Incoherent-Scattering Function, Total Pair-Production Cross Section, and Pair-Production Length for Helium*

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A formula for the incoherent-scattering function using either the correlated or the uncorrelated wave-function model of atomic helium is presented. With this the total triplet-production cross section is calculated, and thus the total pair-production cross section is evaluated (triplet+coherent pair). The pair-production length of helium is calculated to be 963.64±1.5 cm for saturated liquid helium at 4.2°K. The pair-production length is defined as $l_{He}=A/N_0\sigma_{He}\rho_{He}$. The systematic uncertainty of about ±0.15% comes from the choice of screening functions. This is a factor of 10 more accurate than the known pair-production lengths of any other element. It is suggested that this increased accuracy for the pair-production length of helium may find application as a standard in experimental measurements of absolute cross sections for high-photon-energy induced reactions.

THE prediction of accurate total cross sections for electron-positron pair production by energetic photons is hampered by uncertainties in the screening correction. These originate in the choice of the proper atomic or molecular model for the system in whose electric field the pair is produced.

While the simplest atomic system is that of hydrogen, the molecular corrections for this element are not understood sufficiently well to permit an accurate prediction of the coherent and incoherent form factors. The next more complex atomic system, that of helium, can be represented by a combination of spherically symmetric (hydrogenlike) wave functions. Furthermore, helium exists in an atomic state, thus eliminating any molecular corrections. At present, several accurate wave functions of not excessive complexity exist for helium. These are, in the order of increasing accuracy of prediction of the ionization energy (shown in parentheses), the Thomas-Fermi-Molière model (+33%), the uncorrelated model (−1.96%), the Hartree-Fock-Slater model (−1.48%), and the correlated model (−0.94%). We therefore expect that the predictions of other physical quantities based on these models will be reliable in the same proportion as the stated accuracies for the ionization energy. To be able to predict the total pair-production cross section, the wave functions must be Fourier-transformed over the spatial variables to form coherent- and incoherent-scattering functions.

The first two models chosen for this procedure are the radially correlated and the uncorrelated models of Shull and Löwdin[1] which, for the ground state of atomic helium, are given by

$$\psi_0=N\{\exp[-\eta(1+\nu)r_1]\exp[-\eta(1-\nu)r_2]+\exp[-\eta(1+\nu)r_2]\exp[-\eta(1-\nu)r_1]\}, \quad (1)$$

where N is the normalization factor, $\eta=1.6875$, $\nu=0$ for the uncorrelated model, and $\eta=1.6859$, $\nu=0.295$ for the correlated model. If we rewrite (1) as

$$\psi_0/N=a(1)b(2)+b(1)a(2), \quad (2)$$

with

$$a(i)=(A^3/\pi)^{1/2}\exp(-Ar_i),$$
$$b(i)=(A^3/\pi)^{1/2}\exp(-Br_i),$$

we can make the identification

$$A=\eta(1+\nu), \quad B=\eta(1-\nu). \quad (3)$$

The coherent- and incoherent-scattering functions needed in the calculation of the screening to pair production are given by the well-known formulas[2]

$$ZF(q)=\frac{1}{C}\int\psi_0^*\Big(\sum_{i=1}^{2}e^{iq\cdot r_i}\Big)\psi_0 d\tau, \quad (4)$$

$$ZS(q)=\frac{1}{C}\sum_{i=1}^{2}\sum_{j=1}^{2}\int\psi_0^*e^{iq\cdot(r_i-r_j)}\psi_0 d\tau-|ZF(q)|^2, \quad (5)$$

with

$$C=\int\psi_0^*\psi_0 d\tau,$$

for the atomic charge Z times the coherent- and incoherent-scattering functions, respectively. Using (2) in (4) and (5), one obtains

$$ZF(q)=\frac{1}{C}\left[\frac{16A^4}{(4A^2+q^2)^2}+\frac{128(AB)^3}{[(A+B)^2+q^2]^2(A+B)^2}+\frac{16B^4}{(4B^2+q^2)}\right], \quad (6)$$

$$C=[1+64(AB)^3/(A+B)^6],$$

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[1] H. Shull and P. Löwdin, J. Chem. Phys. **25**, 1035 (1956).

