Regge Poles for Spinless Particles in the Backward Direction*

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A new Regge representation is introduced which involves Legendre functions of a modified argument. The analytic and asymptotic properties of the new partial waves are studied. It is shown that the new representation is valid in extended regions of the Mandelstam plane where the background integral converges. In these regions the new representation explicitly possesses the correct analyticity in the new plane corresponding to the $\cos\theta_u$ plane. In regions of the Mandelstam plane where the conventional Regge representation is also valid, we can relate the new partial waves to the conventional Regge poles. The compatibility of our results with those of other groups is discussed.

I. INTRODUCTION

Recent experimental and theoretical work on the backward peak has developed new interest in the question of whether Regge asymptotic behavior holds in the backward direction.[1] For the unequal-mass case, for instance, there is a region in the backward direction where the crossed-reaction angle is small even when s is large. This has created considerable uncertainty[2] in the literature as to whether Regge behavior should be expected for backward πN scattering, and it has also created suspicion on the validity of any representation $A(u,s)=g(u,z_u)$ of the scattering amplitude at $u=0$.

To remedy the difficulty several approaches have been proposed[3,13] satisfying consistency requirements of Mandelstam and angular-momentum analyticity, but these approaches either rely heavily on the Khuri representation (which in turn depends on the Regge representation), or they introduce a new representation whose validity is doubtful. In particular, any representation which follows from the conventional Regge representation is questionable in the neighborhood of the line $u=0$, because the Froissart-Gribov partial waves are not defined there. Therefore the conventional Regge representation is useful to the extent that it allows us to establish connections between conventional Regge poles and poles of the new partial waves. Furthermore, there still remains unanswered the question of whether a representation involving Legendre functions could give Regge asymptotic behavior in the backward direction.

In this paper we derive, under the assumption of the validity of the Mandelstam representation and of two Regge-type properties, Sommerfeld-Watson formulas which are useful in restricted regions of the Mandelstam plane. In these particular regions they possess two desirable properties: (i) they explicitly exhibit Mandelstam analyticity in the plane corresponding to the $\cos\theta_u$ plane; (ii) they possess Regge asymptotic behavior. It should be noted that the approach is general enough to be extended to other regions of the Mandelstam plane. It can be shown that the new formulas can be obtained using the Khuri representation, but the existence of such a representation is not required. The existence of the conventional Regge representation is sufficient.

Section II reviews the relativistic formulation of the problem and provides the necessary background for the following sections. In Sec. III we introduce the new Regge representation. In Sec. IV we summarize our results and compare them with the results obtained recently by two other groups.

This is the first of two papers dealing with Regge poles in the backward direction. The objective is to develop a theoretical formalism for π-N scattering in the backward direction, which incorporates the unequal-mass complications and also the nontrivial constraints due to spin. Finally, we will use the formalism in analyzing the available data.[1]

II. RELATIVISTIC FORMALISM

The invariant amplitude satisfies a fixed-u dispersion relation:

$$A(s,u,t)=\frac{g^2}{M-s}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\frac{\rho_u(u')}{u'-u}du'+\frac{1}{\pi}\times\int_{k_0(u)}^{-\infty}\frac{A_s(u,s')}{z'-z}dz'+\frac{1}{\pi}\int_{k_1(u)}^{\infty}\frac{A_t(u,t')}{z'-z}dz',\quad(1)$$

where $z=\cos\theta_u$; $A_s(u,s')$ and $A_t(u,t')$ are the absorptive parts for the s and t channels, respectively; $\rho_u(u')$ is the single spectral function; and $k_1(u)$, $k_0(u)$ are the branch points of the right- and left-hand cuts, respectively, given by

$$k_1(u)=1+8\mu^2u/[u-(M+\mu)^2][u-(M-\mu)^2],\quad(2)$$

$$-k_0(u)=1+2(M+\mu)^2/[u-(M+\mu)^2].\quad(3)$$

Curves for these functions are plotted in Fig. 1. The effects of subtractions, of the pole term, and of the in-

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[1] V. Barger and D. Cline, Phys. Rev. Letters **16**, 913 (1966); H. Brody *et al.*, *ibid.* **16**, 828 (1966); J. Orear *et al.*, Phys. Rev. **152**, 1162 (1966); A. Ashmore *et al.*, Phys. Rev. Letters **19**, 460 (1967).

