5. CONCLUSION

We have demonstrated that the method of I can be extended to the point where the three-baryon problem is reduced to the solution of nonsingular integral equations for energies all the way up to the first inelastic threshold, in complete analogy with the results of Tiktopoulos[4] for the two-body Bethe-Salpeter equation. It becomes more and more likely that analogous results hold for the n-particle problem.

When coupled with a variational principle, the technique has been shown by Schwartz and Zemach[5] to provide an efficient way of solving the two-body Bethe-Salpeter equation. No doubt this could be extended to the three-body case, but it remains to be seen whether the alternative methods that have been suggested for numerical work might prove to be more effective. These include the subtraction method and the method of separable approximations.

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[5] C. Schwartz and C. Zemach, Phys. Rev. **141**, 1454 (1966).

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Two Simple Approximation Schemes for Lippmann-Schwinger–Type Equations*

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A pair of simple approximation schemes is proposed for generalized Lippmann-Schwinger equations. The equation is first reduced to the nonsingular Noyes form; this is essentially an equation for the "form factor" which takes the external particles off the shell. In the first scheme, a pole approximation is then made for the form factor. The parameters are determined by expanding the Noyes equation about the on-shell point. The method is applied in $\pi\pi$ scattering to the Logunov-Tavkhelidze equation, which was shown by Blankenbecler and Sugar to be an approximation to the Bethe-Salpeter equation. In addition to the form-factor method, these equations were also investigated using an extension of the Pagels approximation. Possible applications to the Bethe-Salpeter equation are discussed.

I. INTRODUCTION

THERE has recently been a revival of interest in those equations of the Lippmann-Schwinger type which can be applied to relativistic problems[1–5]—we are including the Bethe-Salpeter equation in this category.[6–11] Unfortunately such equations are usually more complicated than dispersion methods. It would therefore be desirable to find simple approximations for them. One such approximation was proposed by Biswas and Balázs,[9] who made a Pagels-type approximation[12] to the static-model Bethe-Salpeter equation. The main problem with this approximation is that it has parameters which cannot be determined unambiguously. Nevertheless it is probably the simplest approximation and so we will use it in Secs. IV and V.

Because of the ambiguities of the Pagels approach, we were led to consider another approximation. Here one first writes an equation for the "form factor" which takes the external particles off the shell (using the Noyes technique[13]). This can be approximated by making a multipole expansion at some point. The resulting parameters can then be determined by making a Taylor expansion of the Noyes equation about the on-shell momentum. This sort of approach is particularly adaptable to a scheme in which one wishes to impose

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[1] A. A. Logunov and A. N. Tavkhelidze, Nuovo Cimento **29**, 380 (1963).
[2] L. A. P. Balázs, Phys. Rev. **137**, B1510 (1965); **139**, B1646 (1965).
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[13] H. P. Noyes, Phys. Rev. Letters **15**, 538 (1965).

vertex symmetry, as proposed by Cutkosky and Leon.[6] One such scheme is discussed briefly in Sec. III.

The above methods are applied in P-wave $\pi\pi$ scattering both to the equivalent-potential approach[2] and to the Logunov-Tavkhelidze equation[1] (as used by Blankenbecler and Sugar[3]). The former approach, at least in first approximation, does not require a cutoff for the scattering of spinless particles. It was also found to lead to the most reasonable numerical results, especially when higher-spin particles such as the f^0 and g mesons were exchanged. Because of the complications involved, it was not applied to the full Bethe-Salpeter equation. A Pagels approximation, however, seems to confirm the fact that this equation gives results not too different from those of the Logunov-Tavkhelidze equation.

II. EQUIVALENT POTENTIAL APPROACH AND THE POLE APPROXIMATION FOR THE FORM FACTOR

Perhaps the simplest approach using an off-shell equation in strong interactions is the one using the Schrödinger equation with a local but energy-dependent equivalent potential.[2] The potential can be constructed iteratively by using the strip approximation or any other scheme which leads to fixed-energy dispersion relations. In lowest order, we are led to a particularly simple prescription. We just demand that the potential give the same on-shell amplitude in Born approximation as the one-particle exchange graphs at each energy. This can be shown to give at least the long-range force correctly. If for example we have $\pi\pi$ scattering with ρ exchange (see Fig. 1), we obtain (in momentum space) a potential

$$W(t,s)=12s^{-1/2}\pi\Gamma_1 q_R^2 \times\left[\frac{1}{m^2-t}+\frac{(-1)^I}{m^2-u}\right]P_1\left(1+\frac{2s}{m^2-4}\right), \quad (1)$$

where $t=-(\mathbf{q}'-\mathbf{q})^2$, $\mathbf{q}$ and $\mathbf{q}'$ are the initial and final three-momenta in the c.m. system, s is the square of the total (relativistic) c.m. energy, $u=4-s-t$, $2(q_R^3\Gamma_1/m)$ is the half-width in the q^2 variable, $q^2=\frac{1}{4}m^2-1$, and m is the mass of the ρ. We are taking the pion mass $=1$. The potential (1) can then be inserted into Lippmann-Schwinger equation:

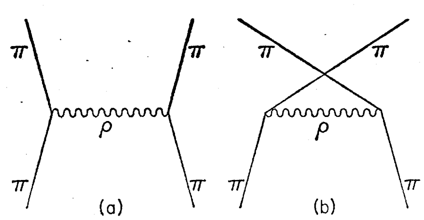

FIG. 1. ρ-exchange graphs.

$$\Phi(\mathbf{q}',\mathbf{q},s)=W[-(\mathbf{q}'-\mathbf{q})^2,s] +\int\frac{d^3p}{2\pi^2}\frac{W[-(\mathbf{q}'-\mathbf{p})^2,s]\Phi(\mathbf{p},\mathbf{q},s)}{p^2-q^2} \quad (2)$$

which is equivalent to the Schrödinger equation and leads to the physical scattering amplitude $F=\Phi(\mathbf{q},\mathbf{q})$ normalized so that the differential cross section $=|F|^2$. From now on we shall restrict ourselves to the on-shell value $q^2=\frac{1}{4}s-1$.