[2] F. Rohrlich, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (John Wiley & Sons, Inc., New York, 1959), Vol. I.

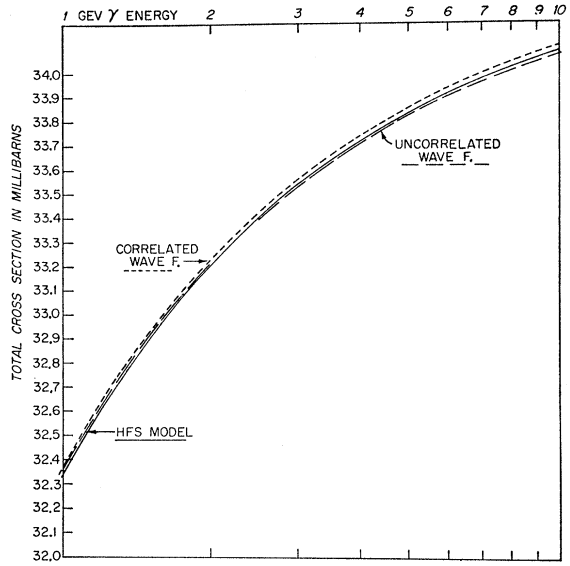

FIG. 1. Coherent pair-production total cross section versus the energy of the photon producing the pair, in helium, under the assumption of three different atomic models.

and

$$ZS(q)=2+\frac{2}{C}\left[\frac{256(AB)^4}{(4A^2+q^2)^2(4B^2+q^2)^2}+\frac{64(AB)^3(A+B)^2}{[(A+B)^2+q^2]^4}\right]-|ZF(q)|^2. \quad (7)$$

Formula (6) has already been obtained by Hurst *et al.*,[3] but they have a typographical error in their version of formula (6). Their tables of $F(q)$ are, however, correct. Table I presents the values of $ZS(q)$ as a function of $\sin\theta/\lambda$ in units of $4\pi/\text{Å}$. Conversion to q in keV/c units is accomplished by noting that 1.973 keV/$c=10^8$ cm$^{-1}=1/\text{Å}$. Accurate tables of both $F(q)$ and $S(q)$ for q in keV/c units may be obtained by writing the author.

The total pair production from helium was evaluated from a formula given by Bethe[4] and subsequently justified by Suh and Bethe[5] regarding the use of the incoherent-scattering function.

The Bethe formula for coherent production is derived in the Born approximation with errors of the order of $1/k^2$. Suh and Bethe find that the use of the Bethe formula for incoherent production is justified at high energies where the error will be of the order $\ln k/k$, and then give references to experimental work which confirms their theoretical findings above about $k=50$ MeV. We therefore expect that above about 1000 MeV the Bethe formula for coherent and incoherent production will be reliable to several tenths of a percent, and above 10 GeV to 0.1%, based on the stated theoretical

[3] R. P. Hurst, J. Miller, and F. A. Matsen, Acta Cryst. **11**, 320 (1958).
[4] H. A. Bethe, Proc. Cambridge Phil. Soc. **30**, 524 (1934).
[5] K. S. Suh and H. A. Bethe, Phys. Rev. **115**, 672 (1959).

TABLE I. Incoherent-scattering function S as a function of $q\equiv(\sin\theta)/\lambda$ (in units of $4\pi\text{Å}^{-1}$) for the correlated and uncorrelated models of helium. $\eta=1.6859$, $\nu=0.295$ for the correlated model; $\eta=1.6875$, $\nu=0.0$ for the uncorrelated model.