[2] G. F. Chew and J. D. Stack, University of California Lawrence Radiation Laboratory Report No. UCRL-16293, 1965 (unpublished); J. D. Stack, Phys. Rev. Letters **16**, 286 (1966); I. Sakmar, Nuovo Cimento **40**, 76 (1965).

[3] C. E. Jones and M. L. Goldberger, Phys. Rev. Letters **17**, 105 (1966); D. Z. Freedman and J.-M. Wang, *ibid.* **17**, 569 (1966).

tegral which depends on u only are discussed in the literature.[4]

To obtain the analytic and asymptotic properties of the partial-wave amplitudes in the angular-momentum plane, we expand

$$g(u,z)=A(s,u,t)-[\text{pole and const term}] \qquad (4)$$

in terms of $P_l(z)$ and then invert the expansion to obtain the partial-wave amplitudes. Furthermore, by interchanging the order of integration and by noticing that one of the integrals is a Legendre function of the second kind, we arrive at the following formula:

$$a(u,l)=\frac{1}{\pi}\int_{k_1}^{\infty} A_t(u,t')Q_l(z')dz' -\frac{1}{\pi}\int_{-k_0}^{\infty} A_s(u,s')Q_l(-z')dz'. \qquad (5)$$

The interchange of the order of integration is justified by Fubini's theorem[5] provided that the integrals in (5) are absolutely convergent. This requirement is satisfied by both integrals provided that $u>(M+\mu)^2$ and l is finite with $\mathrm{Re}l>N$, where N is determined by

$$A(u,z)\sim z^N \quad \text{as} \quad z\to\infty. \qquad (6)$$

The first integral remains finite even when we keep $\mathrm{Re}l>N$ and let $|\mathrm{Im}l|\to\infty$; in fact, it decreases exponentially in l. The second integral has an over-all factor of $(-1)^l$, which will cause an exponential increase as $\mathrm{Im}l$ goes to infinity. To avoid this difficulty we define for $u>(M+\mu)^2$, after Froissart and Gribov,[6] even- and odd-parity partial waves:

$$a^{\pm}(u,l)=\frac{1}{\pi}\int_{z_0}^{\infty}\{A_t(u,t')\pm A_s(u,s')\}Q_l(z')dz', \qquad (7)$$

where $z_0=\min(k_1,-k_0)$. We further define

$$A^{\pm}(u,z)=\sum_{l=0}^{\infty}(2l+1)a^{\pm}(u,l)P_l(z). \qquad (8)$$

The partial waves $a^{\pm}(u,l)$ are well defined by (7) for $\mathrm{Re}l>N$ and $u>(M+\mu)^2$; but they cannot be defined by the same formula for $\mathrm{Re}l\leq N$. We assume that there exists an analytic continuation of the Froissart-Gribov partial waves for $\mathrm{Re}l\leq N$ and such that the linear combinations $a^+(u,l)\pm a^-(u,l)$ reduce, for even and odd values of l, respectively, to the physical values of the