If we project out a given partial wave, Eqs. (2) and (1) become

$$\phi_l(\nu',s)=W_l(\nu',\nu,s)+\frac{1}{\pi}\int_0^\infty d\nu''\frac{(\nu'')^{1/2}}{\nu''-\nu}W_l(\nu',\nu'',s)\phi(\nu'',s) \quad (3)$$

and

$$W_l(\nu',\nu'',s) =\frac{12\Gamma_1\nu_R}{(s)^{1/2}}P_1\left(1+\frac{2s}{m^2-4}\right)\frac{1}{q'q''}Q_l\left(\frac{m^2+\nu'+\nu''}{2q'q''}\right) \quad (4)$$

with $\nu=q^2$, $\nu'=q'^2$, $\nu''=q''^2$, $\nu_R=q_R^2$, and $\phi_l(\nu,s)=F_l(\nu)=(e^{i\delta}\sin\delta)/q$, where δ is the phase shift. It has been shown by Noyes[13] that if we write

$$(q'q)^{-l}\phi_l(\nu',s)=\nu^{-l}F_l(\nu)f(\nu',s), \quad (5)$$

Eq. (3) reduces to the nonsingular equation

$$f(\nu',s)=\frac{V(\nu',\nu,s)}{V(s)}d(s)+\frac{1}{\pi}\int_0^\infty d\nu''\frac{\nu''^{l+1/2}}{\nu''-\nu} \times V(\nu',\nu'',s)f(\nu'',s) \quad (6)$$

with

$$d(s)=1-\frac{1}{\pi}\int_0^\infty d\nu''\frac{\nu''^{l+1/2}}{\nu''-\nu}V(\nu,\nu'',s)f(\nu'',s), \quad (7)$$

$V(s)=V(\nu,\nu,s)$, and $V(\nu',\nu'',s)=(q'q'')^{-l}W_l(\nu',\nu'',s)$. The momentum factors are introduced so as to have a nonsingular threshold behavior. Once we have solved for f and d from Eqs. (6) and (7), we obtain $F_l(\nu)$ from

$$\nu^{-l}F_l(\nu)=V(s)/d(s). \quad (8)$$

So far, Eqs. (6)–(8) are exact consequences of Eq. (3). Suppose for simplicity we now make an asymptotic approximation for the Q_l function in Eq. (4). This is usually a good approximation and has the form

$$Q_l(x)\simeq\frac{1}{x^{l+1}}\sum_{n=0}^{N}\frac{\alpha_n}{x^{2n}}. \quad (9)$$

Now $f(\nu',s)$ is nonsingular for $\nu'>0$ on the real axis, as we have seen. From Eqs. (6), (7), (4), and (9) it then follows that the only singularities of $f(\nu',s)$ in the ν' plane are on the real axis and with $\nu'<\nu'_{\max}$

$=\max[-(m^2+\nu), -m^2]$. In fact we can always write

$$f(\nu',s)=\frac{1}{\pi}\int_{-\infty}^{\nu'_{\max}} dy\frac{h(y,s)}{y-\nu'}. \tag{10}$$

The weight function $h(y,s)$ is not necessarily smoothly varying. It may include delta functions or derivatives of delta functions, for instance. Diagrammatically Eq. (10) is equivalent to saying that the singularities of $f(\nu',s)$ are confined to the shaded region of Fig. 2.

Since we actually need $f(\nu',s)$ only in the nonsingular region $\nu'>0$ we can approximate it by a sum of multipoles

$$f(\nu',s)\simeq\sum_{n=0}^{M}\frac{a_n}{(\lambda+\nu')^{k+n}} \tag{11}$$

at some point $\nu=\lambda$ which we would expect to lie in the region $\nu'<\nu'_{\max}$. This is the only property of Eq. (10) or Fig. 2 that we will use. Actually it is straightforward to extend the above arguments to the case where the exact $Q_n(x)$ is used. Instead of Eq. (9) we could then use the representation

$$Q_l(x)=\frac{1}{2}\int_{-1}^{1} dz\frac{P_l(z)}{x-z} \tag{12}$$

for determining the location of the needed singularities. It turns out that $f(\nu',s)$ has certain singularities (including complex ones) in addition to those shown in Fig. 2. However the distance of these singularities from the region of $\nu'>0$ on the real axis is not substantially smaller than the ones shown in Fig. 2, so the approximation (11) should still be reasonable.

The choice of the integer k determines the asymptotic behavior as $\nu'\to\infty$. While the hope is that the results are not too sensitive to this choice (since we expect the low-energy region to be primarily responsible for the dynamics) it would be desirable to choose k so that the asymptotic behavior is the same as that implied by the exact Eq. (6). This leads to $k=l+1$.

To determine λ and the a_n we will expand Eq. (6) about its on-shell value $\nu'=\nu$, since we expect Eq. (6) to be more reasonable in the region $\nu'\simeq\nu$ than elsewhere. Now at $\nu'=\nu$ we have

$$f(\nu,s)=1, \tag{13}$$

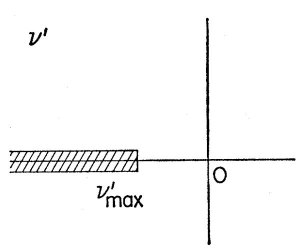

FIG. 2. Singularity structure of $f(\nu',s)$ in the ν' plane if we make the approximation (9).

as can be seen directly from Eq. (5). To obtain further conditions we just differentiate both sides of Eq. (6) to obtain

$$\left[\frac{\partial^{(n)}}{\partial\nu'^{(n)}}f(\nu',s)\right]_{\nu'=\nu}=\frac{d(s)}{V(s)}\left[\frac{\partial^{(n)}}{\partial\nu'^{(n)}}V(\nu',\nu,s)\right]_{\nu'=\nu} +\frac{1}{\pi}\int_0^\infty d\nu''\frac{\nu''^{l+1/2}}{\nu''-\nu}f(\nu'',s)\left[\frac{\partial^{(n)}}{\partial\nu'^{(n)}}V(\nu',\nu'',s)\right]_{\nu'=\nu} \tag{14}$$

for $n=1,\cdots(M+1)$. By substituting Eq. (11) into Eqs. (13) and (14) and using Eq. (7) we can determine all our multipole parameters for any given s. From Eqs. (11), (7), and (8) we can then find the amplitude at that energy.

To check the above method we applied it to the ρ-bootstrap problem (assuming only ρ exchange). This was done exactly by Balázs and Vaidya[14] who obtained a self-consistent ρ with $m=4.2$ and $\Gamma_1=0.47$. We did not attempt to do a fully self-consistent calculation. Instead we took $m=4.2$ for the exchanged ρ and varied the Γ_1, in Eq. (4) until an output P-wave ($I=1$) resonance was produced at the same mass, i.e., until we got

$$\mathrm{Re}\,d(m^2)=0 \tag{15}$$

with $m^2\simeq18$ (i.e., $m\simeq4.2$). Setting $k=0, M=0$, and using the approximation $Q_1(x)\simeq\frac{1}{3}x^{-2}$, this gave an input $\Gamma_1=0.76$. The width of the output resonance was then found from the formula

$$\Gamma_1=\tfrac{1}{4}(\nu_R+1)^{1/2}\frac{V(m^2)}{\mathrm{Re}\,d'(m^2)}, \tag{16}$$

where the derivative of d was calculated numerically by solving our equations for $s\simeq14$ and assuming $d(s)$ to be linear between $s=18$ and $s=14$. This gave an output width $\Gamma_1\simeq0.99$. We thus have only rough self-consistency for Γ_1. The widths also turn out bigger than those obtained exactly. However, if we are only interested in approximate results, it appears that our form-factor approximation is not unreasonable.