q	ZS (correlated model)	ZS (uncorrelated model)
0.00	0.0	0.0
0.01	0.00364402	0.00310263
0.02	0.0145157	0.0123746
0.03	0.0324371	0.0277084
0.04	0.0571194	0.0489283
0.05	0.0881725	0.0757934
0.06	0.125120	0.108004
0.07	0.167413	0.145206
0.08	0.214452	0.187004
0.09	0.265602	0.232961
0.10	0.320208	0.282615
0.11	0.377619	0.335483
0.12	0.437194	0.391074
0.13	0.498323	0.448892
0.14	0.560431	0.508449
0.15	0.622989	0.569268
0.16	0.685517	0.630893
0.17	0.747589	0.692892
0.18	0.808833	0.754862
0.19	0.868933	0.816431
0.20	0.927619	0.877265
0.21	0.984675	0.937064
0.22	1.03993	0.995564
0.23	1.09324	1.05254
0.24	1.14453	1.10780
0.25	1.19371	1.16119
0.26	1.24076	1.21258
0.27	1.28567	1.26189
0.28	1.32843	1.30905
0.29	1.36908	1.35401
0.30	1.40765	1.39677
0.31	1.44420	1.43732
0.32	1.47877	1.47568
0.33	1.51143	1.51190
0.34	1.54225	1.54602
0.35	1.57131	1.57810
0.36	1.59867	1.60820
0.37	1.62442	1.63641
0.38	1.64862	1.66279
0.39	1.67135	1.68744
0.40	1.69269	1.71043
0.41	1.71271	1.73185
0.42	1.73148	1.75178
0.43	1.74907	1.77031
0.44	1.76554	1.78752
0.45	1.78095	1.80349
0.46	1.79538	1.81830
0.47	1.80887	1.83202
0.48	1.82149	1.84472
0.49	1.83328	1.85648
0.50	1.84429	1.86735
0.51	1.85458	1.87740
0.52	1.86420	1.88669
0.53	1.87317	1.89527
0.54	1.88155	1.90320
0.55	1.88936	1.91052
0.60	1.92128	1.93947
0.70	1.95973	1.97181
0.80	1.97898	1.98643
0.90	1.98875	1.99322
1.00	1.99381	1.99648

assumptions.[6] The formula which we shall use is, then,

$$\sigma(k)=\frac{4Z^2\alpha}{k^3}r_0^2\int_\mu^{k-\mu}dE\Bigg[[E^2+(k-E)^2]\times\left(\int_\delta^\mu(q-\delta)^2\{[1-F(q)]^2+S(q)\}\frac{dq}{q^3}+1\right)+\tfrac{2}{3}E(k-E)\left(\int_\delta^\mu\left[q^3-6\delta^2q\ln\frac{q}{\delta}+3\delta^2q-4\delta^3\right]\times\{[1-F(q)]^2+S(q)\}\frac{dq}{q^4}+\frac{5}{6}\right)\Bigg],\quad(8)$$

where $\alpha=1/137$, $\mu=m_ec^2$, k is the photon energy, E is the energy of one member of the pair, q is the momentum transfer, δ is the minimum momentum transfer, given approximately by $\mu^2k/2E(k-E)$, and r_0 is the classical electron radius $(=e^2/\mu)$. This formula was evaluated by applying the definition of an integral as a limit of a sum in order to evaluate (8) numerically. $F(q)$ and $S(q)$ are as given previously. The integrals in (8) were evaluated for an increasing number of steps until convergence of the sum was better than one part in 10^3. Figures 1–3 present the coherent contribution to σ [due to the term $(1-F(q))^2$], the incoherent contribution to σ [due to $S(q)$], and the total cross section, respectively, for the two wave functions discussed.

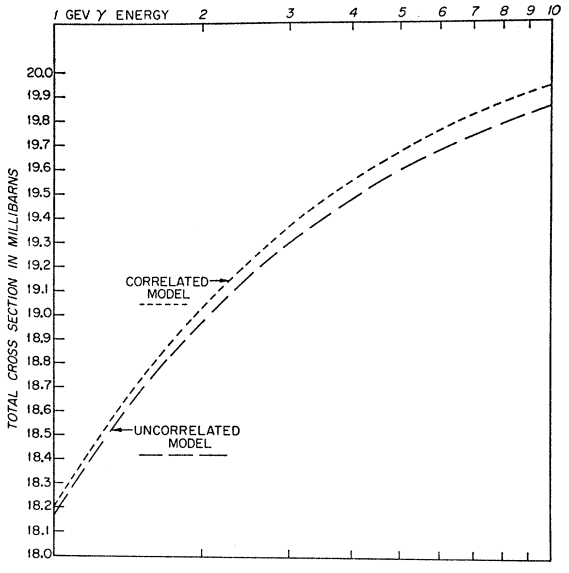

FIG. 2. Incoherent pair-production total cross section as in Fig. 1, for two different atomic models.