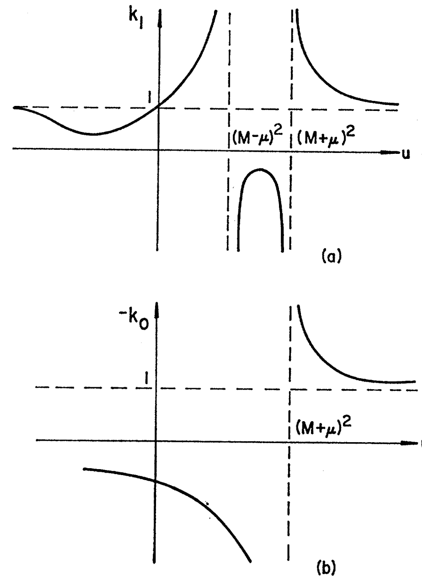

FIG. 1. Functions for the lower limits of integration in the fixed-u dispersion relations.

partial waves. We can now express the invariant amplitude in terms of $A^{\pm}(u,z)$:

$$g(u,z)=\tfrac{1}{2}[A^+(u,z)+A^-(u,z)] +\tfrac{1}{2}[A^+(u,-z)-A^-(u,-z)]. \qquad (9)$$

For $u>(M+\mu)^2$, the first term contains the right-hand cut and the second one contains the left-hand cut. In addition the first term is defined for $0<u<(M-\mu)^2$, but it is not obvious that it is identical there with the last integral in (1) because we have not established that it is the correct analytic continuation of the first term in (9).

Several authors have attempted to prove the analytic properties of the partial-wave amplitudes, but their attempts met only partial success.[7] For our particular case, the polynomial boundedness condition and (7) imply analytic properties and asymptotic behaviors which hold in restricted regions of the Mandelstam plane.[8]

For $u>(M+\mu)^2$ or $(M-\mu)^2>u>0$, and $\mathrm{Re}l>N$:

$$\text{(i)}\quad a^+(u,l)+a^-(u,l)\lesssim\frac{1}{l^{1/2}}e^{-\alpha_1 l} \quad \text{as} \quad l\to\infty,\quad k_1=\cosh\alpha_1; \qquad (10a)$$

[4] R. G. Moorhouse, *Strong Interactions and High Energy Physics* (Oliver and Boyd, London, 1963), p. 141; E. A. Paschos, thesis, Cornell University, 1967 (unpublished).

[5] D. V. Widder, *The Laplace Transform* (Princeton University Press, Princeton, N. J., 1941), p. 26.

[6] Euan J. Squires, *Complex Angular Momentum and Particle Physics* (W. A. Benjamin, Inc., New York, 1963), Chap. 3, and references given therein.

[7] V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 1962 (1961) [English transl.: Soviet Phys.—JETP **14**, 1395 (1962)]; K. Bardakci, Phys. Rev. **127**, 1832 (1962); A. O. Barut and D. E. Zwanziger, *ibid.* **127**, 974 (1962); S. Mandelstam, Ann. Phys. (N. Y.) **21**, 302 (1963).

[8] We do not include in this section attempts for improving the domains of validity by extracting the threshold factors.

(ii) it is an analytic function of l for $\mathrm{Re}l > N$.

For $u > (M+\mu)^2$

(i) $$a^+(u,l) - a^-(u,l) \lesssim \frac{1}{l^{1/2}} e^{-\alpha_0 l}$$
$$\text{as} \quad l \to \infty, \quad -k_0 = \cosh\alpha_0; \quad \text{(10b)}$$

(ii) it is an analytic function of l for $\mathrm{Re}l > N$.

Recent work by Hepp[9] has shown that the polynomial boundedness condition follows from Wightman theory. We can therefore consider (10) as a consequence of the Froissart-Gribov definition of partial waves and field theory.

In establishing Sommerfeld-Watson formulas we must postulate that the partial-wave amplitudes

(i) are meromorphic functions of l for $-\frac{1}{2} \leq \mathrm{Re}l \leq N$,

(ii) have the same asymptotic behaviors given by (10),

whenever u is in the restricted regions described above. With these postulates, $A^+(u,z)$ and $A^-(u,z)$ can be represented by Sommerfeld-Watson formulas in the region $-1 < \mathrm{Re}z < 1$, $\mathrm{Im}z = 0$. Furthermore, they provide, by analytic continuation, a representation for the amplitude at those points of the complex z plane, where both the background and the pole term are analytic functions of z.