It was found by Balázs and Vaidya[14] that the inclusion of f^0 exchange led to an improvement of the ρ width. This seems to suggest that higher-spin exchanges tend to narrow resonances. Recently, various experiments have been performed which seem to suggest the existence of another such resonance, namely, the g meson which has a mass of 1637 MeV and which we assumed to have $I=1$ and spin$=3$. We therefore did a P-wave calculation of the above type (with $M=0$ and $K=2$) in which f^0 and g exchange was assumed in addition to ρ exchange. Experimental values[15] were taken for all the exchanged masses as well as for the ρ and f^0 widths (148 and 110 MeV, respectively). Since the g width is not very well known, it was varied until

[14] L. A. P. Balázs and S. M. Vaidya, Phys. Rev. **140**, B1025 (1965); see also J. Finkelstein, *ibid.* **145**, 1185 (1966).
[15] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **39**, 1 (1967).

the output ρ mass came out at its experimental value $m=(30)^{1/2}$. This led to a width for the g meson of about 80 MeV, which is certainly consistent with experiment.[15] The output ρ width was then calculated from Eq. (16), where $d'(m^2)$ was computed numerically by solving our equations for $s=24$ and assuming $d(s)$ to be linear between $s=30$ and $s=24$. This gave an output ρ width of 152 MeV, which agrees quite well with the experimental input value of 148 MeV.[16]

To see how much the above results depend on our choice of the integer k, we repeated the above calculation with $k=1$. This led to an input g width of 32.5 MeV, which is somewhat narrower than suggested by experiment[15] but is not altogether inconsistent with it. The output ρ width hardly changed at all, increasing only to 160 MeV. All in all, it therefore does not appear that anything is drastically dependent on k.

III. LOGUNOV-TAVKHELIDZE EQUATION AND VERTEX SYMMETRY

One of the difficulties with the equivalent-potential approach of Sec. II is that it breaks down at $s=0$. An equation which does not have this difficulty is one proposed by Logunov and Tavkhelidze[1] (it does, however, have difficulties elsewhere). This equation resembles Eq. (3) but is usually written for the invariant T matrix:

$$\tau_l(\nu',s)=u_l(\nu',\nu,s)+\frac{1}{\pi}\int_0^\infty \frac{d\nu''}{\nu''-\nu}\left(\frac{\nu''}{\nu''+1}\right)^{1/2} \times u_l(\nu',\nu'',s)\tau_l(\nu'',s) \quad (17)$$

with $\tau_l(\nu,s)=A_l(\nu)=(\nu+1)^{1/2}F_l(\nu)=$ invariant amplitude. If one follows the prescription of Blankenbecler and Sugar,[3] the potential $u_l(\nu',\nu'',s)$ is calculated from Fig. 1 with the pion three-momenta off-shell but with the energies of each pion line $=\frac{1}{2}(s)^{1/2}$. If one then takes $g_{\mu\nu}/(k^2-m^2)$ for the ρ propagator (with 4-momentum $=k$) one obtains

$$u_l(\nu',\nu'',s)=\frac{3\Gamma_1}{2q'q''}[s+m^2+2(\nu'+\nu'')]Q_l\left(\frac{m^2+\nu'+\nu''}{2q'q''}\right). \quad (18)$$

It was shown by Blankenbecler and Sugar that Eq. (17) can be thought of as an approximation to the Bethe-Salpeter equation. We shall see in Secs. IV and V that this is indeed reasonable for the ρ bootstrap, at least if we make a Pagels approximation.

One can now follow exactly the same procedure as in the preceding section. The equations corresponding to (5)–(8), (10), (11), and (13)–(16) become

$$(q'q)^{-l}\tau_l(\nu',s)=\nu^{-l}A_l(\nu)g(\nu',s), \quad (19)$$

$$g(\nu',s)=\frac{U(\nu',\nu,s)}{U(s)}D(s)+\frac{1}{\pi}\int_0^\infty d\nu''\left(\frac{\nu''}{\nu''+1}\right)^{1/2}\frac{\nu''^l}{\nu''-\nu} \times U(\nu',\nu'',s)g(\nu'',s), \quad (20)$$

$$D(s)=1-\frac{1}{\pi}\int_0^\infty d\nu''\left(\frac{\nu''}{\nu''+1}\right)^{1/2}\frac{\nu''^l}{\nu''-\nu} \times U(\nu,\nu'',s)g(\nu'',s), \quad (21)$$

with $U(s)=U(\nu,\nu,s)$ and $U(\nu',\nu'',s)=(q'q'')^{-l}u_l(\nu',\nu'',s)$,

$$\nu^{-l}A_l(\nu)=U(s)/D(s), \quad (22)$$

$$g(\nu',s)=\frac{1}{\pi}\int_{-\infty}^{\nu'_{\max}} dy\frac{h(y,s)}{y-\nu'}, \quad (23)$$

$$g(\nu',s)=\sum_{n=0}^{M}\frac{a_n}{(\lambda+\nu'')^{k+n}}, \quad (24)$$

$$g(\nu,s)=1, \quad (25)$$

$$\left[\frac{\partial^{(n)}}{\partial\nu'^{(n)}}g(\nu',s)\right]_{\nu'=\nu}=\frac{D(s)}{U(s)}\left[\frac{\partial^{(n)}}{\partial\nu'^{(n)}}U(\nu',\nu,s)\right]_{\nu'=\nu} +\frac{1}{\pi}\int_0^\infty d\nu''\left(\frac{\nu''}{\nu''+1}\right)^{1/2}\frac{\nu''^l}{\nu''-\nu}g(\nu'',s) \times\left[\frac{\partial^{(n)}}{\partial\nu'^{(n)}}U(\nu',\nu'',s)\right]_{\nu'=\nu} \quad (26)$$

$$\mathrm{Re}D(m^2)=0, \quad (27)$$

$$\Gamma_1=\frac{1}{4}\frac{U(m^2)}{\mathrm{Re}D'(m^2)}. \quad (28)$$

The only calculations which were attempted were for $l=1$. Equation (18) leads to a marginally singular force so it is not clear what we should take for k. In practice we just took $k=1$ and $M=0$.