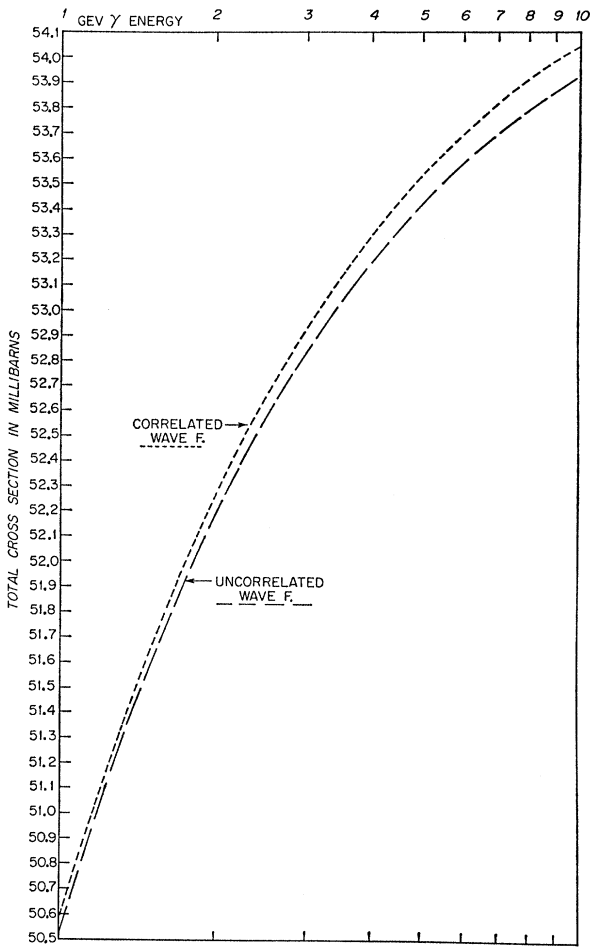

FIG. 3. Sum of coherent and incoherent pair production as in Fig. 2.

The Hartree-Fock-Slater model wave function, of -1.48% binding-energy error, has already been evaluated for $F(q)$,[7] the coherent-scattering function for helium. This was obtained as a table of values, read into a program to integrate (8). This table was interpolated as needed in the evaluation of σ. Since the HFS (Hartree-Fock-Slater) $F(q)$ values are only available to $\sin\theta/\lambda=2.00$, the correlated wave-function model was used beyond this point in the table. Comparison of the overlap of the two functions shows that the error so introduced is much smaller than one part in 10^3 in σ. Care was taken to assure that the convergence of the numerical evaluation was to better than one part per 1000, and different step sizes in the table of F were utilized to assure that this feature introduced no additional error. These results are also presented in

[6] Comparison of the Bethe formula given here and the relativistic covariant calculation of the same process by R. Jost, J. M. Luttinger, and M. Slotnick [Phys. Rev. 80, 189 (1950)] shows agreement to 0.4% above 1-GeV photon energy and agreement to 0.1% above 10 GeV. This error is somewhat larger than would be anticipated from the approximations used in the derivation of the Bethe formula. A thorough study of both formulas will be published shortly. This study will show that the errors of the Bethe formula are of order $1/k$.

[7] H. P. Hanson, F. Herman, J. D. Lea, and S. Skillman, Acta Cryst. 17, 1040 (1964).

TABLE II. Cross section (mb/atom) for coherent and incoherent pair production and the total cross section for pair production in helium for various atomic models, as a function of energy. The errors in the values reported are less than 0.5% above 10^3 MeV and less than 0.1% above 10^4 MeV. The values for the cross section below 1000 MeV are accurate predictions of the Bethe formula, but that formula is expected to be in greater disagreement because of the high-energy approximations used in its derivation. Below 100 MeV the errors are expected to be several percent.