We investigate the analytic properties of $A^+(u,z) + A^-(u,z)$ for $u > (M+\mu)^2$.

$$A^+(u,z) + A^-(u,z) = -\frac{1}{2i} \int_{-1/2-i\infty}^{-1/2+i\infty} (2l+1) \times [a^+(u,l) + a^-(u,l)] \frac{P_l(-z)}{\sin\pi l} dl - \pi \sum_{i=1}^{M} (2\alpha_i'+1)\beta_i'(u) \frac{P_{\alpha_i'}(-z)}{\sin\pi\alpha_i'}, \quad (11)$$

where $\alpha_i'(u)$, $\beta_i'(u)$ are the position and residue of the ith pole of the partial wave. We used primes to account for the singularities of both even- and odd-parity waves. The possible singular points are $1 < \mathrm{Re}z < \infty$, $\mathrm{Im}z = 0$. Not all of these points are singular, since for $1 < \mathrm{Re}z < k_1$, $\mathrm{Im}z = 0$ the cuts of the background and the pole term cancel each other. This is easily seen by adding to (11) the integral

$$\int_C (2l+1)[a^+(u,l) + a^-(u,l)] \frac{P_l(-z)}{\sin\pi l} dl, \quad (12)$$

C being the semicircle at infinity, shown in Fig. 2. This integral, as it is shown in the Appendix, is zero for

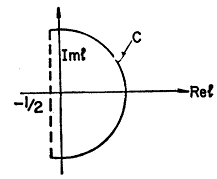

FIG. 2. Semicircle at infinity.

$1 < \mathrm{Re}z < k_1$, $\mathrm{Im}z = 0$. We next deform the contour of integration and write (11) as

$$A^+(u,z) + A^-(u,z) = -\frac{1}{2i} \int_{C_1} (2l+1) \times [a^+(u,l) + a^-(u,l)] \frac{P_l(-z)}{\sin\pi l} dl, \quad (13)$$

where C_1 is shown in Fig. 3. It follows now that the discontinuity of $A^+(u,z) + A^-(u,z)$ across $\mathrm{Im}z = 0$, $1 < \mathrm{Re}z < k_1$ is zero:

$$A^+(u, z+i\epsilon) + A^-(u, z+i\epsilon) - A^+(u, z-i\epsilon) - A^-(u, z-i\epsilon) = \int_{C_1} (2l+1) \times \left[a^+(u,l) + a^-(u,l) \right] P_l(z) dl = 0. \quad (14)$$

We conclude that $A^+(u,z) + A^-(u,z)$ can be represented by (11) in the entire z plane except for the segment $\mathrm{Re}z > k_1$, $\mathrm{Im}z = 0$. We note further that at those values of u in the Mandelstam plane where $k_1 < 1$ the analyticity argument given above will not hold.

We can similarly show that $A^+(u, -z) - A^-(u, -z)$ can be represented for $u > (M+\mu)^2$ by a Sommerfeld-Watson formula which contains the left-hand cut.

III. GENERAL IDEA

In this section we are going to expand $A^+(u,z) + A^-(u,z)$ in terms of $P_l[h(u)z + g(u)]$, where $h(u)$ and $g(u)$ are functions of u only. The idea is to choose $h(u)$ and $g(u)$ in such a way that $(hz+g)$ is large in some region of the crossed channel and also prove that under the Regge-type assumptions of the previous section, the new partial-wave amplitudes have similar properties in this region. Under such conditions we will certainly obtain Regge asymptotic behavior in this particular region of the crossed channel.

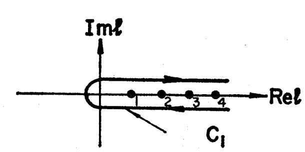

FIG. 3. Deformed contour of integration for Eqs. (13) and (14).