If one makes the approximation $Q_1(x)\simeq\frac{1}{3}x^{-2}$, the equation becomes convergent with the potential (18). If one now adjusts the width Γ_1 in (18) so that Eq. (27) is satisfied with $m^2=30$, i.e., the experimental value, one obtains an input $\Gamma_1\simeq0.46$.[17] One can now obtain the width of the output ρ resonance from Eq. (28). The value of ReD' can be estimated by calculating $D(s)$ at $s=4$, where the equations simplify considerably, and assuming that ReD is linear between $s=4$ and $s=m^2$. This leads to an output $\Gamma_1\simeq4.5$, an extremely poor value for this parameter (experimentally $\Gamma_1\simeq0.24$).

[16] Recent experimental data in which the ρ resonance was measured directly give a lower value for its width. See V. L. Auslander *et al.*, Phys. Letters **25B**, 433 (1967). However, the results of the preceding paragraph suggest that a more accurate calculation may lead to a narrower ρ width.

[17] In these calculations the extreme-relativistic approximation $[\nu''/(\nu''+1)]^{1/2}\simeq1$ was used. This approximation seems to be quite reasonable for the $\pi\pi$ problem since $m^2\gg1$.

This may, however, be due to our crude way of estimating $\mathrm{Re}D'(m^2)$, or to our choice of the ρ propagator.

The results are improved if one includes the effect of the f^0 meson exchange. For the propagator of the f^0, we neglect all the k-dependent terms in the numerator, and take it to be

$$(g_{\mu\alpha}g_{\nu\beta}+g_{\mu\beta}g_{\nu\alpha}-\tfrac{2}{3}g_{\mu\nu}g_{\alpha\beta})/(k^2-m_f^2)\,,$$

where m_f is the mass of the f^0. Then the potential U defined before has an additional term

$$U_f=\frac{\gamma_f}{q^2q'^2}[(s+m_f^2+2q^2+2q'^2)^2-\tfrac{1}{3}(s-m_f^2-2q^2-2q'^2)^2]\times Q_1\left(\frac{q^2+q'^2+m_f^2}{2qq'}\right),$$

where $\gamma_f=2.8\times10^{-3}$ for an f^0 width of $\Gamma_f=100$ MeV. With the inclusion of the f^0 exchange, the equations are now divergent. We introduce a cutoff in the integrals to make them meaningful. We took a cutoff of $\nu''=150$ in the variable ν'', and $k=1$ and $M=0$ for the parametrization of g in (24). With the approximation $Q_1(x)\simeq\frac{1}{3}x^{-2}$, the calculations are straightforward though tedious, and we find that with the experimental value of $\Gamma_f=100$ MeV, $m_\rho^2=30$, the output m_ρ^2 is 30 if $\Gamma_1\simeq0.60$. Furthermore the output Γ_1 comes out to be $\simeq0.80$. These results are considerably better than the ones without the f^0 but still about 3.5 times larger than the experimental value of $\Gamma_1\simeq0.24$.

We can also include the effect of the Pomeranchuk repulsion as suggested by Chew,[18] but this introduces rather serious ambiguities about continuation for off-shell values of q^2 and q'^2. In any case, qualitatively one expects the repulsion to decrease the output width.[18]

One of the advantages of our form-factor approximation is that it permits us to set up a scheme which satisfies vertex symmetry. In particular, we might expect better convergence in such a scheme. The importance of this has been particularly emphasized by Cutkosky and Leon.[6] Of course they start from the full Bethe-Salpeter equation. However, we have seen that the Logunov-Tavkhelidze equation can be thought of as an approximation to the Bethe-Salpeter equation. Moreover, it should be quite straightforward to generalize our results to the full Bethe-Salpeter case. Equation (17) gives $\tau_l(\nu',s)$ with the initial particles on the shell but with the final particles of the shell. Suppose we look at $s\simeq m^2$. Since we have the ρ pole at $s=m^2$, we must have, from Fig. 3,

$$\frac{\tau_l(\nu',s)}{(q'q)^l}\simeq4\Gamma_1\frac{h(P-q')h(P+q')}{m^2-s} \tag{29}$$

if we make the extra assumption that the function

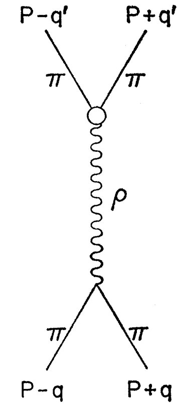

FIG. 3. Output ρ resonance. Here $P=(\frac{1}{2}m,0,0,0)$ and $q=(0,\mathbf{q})$, where the first component is the time component.

describing the off-shell behavior factors. But $\nu^{-l}A_l(\nu)\simeq4\Gamma_1/(m^2-s)$, so from Eq. (19),

$$g(\nu',m^2)=h(P-q')h(P+q')\,. \tag{30}$$

Now, if in calculating the potentials from Fig. 1 we insert the same vertex functions in the appropriate places, we have

$$\begin{aligned}u_l(\nu',\nu'',s)&\rightarrow u_l'(\nu',\nu'',s)\\&=h(P-q')h(P+q')h(P-q'')h(P+q'')u_l(\nu',\nu'',s)\\&=g(\nu',m^2)g(\nu'',m^2)u_l(\nu',\nu'',s)\,.\end{aligned} \tag{31}$$

If we substitute this into Eq. (20) at $s=m^2$, where $\mathrm{Re}D(s)=0$, we obtain

$$1=\frac{P}{\pi}\int_0^\infty d\nu''\left(\frac{\nu''}{\nu''+1}\right)^{1/2}\frac{\nu''^l}{\nu''-\nu}U(\nu',\nu'',m^2)g^2(\nu'',m^2)\,, \tag{32}$$

while the Eqs. (26) become

$$0=\frac{P}{\pi}\int_0^\infty d\nu''\left(\frac{\nu''}{\nu''+1}\right)^{1/2}\frac{\nu''^l}{\nu''-\nu}g^2(\nu'',m^2)\times\left[\frac{\partial^{(n)}}{\partial\nu'^{(n)}}U(\nu',\nu'',m^2)\right]_{\nu'=\nu}. \tag{33}$$

This together with (25) can be used to determine the parameters λ and a_n if we take Eq. (24) for $g(\nu',m^2)$. Once we know $g(\nu',m^2)$, we can use the potential given by Eq. (31) to find the amplitude in the usual way at other values of s.

Unfortunately, preliminary calculations with the simple scheme just described do not seem to lead to physically admissible solutions. This, however, may be due to the drastic approximation of dropping everything except the ρ meson. It is quite possible that the inclusion of other effects may lead to more sensible results.

