{2}) + A(\frac{3}{2})}{A(\frac{1}{2}) - 2^{1/2}A(\frac{3}{2})}\right|^2. \tag{41}$$

There is good reason to believe that the $\Delta S = -\Delta Q$ amplitude and the $\Delta I = \frac{3}{2}$ amplitude are both small compared to the $\Delta S = \Delta Q$ and $\Delta I = \frac{1}{2}$ amplitudes, respectively. In that case the right-hand side of Eq. (40) may be expanded to first order in small quantities with the result that

$$1.73 \pm 0.10 = 2[1 - 2\,\text{Re}\chi + 3\sqrt{2}\,\text{Re}(A(\tfrac{3}{2})/A(\tfrac{1}{2})) + \delta]. \tag{42}$$

It is clear that Eq. (42) does not furnish a value of both $\text{Re}\chi$ and $\text{Re}(A(\frac{3}{2})/A(\frac{1}{2}))$ separately; however, the following crude estimates can be made. If the $\Delta S = -\Delta Q$ amplitude is neglected, then

$$\text{Re}(A(\tfrac{3}{2})/A(\tfrac{1}{2})) \simeq -0.03 \pm 0.01, \quad (\text{no } \Delta S = -\Delta Q) \tag{43}$$

and if the $\Delta I = \frac{3}{2}$ amplitude is neglected[12]

$$\text{Re}\chi \simeq 0.07 \pm 0.03, \quad (\text{no } \Delta I = \tfrac{3}{2}). \tag{44}$$

A further application of Eq. (40) to the existing experimental data is discussed elsewhere.[13]

## ACKNOWLEDGMENTS

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[8] In Ref. 2, the sign of $\Delta\tau/\tau$ should be positive.

[9] T. D. Lee, R. Oehme, and C. N. Yang, Phys. Rev. **106**, 340 (1957).

[10] T. T. Wu and C. N. Yang, Phys. Rev. Letters **13**, 380 (1964).

[11] A. H. Rosenfeld, N. Barash-Schmidt, A. Barbaro-Galtieri, W. J. Podolsky, L. R. Price, M. Roos, P. Soding, W. Willis, and C. Wohl, Rev. Mod. Phys. **40**, 77 (1968).

[12] Note that the $\Delta I = \frac{1}{2}$ rule does not imply $\Delta S = \Delta Q$ unless it is further assumed that $|\Delta S| < 2$.

[13] E. S. Ginsberg (to be published).