[9] K. Hepp, Helv. Phys. Acta 37, 639 (1964).

A suitable choice of functions for the unequal-mass case is

$$h(u)=\frac{q_u^2}{s_0},\quad g(u)=\frac{1}{s_0}\times\{[(M^2+q_u^2)(\mu^2+q_u^2)]^{1/2}+M^2-3\mu^2\},\quad (15)$$

$$h(u)z+g(u)=-\frac{1}{2s_0}(s-M^2-\mu^2)+\frac{M^2-3\mu^2}{s_0},$$

where s_0 is an arbitrary scaling factor having the dimensions of s. On any fixed-u line which passes through the backward strip there is always a segment where $-1<hz+g<1$. By a suitable choice of s_0 we can always find a part of this segment which lies outside the right-hand cut. For the line $u=0$, for example, the right-hand cut starts at $s=2(M^2-\mu^2)$ and by choosing $s_0=\frac{1}{4}M^2$ we obtain $hk_1+g=2(1-3\mu^2/M^2)$. Therefore, for $-1<hz+g<1$ the expansion

$$A^+(u,z)+A^-(u,z)=2\sum_{l=0}^{\infty}(2l+1)b(u,l)P_l(hz+g)\quad (16)$$

converges uniformly. Using the analyticity properties of $A^+(u,z)+A^-(u,z)$ we can write the dispersion relation

$$\tfrac{1}{2}[A^+(u,z)+A^-(u,z)]=\frac{1}{\pi}\int_{k_1}^{\infty}\frac{A_t(u,z')}{z'-z}dz'.\quad (17)$$

We can invert (16) to obtain

$$b(u,l)=\frac{1}{\pi}\int_{k_1}^{\infty}A_t(u,z)Q_l(hz+g)hdz.\quad (18)$$

For $u>-\frac{1}{2}M^2+3\mu^2$ and $\mathrm{Re}\,l>N$,

$$b(u,l)\lesssim\frac{1}{l^{1/2}}e^{-\beta_1 l}\quad\text{as}\quad |l|\to\infty,\quad hk_1+g=\cosh\beta_1;\quad (19)$$

in addition, it is an analytic function of l.

We can now proceed in two ways:
(1) We can either assume that $b(u,l)$ has "Regge-type" properties for $u>-\frac{1}{2}M^2+3\mu^2$ and $0<\mathrm{Re}\,l<N$ and write down a Sommerfeld-Watson transform, or
(2) we can assume Regge-type properties to hold for the partial waves of another representation (like the conventional Regge representation) for restricted regions of u, and then proceed to prove that for these particular regions of u the partial waves of the modified representation have Regge-type properties. In this approach, as it has been mentioned already in Sec. I, the Regge representation is useful to the extent that it allows us to find relations between Regge poles and singularities of the new partial waves. We proceed to establish these relations and then discuss the two procedures.

In the following theorems we need $A_t(u,z)$, which can be obtained from (11):

$$A_t(u,z)=D(u,z)+2\pi i\sum_{i=1}^{M}[2\alpha_i'(u)+1]\times\beta_i'(u)P_{\alpha'(u)}(z),\quad (20)$$

where $D(u,z)$ is the discontinuity of the background term, and it goes like $z^{-1/2}$ as $z\to\infty$, at those regions of u where (11) holds. Equation (20) is defined for $u>(M+\mu)^2$ and $(M-\mu)^2>u>0$. For $u>(M+\mu)^2$ it is identical with the discontinuity of (17), but for $(M-\mu)^2>u>0$ it has not been established that it gives the correct analytic continuation of the discontinuity of (17). The following three theorems[10] establish that $b(u,l)$ has Regge-type properties for u in the regions described above.