Cross section	Coherent production			Incoherent production		Total	
Energy (MeV) \ Model	Correlated	Uncorrelated	HFS	Correlated	Uncorrelated	Correlated	Uncorrelated
100	24.24	24.26		12.37	12.37	36.60	36.63
200	27.74	27.77	27.71	14.56	14.57	42.30	42.34
400	30.27	30.28	30.22	16.44	16.46	46.71	46.70
600	31.34	31.33	31.31	17.30	17.30	48.64	48.63
1000	32.33	32.32	32.30	18.18	18.14	50.51	50.47
2000	33.23	33.20	33.20	19.03	18.97	52.26	52.17
6000	33.94	33.90	33.91	19.76	19.68	53.69	53.58
10 000	34.10	34.06	34.08	19.94	19.84	54.04	53.90
100 000	34.36	34.32	34.34	20.24	20.14	54.60	54.46
∞	34.39	34.35	34.37	20.28	20.18	54.67	54.53
	(Values below are less accurate than those above)						
6	7.85	7.85		3.93	3.93	11.78	11.78
10	10.44	10.44		5.22	5.22	15.66	15.66
20	14.38	14.38		7.19	7.19	21.57	21.57
40	18.65	18.66		9.35	9.35	28.01	28.02
60	21.20	21.21		10.68	10.68	31.87	31.89
80	22.95	22.97		11.63	11.63	34.57	34.60

Fig. 1.[8] No calculation of the Hartree-Fock-Slater incoherent-scattering function for helium was available at the time of this work.

In addition, one can consider the Thomas-Fermi-Molière wave function whose ionization energy is given approximately by $E=20.8Z^{7/3}$ eV, which represents an error of approximately +33% in the case of helium. Values of the coherent part of σ for this model are given with sufficient accuracy by Sørenssen.[9] He uses a slightly more approximate formula than (8), but which equals it for large k. His results at high energies, which were independently verified as part of the present work, show a $\sigma_{\text{coherent}} \simeq 3\%$ higher than the coherent cross sections reported here. This difference is due solely to the use of the Thomas-Fermi-Molière screening.

The striking feature of the results is the close agreement of the models for the theoretical prediction of the total pair-production cross section. There exists a 10-to-1 reduction between the difference in ionization energy from model to model and the difference in σ. To the extent that we can determine, even more accurate wave functions would not change the value of σ by more than 0.1%. This is in contrast to other elements where typically these uncertainties are much larger than the total difference for helium. (See, for example, Ref. 8.) This suggests that helium would be an excellent target material for an absolute experimental verification at very high energy of the quantum-electrodynamic description of the production of electron pairs. Alternatively, if the Bethe formula is assumed to be correct, the pair yield from helium could be used as an absolute monitor of bremsstrahlung radiation, and furthermore, the cross section for other photon-induced reactions could be normalized to pair cross section for helium. This would allow high precision in an absolute determination of a cross section. The Appendix discusses photon-flux monitoring. The normalization of the pair-production total cross section in helium to that of other elements would allow a choice of the proper wave function for those elements. Table II shows the total cross section at various energies for helium. Above 10 GeV, we have shown that these results are valid to 0.1% because of calculational uncertainty.

It should be noted that the cross section reported here has to be modified for various small effects to be able to be compared directly with experimental data.[10,11] These are radiative corrections, Coulomb correction, Compton process, and photonuclear absorption. For helium at high energy, only the radiative correction is important. By applying a radiative correction of +0.93% to the total cross section[12] reported here for the correlated wave-function model of helium, in the limit of high energies we obtain a value of 55.18 mb, which yields a pair-production length of $l_{\text{He}}=(120.46/\rho_{\text{He}})$

[8] The values for the total pair cross section with the HFS $F(q)$ are more accurate than the values reported by A. Sørenssen [Nuovo Cimento **38**, 745 (1965); **41**, 543 (1966)] since the continuation there of $F(q)$ beyond $\sin\theta/\lambda=2.00$ was done by the much less accurate Thomas-Fermi-Molière scattering functions. One should note that his statement of the errors introduced by this process is incorrect, especially for the light elements. This is because the $F(q)$ for the light elements is not carried out as far by Hanson (Ref. 7) as for the heavy elements, and in addition, the difference between the F values for the HFS and TFM models is larger for lighter elements.