IV. PAGELS APPROXIMATION

The form-factor approximation we have used is based on Eqs. (10) or (23). Another approximation which can be justified on the basis of these expressions is one first

[18] G. F. Chew, Phys. Rev. 140, B1427 (1965).

proposed by Pagels[12] in connection with the N/D formalism and later extended to the static-model Bethe-Salpeter equation by Biswas and Balázs.[9] Actually the procedure followed by the latter authors applies with minor modifications to any equation of the Lippmann-Schwinger type. It seems to be particularly appropriate when we have an equation which requires a cutoff. This would arise in the formalism of the preceding section if, for example, we used Eq. (18) without any approximations or if we took $(g_{\mu\nu}-p_\mu p_\nu/m^2)/(k^2-m^2)$ for the ρ propagator. In the latter case we would obtain a potential

$$u_l(\nu',\nu'',s)=\frac{3\Gamma_1}{2q'q''}\left[s+m^2+2(\nu'+\nu'')+\frac{(\nu'-\nu'')^2}{m^2}\right]\times Q_l\left(\frac{m^2+\nu'+\nu''}{2q'q''}\right). \quad (34)$$

Inserting therefore a straight cutoff at $\nu''=\Lambda^2$ and substituting Eq. (23) into (20) we obtain, using Eq. (21),

$$g(\nu',s)=\frac{U(\nu',\nu,s)}{U(s)}+\frac{1}{\pi^2}\int_{-\infty}^{\nu'_{\max}} dyh(y,s)K(y,\nu',s)\,, \quad (35)$$

where

$$K(y,\nu',s)=\int_0^{\Lambda^2} d\nu''\left(\frac{\nu''}{\nu''+1}\right)^{1/2}\frac{\nu''^l}{\nu''-\nu}\frac{1}{y-\nu''}\times\left[U(\nu,\nu'',s)-\frac{U(\nu',\nu,s)}{U(s)}U(\nu,\nu'',s)\right]. \quad (36)$$

If we now use either the exact representation (12) or the approximate expression (9) in evaluating Q_l within the potential, and if we make a partial-fraction decomposition of all denominators of the form $(\nu''-x)$ in Eq. (36), we find that we can express $K(y,\nu',s)$ as a linear combination of the integrals

$$I(x)=\int_0^{\Lambda^2} d\nu''\left(\frac{\nu''}{\nu''+1}\right)^{1/2}\frac{\nu''^l}{\nu''-x}. \quad (37)$$

These integrals, moreover, are found to involve x only in the range $\nu'_{\max}>x>-\infty$, well away from the singularities of $I(x)$, which are restricted to $0<x<\Lambda^2$. We can therefore make a pole (or multipole) approximation

$$I(x)\simeq\sum_{i=1}^{L}\frac{c_i}{b_i-x} \quad (38)$$

with $0<b_i<\Lambda^2$. The constants c_i and b_i are adjusted so as to make Eq. (38) a good approximation for $-\infty<x<\nu'_{\max}$, the only region where $I(x)$ is needed.

Now Eq. (38) is equivalent to setting

$$\nu''^l\left(\frac{\nu''}{\nu''+1}\right)^{1/2}=\sum_{i=1}^{L}c_i\delta(\nu''-b_i) \quad (39)$$

in Eq. (37) and hence also in Eqs. (36) and the combination of (20) and (21). The latter thus becomes

$$g(\nu',s)=\frac{U(\nu',\nu,s)}{U(s)}+\frac{1}{\pi}\sum_{i=1}^{L}\frac{c_i}{b_i-\nu}\times\left[U(\nu',b_i,s)-\frac{U(\nu',\nu,s)}{U(s)}U(\nu,b_i,s)\right]g(b_i,s). \quad (40)$$

If we set $\nu'=b_i$, for $i=1,\cdots L$ we have L linear equations which we can solve for the $g(b_i,s)$. These can then be substituted back into Eq. (40) to give $g(\nu',s)$ for any ν'.

Once we have $g(\nu',s)$ we can find $D(s)$ from Eq. (21). Here again, if we substitute (23) into (21) we find that we can express $D(s)$ in terms of $I(x)$. This time, however, $I(x)$ is needed for $x=\nu$ as well as $-\infty<\nu<\nu'_{\max}$. Thus, unless we are at some large negative ν, the approximation (38) is unwarranted. If, however, we modify Eq. (21) to read

$$D(s)=1-\frac{1}{\pi}U(s)I(\nu)-\frac{1}{\pi}\int_0^{\Lambda^2} d\nu''\left(\frac{\nu''}{\nu''+1}\right)^{1/2}\frac{\nu''^l}{\nu''-\nu}\times[U(\nu,\nu'',s)g(\nu'',s)-U(s)] \quad (41)$$

and now substitute Eq. (23) we find that the integral in (41) can be expressed in terms of $I(x)$ with x needed only for $-\infty<x<\nu'_{\max}$. Thus, we can make the approximation (38) to obtain

$$D(s)=1-\frac{1}{\pi}U(s)I(\nu)-\frac{1}{\pi}\sum_{i=1}^{L}\frac{c_i}{b_i-\nu}\times[U(\nu,b_i,s)g(\nu_i,s)-U(s)]. \quad (42)$$

Equations (40), (42), (27), and (28) can now be applied to the ρ resonance problem. Equation (34) was taken for the potential with experimental values for Γ_1 and m^2, and the cutoff Λ^2 was varied until Eq. (27) was satisfied with $m^2=30$. The parameters c_1 and b_1 were fixed by the requirement that Eq. (38) be exact at $x=-\infty$ and $x=\nu'_{\max}$. (Only the case $L=1$ was considered.) The slope $\mathrm{Re}D'(m^2)$ was determined numerically by solving our equations at $s=4$ and assuming that $\mathrm{Re}D(s)$ is linear between $s=30$ and $s=4$. This led to an output width, as given by Eq. (28), of $\Gamma_1=1.08$. This value is, of course, much larger than the experimental value $\Gamma_1=0.24$ but is not too different from the outcome of the calculation described in Sec. II, in which only ρ exchange was assumed.

The inclusion of the f^0 exchange once again reduces the ρ-meson width. As in Sec. III, we fix the cutoff at $\nu''=150$. Then for an input of $\Gamma_1\simeq 0.80$, and the experimental values $\Gamma_f=100$ MeV and $m_f=1250$ MeV, we have an ouput $m^2=30$ and an output $\Gamma_1\simeq 0.80$. Thus we have an approximate bootstrap in this case. The numbers obtained here agree fairly well with the ones obtained in Sec. III, thus lending credence to our approximations.