Theorem 1: For $u>(M+\mu)^2$ and $(M-\mu)^2>u>0$,

$$I_l(u)=\int_{k_1}^{\infty}A_t(u,z)Q_l(hz+g)hdz$$

defines a function analytic for $\mathrm{Re}\,l>\alpha'$, α' being the pole of $a^+(u,l)$ or $a^-(u,l)$ with the largest real part. Asymptotically it is bounded by

$$|I_l(u)|\leq\frac{\text{const}}{|l-N|\,l^{1/2}}[2(hk_1+g)^{-l}]\quad (21)$$

as $l\to\infty$ with $-\frac{1}{2}\pi-\delta<\arg l<\frac{1}{2}\pi+\delta$, $\delta>0$.

Proof: The analyticity follows from two facts:

1. $Q_l(hz+g)A_t(u,z)$ is analytic for $\mathrm{Re}\,l>\mathrm{Re}\,\alpha'$.
2. The integral is uniformly convergent.

The asymptotic estimate is obtained by expanding the Legendre polynomials in terms of a hypergeometric series and then estimating the integral.[11] In this estimate we notice that $A_t(u,z)(hz+g)^{-l}$ is bounded by $M(u)$

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[10] Similar theorems have been proven by N. N. Khuri, Phys. Rev. **132**, 914 (1963).

[11] In performing the estimate we use

$$Q_l(z)=\frac{\pi^{1/2}\Gamma(l+1)}{2^{l+1}\Gamma(l+\frac{3}{2})}\frac{1}{z^{l+1}}F\left(1+\tfrac{1}{2}l,\ \tfrac{1}{2}+\tfrac{1}{2}l,\ l+\tfrac{3}{2};\ \frac{1}{z^2}\right),$$

and the hypergeometric function can be expanded as

$$F\left(1+\tfrac{1}{2}l,\ \tfrac{1}{2}+\tfrac{1}{2}l,\ l+\tfrac{3}{2};\ \frac{1}{z^2}\right)=\sum_{b=0}b_n\frac{1}{z^{2n}}.$$

$\times(hz+g)^{N-l}$, where $M(u)$ is a bounded function of u:

$$|I_l(u)|\leq\frac{\pi^{1/2}}{2^{l+1}}\frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})}\,\text{const}\left|\sum_{n=0}^{\infty}b_n\int_{k_1}^{\infty}A_l(u,z)(hz+g)^{-l-2n-1}hdz\right|$$
$$\leq\frac{\pi^{1/2}}{2^{l+1}}\frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})}\frac{|M(u)|\,\text{const}}{|l-N|}\left|\sum_{n=0}^{\infty}\frac{b_n}{(hk_1+g)^{l+2n-N}}\right|$$
$$\leq\frac{|M(u)|}{|l-N|}\,\text{const}Q_l(hk_1+g)\to\frac{|M(u)|}{|l-N|\,l^{1/2}}\,\text{const}[2(hk_1+g)]^{-l}.\quad(22)$$

Theorem 2: For $u>(M+\mu)^2$ and $(M-\mu)^2>u>0$,

$$I_l^{(1)}(u)=\int_{k_1}^{\infty}Q_l(kz+g)D(z)hdz$$

is analytic for Re$l>-\frac{1}{2}$, and it is bounded by

$$I_l^{(1)}(u)\leq\frac{\text{const}}{l^{3/2}}[2(hk_1+g)]^{-l}\quad\text{as}\quad l\to\infty\,,$$

with $-\frac{1}{2}\pi-\delta<\arg l<\frac{1}{2}\pi+\delta$. The proof is similar to that of Theorem 1 and it is omitted.

Theorem 3: For $u>(M+\mu)^2$ and $(M-\mu)^2>u>0$,

$$I_l^{(2)}(u)=\int_{k_1}^{\infty}P_\alpha(z)Q_l(hz+g)hdz$$

is meromorphic for Re$l>-\frac{1}{2}$, with poles at $=\alpha$, $\alpha-1$, $\cdots$, $\alpha-n$, where $-\frac{1}{2}<\text{Re}(\alpha-n)<\frac{1}{2}$. Its asymptotic behavior is

$$I_l^{(2)}(u)\to\frac{\text{const}}{l^{\alpha+3/2}}[2(hk_1+g)]^{-l}\quad\text{as}\quad l\to\infty\,,\quad(23)$$

with $-\frac{1}{2}\pi-\delta<\arg l<\frac{1}{2}\pi+\delta$.