[9] A. Sørenssen, Nuovo Cimento **38**, 745 (1965); **41**, 543 (1966).

[10] J. K. Walker *et al.*, Phys. Rev. **144**, 1126 (1966).

[11] J. Moffatt, J. J. Thresher, G. C. Weeks, and R. Wilson, Proc. Roy. Soc. (London) **A244**, 245 (1958).

[12] K. Mork and H. Olsen, Phys. Rev. **140**, B1661 (1965).

g/cm², with

$$l_{\mathrm{He}}=A/N_0\sigma_{\mathrm{He}}\rho_{\mathrm{He}},$$

where σ_{He} is the total cross section for pair production in the high-energy limit, corrected for radiative effects, A is the atomic weight of helium (physical) (=4.003), and N_0 is Avogadro's number of atoms per gram (6.0225×10^{23}). Using a value of density of 0.125 g/cm³ for saturated liquid helium at 4.2°K, we arrive at 963.64 cm as a value of the pair-production length. The uncertainty in the pair-production length is $\pm0.15\%$, because of the uncertainty in the screening correction only.

As a check, the coherent and incoherent pair-production cross sections for atomic hydrogen were also computed in an identical fashion. In the complete-screening limit, this computation agreed to better than one part in 10^3 with an analytical integration of (8), and with the results of Wheeler and Lamb.[13]

Professor James K. Walker suggested this problem and provided advice in the work; Professor A. E. Brenner and Jon R. Sauer read the manuscript and made valuable suggestions; the numerical evaluation of formula (8) was done with the aid of staff members of the Harvard Computation Center.

APPENDIX: PHOTON-FLUX MEASUREMENT

Currently, the best means for the monitoring of photon flux is by the use of a quantameter which has been previously intercalibrated with a Faraday cup[14,15] in an electron beam. With care, a $\pm0.1\%$ calibration of the quantameter can be made, but this method has the disadvantage of making necessary a correction to the calibration of the quantameter due to the difference in ionization produced by photon and electron showers. One main result of the present work is the possibility of elimination of the error due to differences in shower development. This would be accomplished by calibration of the quantameter in a photon beam produced in a known radiation length of helium. The radiation length of helium is obtained from the results in this paper by a reduction of 7/9,[16] giving 738.54 ± 0.12 cm for saturated liquid helium at 4.2°K.

The use of a quantameter has the following disadvantages:

(a) Strictly speaking, only the number of equivalent quanta are measured, not the actual number of photons. Knowledge of the bremsstrahlung spectra to $\pm0.1\%$ is needed to determine the number of photons.

(b) The energy of the electron beam must be known to $\pm0.1\%$ also.

(c) The method works less well at low rates. This is because of the necessary measurement of both the Faraday-cup and quantameter signals by our integrating electrometer, which is subject to systematic errors such as drift, etc.

An alternate method of similar accuracy is to measure the photon flux by its interaction in a liquid-helium target. Cryogenic targets of this type can be maintained to 0.1% density variations. By counting the number of electron-positron pairs, using the known helium pair cross section, the photon flux is calculated. This method has the following advantages:

(a) The number of photons is determined directly.

(b) No intracalibration is necessary.

(c) Compensation for energy variations of the photon spectrum is easier, since the cross section is not a strong function of energy.

(d) The result comes as discrete counts, not as an integrated current. Thus ion flux can be more easily measured, and coincidence requirements (e.g., recoil electron in bremsstrahlung) can be added with ease.

Lastly, it should be mentioned that in the case of a liquid-helium bubble chamber in a photon beam, only the number of pairs plus triplets need be counted to determine the original photon flux.

[13] J. A. Wheeler and W. E. Lamb, Phys. Rev. **55**, 858 (1939).

[14] J. de Pagter and M. Fotino, in *Proceedings of the International Symposium on Electron and Photon Interactions at High Energies*, edited by G. Höhler *et al.* (Deutsche Physikalische Gesellschaft, Hamburg, 1965).

[15] D. Yount, Nucl. Instr. Methods **52**, 1 (1967).

[16] For this constant the value 0.7664 was used as a more accurate representation of the ratio $l_{\mathrm{rad}}/l_{\mathrm{pair}}$.