If we do not require an explicit cutoff in the above calculations—which would be the case, for example, in the formalisms of Sec. II—we obviously could not consider an integral of the type given by (37). In this case there is actually a natural cutoff provided by the other factors in the integrals of (20) or (36) or (41). These are generally of the form $(\nu''+\eta)^{-h}$, where h is an integer. Instead of using Eq. (38) we could therefore make the approximation

$$\int_0^\infty d\nu''\left(\frac{\nu''}{\nu''+1}\right)^{1/2}\frac{\nu''^l}{\nu''-x}\frac{1}{(\nu''+\eta)^h} \simeq\sum_{i=1}^{L}\frac{c_i}{b_i-x}\frac{1}{(b_i+\eta)^h} \quad (43)$$

with the c_i and b_i now adjusted to make Eq. (43) a good approximation in the range $-\infty<x<\nu'_{\max}$. The quantity η can be estimated from the rate of falloff of the potential terms in (20), (36), and (41).

Thus repeating the calculation of Sec. II, but with the approximation just mentioned, we find that for $\eta\simeq 80$, $\Gamma_1\simeq 0.23$, $\Gamma_f\simeq 100$ MeV, and the experimental masses for input, we have an output $m^2=30$ if input $\Gamma_g=87$ MeV. Also the output ρ-meson width comes out to be $\simeq 200$ MeV. These numbers agree reasonably well with those of Sec. II.

V. BETHE-SALPETER EQUATION USING A PAGELS-TYPE APPROXIMATION

In the Bethe-Salpeter equation the energies as well as the three-momenta are taken off the shell. This complicates the problem considerably, since we have a two-variable integral equation to deal with. The actual equation has the form

$$T_l(q',\omega';q,\omega;s)=V(q',\omega';q,\omega;s) +i\frac{4}{\pi^2}\int_0^\infty q''^2dq''\int_{-\infty}^{\infty}d\omega''V(q',\omega';q'',\omega'';s) \times G(q'',\omega'',s)T_l(q'',\omega'',q,\omega s) \quad (44)$$

in a given partial wave, with

$$G^{-1}(q'',\omega'',s)=[(\omega''+\tfrac{1}{2}s^{1/2})^2-q''^2-1] \times[(\omega''-\tfrac{1}{2}s^{1/2})^2-q''^2-1]. \quad (45)$$

Here $(\frac{1}{2}s^{1/2}+\omega)$ and $(\frac{1}{2}s^{1/2}-\omega)$ are the energies of the initial pion lines, and $(\frac{1}{2}s^{1/2}+\omega')$ and $(\frac{1}{2}s^{1/2}-\omega')$ the energies of the final pion lines. The T matrix is again normalized so that $T_l(q,0;q,0;s)=A_l(\nu)$. We shall restrict ourselves to potentials coming only from ρ exchange. From Fig. 1 we then obtain

$$V(q',\omega';q'',\omega'';s) =\frac{3\Gamma_1}{2q'q''}\left[s-2(\omega'^2+\omega''^2-q'^2-q''^2)+m^2 -\frac{s(\omega'-\omega'')^2-(\omega'^2-\omega''^2-q'^2+q''^2)^2}{m^2}\right] \times Q_l\left[\frac{m^2-(\omega'-\omega'')^2+q'^2+q''^2}{2q'q''}\right]. \quad (46)$$

The Noyes procedure has been generalized by Levine, Tjon, and Wright[11] to the Bethe-Salpeter equation. If we write

$$(q'q)^{-l}T_l(q',\omega';q,0;s)=q^{-2l}f(q',\omega',s), \quad (47)$$

Eq. (45) reduces to the equation

$$f(q',\omega',s)=\frac{W(q',\omega';q,0;s)}{W(s)}\Delta_l(s)-i\frac{4}{\pi^2}\int_0^\infty q''^{2l+2}dq''\int_0^\infty d\omega'' \times[W(q',\omega';q'',\omega'';s)+W(q',\omega';q'',-\omega'';s)] \times G(q'',\omega'',s)f(q'',\omega'',s) \quad (48)$$

with

$$\Delta_l(s)=1-i\frac{8}{\pi^2}\int_0^\infty q''^{2l+2}dq''\int_0^\infty d\omega''W(q,0;q'',\omega'';s) \times G(q'',\omega'',s)f(q'',\omega'',s), \quad (49)$$

where $W(s)=W(q,0;q,0;s)$ and $W(q',\omega';q'',\omega'';s) =(q'q'')^{-l}V(q',\omega';q'',\omega'';s)$. We have used the symmetry property $f(q',\omega',s)=f(q',-\omega',s)$ which follows directly from the original Bethe-Salpeter equation combined with Eq. (47). Once we have solved for f and Δ_l we obtain $A_l(\nu)$ from

$$\nu^{-l}A_l(\nu)=W(s)/\Delta_l(s). \quad (50)$$

The above equations are not yet in tractable form. We must first make a Wick rotation to the imaginary axis. Equations (48) and (49) then give[11]

$$f(q',i\omega',s)=\frac{W(q',i\omega';q,0;s)}{W(s)}+\frac{4}{\pi^2}\int_0^\infty q''^{2l+2}dq''\int_0^\infty d\omega''\Big[W(q',i\omega';q'',i\omega'';s)+W(q',i\omega';q'',-i\omega'';s)$$
$$-2W(q,0;q'',i\omega'';s)\frac{W(q',i\omega';q,0;s)}{W(s)}\Big]G(q'',i\omega'',s)f(q'',i\omega'',s)-\frac{2s^{-1/2}}{\pi}\int_0^q q''^{2l+2}dq''$$
$$\times\Big[W(q',i\omega';q'',\bar\omega(q'');s)+W(q',i\omega';q'',-\bar\omega(q'');s)-2W(q,0;q'',\bar\omega(q'');s)\frac{W(q',i\omega';q,0,s)}{W(s)}\Big]$$
$$\times\frac{\bar f(q'',s)}{\bar\omega(q'')(q''^2+1)^{1/2}},\quad(51)$$

$$f(q',s)=\frac{W(q',\bar\omega(q');q,0;s)}{W(s)}+\frac{4}{\pi^2}\int_0^\infty q''^{2l+2}dq''\int_0^\infty d\omega''\Big[W(q',\bar\omega(q');q'',i\omega'';s)+W(q',\bar\omega(q');q'',-i\omega'';s)$$
$$-2W(q,0;q'',i\omega'';s)\frac{W(q',\bar\omega(q');q,0;s)}{W(s)}\Big]G(q'',i\omega'',s)f(q'',i\omega'',s)-\frac{2s^{-1/2}}{\pi}\int_0^q q''^{2l+2}dq''$$
$$\times\Big[W(q',\bar\omega(q');q'',\bar\omega(q'');s)+W(q',\bar\omega(q');q'',-\bar\omega(q'');s)-2W(q,0;q'',\bar\omega(q'');s)\frac{W(q',\bar\omega(q');q,0;s)}{W(s)}\Big]$$
$$\times\frac{\bar f(q'',s)}{\bar\omega(q'')(q''^2+1)^{1/2}},\quad(52)$$

where $\bar\omega(q')=\frac{1}{2}s^{1/2}-(q'^2+1)^{1/2}+i\epsilon$ and

$$\bar f(q',s)=f(q',\bar\omega(q'),s).$$

Before making a Wick rotation for Δ_l, we first rewrite it as

$$\Delta_l(s)=1+i\frac{8}{\pi^2}\int_0^\infty q''^{2l+2}dq''\int_0^\infty d\omega''G(q'',\omega'',s)$$
$$\times[W(q,0;q'',\omega'';s)f(q'',\omega'',s)-W(s)]$$
$$+W(s)H(s);\quad(53)$$

we have here added and subtracted $W(s)H(s)$, where

$$H(s)=i\frac{8}{\pi^2}\int_0^\infty q''^{2l+2}dq''\int_0^\infty d\omega''G(q'',\omega'',s),\quad(54)$$

an expression which can be evaluated analytically. For general l, H, as well as all our other equations, are divergent, so it will be understood that we replace q''^{2l+2} everywhere by $q''^{2l+2}\theta(\Lambda-q'')$, where Λ is a cutoff. If we make a Wick rotation for the integral term in Eq. (53), which is now nonsingular, we obtain

$$\Delta_l(s)=1-\frac{8}{\pi^2}\int_0^\infty q''^{2l+2}dq''\int_0^\infty d\omega''G(q'',i\omega'',s)$$
$$\times[W(q,0;q'',i\omega'';s)f(q'',i\omega'',s)-W(s)]$$
$$+\frac{4s^{-1/2}}{\pi}\int_0^q\frac{q''^{2l+2}dq''}{\bar\omega(q'')(q''^2+1)^{1/2}}$$
$$\times[W(q,0;q'',\bar\omega(q'');s)\bar f(q'',s)-W(s)]$$
$$+H(s)W(s).\quad(55)$$

Suppose for the time being we neglect the last (single-integral) term on the right side of Eq. (51). This is exact for $s<4$ and is a good approximation even for $s>4$. Now for $\omega''>0$, $q''>0$, $f(q'',i\omega'',s)$ is nonsingular for $s<(m+2)^2$, i.e., below the inelastic threshold. In practice it may be almost nonsingular for considerably larger values of s. From Eq. (51), $f(q',i\omega',s)$ thus develops singularities in this range of s only when the W's are singular. If we make the approximation (9) it is then straightforward to show that, no matter what ω' is, we can write an expression analogous to Eq. (10), namely,

$$f(q',i\omega',s)=\int_{-\infty}^{\nu'_{\max}}dy\frac{h(y,i\omega',s)}{y-q'^2},\quad(56)$$

where $\nu'_{\max}$ is defined as in Eq. (10). If we substitute Eq. (56) into (51) and follow the same sort of procedure we did in Sec. IV, we find that we can make the approximation

$$q''^{2l+2}\simeq c\delta(q''-a),\quad(57)$$

where c and a are such that

$$\int_0^\infty\frac{q''^{2l+2}}{q''^2-x}dq''\simeq\frac{c}{a^2-x}\quad(58)$$

is a good approximation in the range $-\infty<x<\nu'_{\max}$.[19] In practice we determined c and a by requiring that

[19] Strictly speaking, there are terms for which the approximation has to be good even for x close to the singular region $0<x<\infty$. However, it can be argued that such contributions are small.

Eq. (58) be exact at $x=\nu'_{\max}$ and $x=-\infty$. Of course, in general we could have a sum of several delta functions in (57) and a corresponding sum of poles in (58). However, it is simpler to discuss the problem with only one pole.

If we substitute Eq. (57) into (51), remembering that we are ignoring the contribution of $\bar{f}$ for the moment, we obtain a single variable integral equation on setting $q'=a$:

$$f(a,i\omega',s)=\frac{W(a,i\omega';q,0;s)}{W(s)}+\frac{4c}{\pi^2}\int_0^\infty d\omega''\bigg[W(a,i\omega';a,i\omega'';s)+W(a,i\omega';a,-i\omega'';s)-2W(q,0;a,i\omega'';s)\times\frac{W(a,i\omega';q,0;s)}{W(s)}\bigg]G(a,i\omega'',s)f(a,i\omega'',s). \tag{59}$$

We can now go through the above procedure all over again, but this time dealing with ω' and ω'' instead of q' and q''. This time we are dealing with integrals which are rapidly convergent so it is necessary to consider integrals of the type given by (43). We found it convenient to make the approximation

$$\int_0^\infty \frac{d\omega''}{\omega''^2-x}G(a,i\omega'',0)\simeq\frac{c'}{b^2-x}, \tag{60}$$

which has to be a good approximation for $-\infty<x<-(m^2-1+a^2)$ as well as certain regions of the complex plane. In practice we required that it be exact at $x=-\infty$ and at $x=-(m^2-1+a^2)$ and then checked to see whether it is reasonable elsewhere. This determines c' and b. Of course, as before, Eq. (60) makes it possible for us to set

$$G(a,i\omega'',0)\simeq c'\delta(\omega''-b) \tag{61}$$

or

$$G(a,i\omega'',s)\simeq\frac{G(a,ib,s)}{G(a,ib,0)}c'\delta(\omega''-b). \tag{62}$$

If the latter is substituted into Eq. (59) we have a purely algebraic equation without any integrals.

If we do not ignore the contributions of $\bar{f}$, we naturally have a more complicated situation, especially since we have to consider Eq. (52) in addition to (51). The double integral in (51) can be treated in essentially the manner described in the last two paragraphs, except that now the approximation (58) has to be valid in the region $-\infty<x<-1$. In practice it turns out that (58) is reasonable even over this larger range. Of course it does not have to be as accurate as in (51) because the contribution of $\bar{f}$ seems to be comparatively unimportant.

Turning next to the single integrals in (51) and (52), we find that we can take

$$\frac{q''^{2l+2}}{(q''^2+1)^{1/2}}\simeq\lambda\delta(q''-\eta), \tag{63}$$

provided that

$$\int_0^q \frac{q''^{2l+2}}{(q''^2+1)^{1/2}}\frac{dq''}{q''^2-x}\simeq\frac{\lambda}{\eta^2-x} \tag{64}$$

is a good approximation in the range $-\infty<x<-1$. If we are above the inelastic threshold it is also required for certain additional values of x, but insofar as inelastic effects can be assumed to be unimportant we can presumably ignore this difficulty. In practice λ and η were determined by requiring that the value and derivative are exact at $x=-\infty$. It was then found that the resulting approximation (64) is reasonable even at the other values of x where it is required.

If we now insert the approximations (57), (62), and (63) into Eqs. (51) and (52), as well as into (55), for which the above considerations also apply, we obtain bthe algeraic equations

$$f(a,ib,s)=\frac{W(a,ib;q,0;s)}{W(s)}+\frac{4cc'}{\pi^2}\bigg[W(a,ib;a,ib;s)+W(a,ib;a,-ib;s)-2W(q,0;a,ib;s)\frac{W(a,ib;q,0;s)}{W(s)}\bigg]\times\frac{G(a,ib,s)}{G(a,ib,0)}f(a,ib,s)-\frac{2\lambda}{\pi s^{1/2}}\bigg[W(a,ib;\eta,\bar{\omega}(\eta);s)+W(a,ib;\eta,-\bar{\omega}(\eta);s)-2W(q,0;\eta,\bar{\omega}(\eta);s)\frac{W(a,ib;q,0;s)}{W(s)}\bigg]\frac{\bar{f}(\eta,s)}{\bar{\omega}(\eta)}, \tag{65}$$

$$\bar{f}(\eta,s)=\frac{W(\eta,\bar{\omega}(\eta);q,0;s)}{W(s)}+\frac{4cc'}{\pi^2}\bigg[W(\eta,\bar{\omega}(\eta);a,ib;s)+W(\eta,\bar{\omega}(\eta);a,-ib;s)-2W(q,0;a,ib;s)\frac{W(\eta,\bar{\omega}(\eta);q,0;s)}{W(s)}\bigg]\times\frac{G(a,ib,s)}{G(a,ib,0)}f(a,ib,s)-\frac{2\lambda}{\pi s^{1/2}}\bigg[W(\eta,\bar{\omega}(\eta);\eta,\bar{\omega}(\eta);s)+W(\eta,\bar{\omega}(\eta);\eta,-\bar{\omega}(\eta);s)-2W(q,0;\eta,\bar{\omega}(\eta);s)\frac{W(\eta,\bar{\omega}(\eta);q,0;s)}{W(s)}\bigg]\frac{\bar{f}(\eta,s)}{\bar{\omega}(\eta)}, \tag{66}$$

and

$$\Delta_l(s)=1-\frac{8cc'}{\pi^2}\frac{G(a,ib,s)}{G(a,ib,0)}[W(q,0;a,ib;s)f(a,ib,s)-W(s)]$$
$$+\frac{4\lambda s^{-1/2}}{\pi\bar{\omega}(\eta)}[W(q,0;\eta,\bar{\omega}(\eta);s)f(\eta,s)-W(s)]-W(s)H(s). \quad (67)$$

A calculation based on the above equations was made with the exchange potential (46) in the P wave. The experimental values $\Gamma_1=0.24$ and $m^2=30$ were taken for the input and Λ varied until an output resonance was produced at the same mass, i.e., until we obtained

$$\mathrm{Re}\Delta_1(m^2)=0 \quad (68)$$

with $m^2=30$. The width of this output resonance was then computed from the formula

$$\Gamma_1=\frac{1}{4}\frac{W(m^2)}{\mathrm{Re}\Delta_1'(m^2)}, \quad (69)$$

where $\mathrm{Re}\Delta_1'(m^2)$ was estimated by solving our equations at $s=4$, where they simplify considerably, and assuming that $\mathrm{Re}\Delta_1$ is linear between $s=4$ and $s=30$. This leads to an output $\Gamma_1\simeq1.08$, a value almost identical with the one obtained in Sec. IV when a similar procedure was followed.

VI. CONCLUSION

The analytic structure of the Noyes form factor, as expressed in Eqs. (10) and (23) enabled us to set up two fairly simple approximation schemes. One of them is equivalent to making a delta-function approximation for the phase-space factor as in Eq. (39). This is clearly the simplest method but is somewhat ambiguous in that the answers depend on our choice of c_i and b_i. Obviously, there is a wide range of c_i and b_i which permit Eq. (38) to be a good approximation.

The other method, which we actually considered first, does not have this difficulty. Here, one just makes a multipole approximation for the Noyes form factor and then determines the parameters introduced in this way by expanding the Noyes equations about the on-shell value of the momentum. This method was applied to the calculation of the ρ and gave quite good results within the equivalent potential approach when the exchange of the f^0 and g mesons was included.

It is quite conceivable that the form-factor approximation could give meaningful results even if the original Lippmann-Schwinger equation on which it is based fails as a result of divergences. All we have to do is to assume a sufficiently large k in (11) or (24). The Eqs. (7) and (14) or (21) and (26) would then converge and it is still possible to determine the a_n and λ. Of course we have to keep the number M fairly small, since otherwise we have to take too many derivatives in Eqs. (14) or (26). This, in turn, involves high-ν' effects which are presumably responsible for the original divergences in the first place.

The attitude expressed in the preceding paragraph would be particularly applicable if, for example, we had a strong repulsive core at small distances (see, for example, Ref. 4). It is well known that such a core leads to meaningful results in configuration space even though the Lippmann-Schwinger equation has divergences. In particular, it is known that the wave function (and hence the Noyes form factor, which is simply related to it) is not too different from what it would have been with a well-behaved potential. This means that an expression of the type given in Eqs. (11) or (24) is not unreasonable. Also, the low derivatives in the Eqs. (14) or (26) are presumably related to long-range effects and so would not be too sensitive to the repulsive core. We might thus expect reasonable results from them even if we leave out the contribution of such a core.

Because of the practical difficulties involved, we have not attempted a form-factor approximation for the full Bethe-Salpeter equation. However, it should be relatively straightforward to set up such an approximation scheme. One procedure might be to take for the Noyes form factors (which essentially represent the off-shell behavior of the final pion lines) functions of the type suggested by Cutkosky and Leon,[6] who took products of two functions of the form $[(1-\lambda)/(p^2-\lambda)]^{1/2}$, one for each pion line; here p is the four-momentum of the pion line. We could then expand the Bethe-Salpeter equation in the (p^2-1), i.e., about the on-shell values of the final pion lines. This should determine λ, or any other constants we might introduce as a result of a more complicated parametrization.