Proof: We expand $P_\alpha(z)$ in descending powers of z:

$$P_\alpha(z)=g_0z^\alpha+g_1z^{\alpha-2}+\cdots+g_nz^{\alpha-2n}+G_\alpha(z)\,.\quad(24)$$

n is determined by $\frac{3}{2}>\text{Re}(\alpha-2n)>-\frac{1}{2}$ and $G_\alpha(z)$ decreases at least as fast as $z^{-1/2}$ for $z\to\infty$. The integral containing $G_\alpha(z)$ has the same properties with $I_l^{(1)}(u)$. It is sufficient therefore to investigate one of the remaining integrals.

$$\int_{k_1}^{\infty}Q_l(hz+g)z^\alpha hdz$$
$$=\frac{\pi^{1/2}}{2^{l+1}}\frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})}\left\{\sum_{n=0}^{N}b_n\int_{k_1}^{\infty}\frac{z^\alpha hdz}{(hz+g)^{2n+l+1}}+\sum_{n=N+1}^{\infty}b_n\int_{k_1}^{\infty}\frac{z^\alpha hdz}{(hz+g)^{2n+l+1}}\right\}.\quad(25)$$

N is defined by $2N>\text{Re}\alpha>2(N-1)$. With this condition the second term in (25) is uniformly convergent, and it has as an asymptotic upper bound, the bound appearing in (21). There remains a finite number of terms like

$$\int_{k_1}^{\infty}\frac{z^\alpha hdz}{(hz+g)^{2n+l+1}}=\frac{1}{h^\alpha}\int_0^{\infty}\frac{(x+hk_1)^\alpha d(x)}{(x+hk_1+g)^{2n+l+1}}.\quad(26)$$

We can again split the integral in (26) into two terms: one behaving like $I_l^{(1)}(u)$ and a finite number of terms like

$$\int_0^{\infty}\frac{x^\alpha dx}{(x+hk_1+g)^{2n+l+1}}=(hk_1+g)^{\alpha-2n-l}\int_0^{\infty}\frac{t^\alpha dt}{(1+t)^{2n+l+1}}$$
$$=(hk_1+g)^{\alpha-2n-l}\frac{\Gamma(\alpha+1)\Gamma(2n+l-\alpha)}{\Gamma(2n+l+1)}.\quad(27)$$

The analytic properties of such a term are: For Re$l>-\frac{1}{2}$ there are poles at $l=\alpha-2n$, $\alpha-2n-1$, $\alpha-2n-2$, $\cdots$, $\alpha-2n-m$, where $-\frac{1}{2}<\text{Re}(\alpha-2n-m)<\frac{1}{2}$. The asymptotic behavior is

$$I_l^{(2)}(u)\to\frac{\text{const}}{l^{\alpha+3/2}}[2(hk_1+g)]^{-l}\quad(28)$$

as $l\to\infty$, with $-\frac{1}{2}\pi-\delta<\arg l<\frac{1}{2}\pi+\delta$.

From the properties of $b(u,l)$, which we have proved so far, we are guaranteed the existence of the following Sommerfeld-Watson formula for $(M+\mu)^2<u$:

$$\tfrac{1}{2}[A^+(u,z)+A^-(u,z)]$$
$$=-\frac{1}{2i}\int_{-1/2-i\infty}^{-1/2+i\infty}(2l+1)b(u,l)\frac{P_l(-hz-g)}{\sin\pi l}dl$$
$$-\pi\sum_i[2\gamma_i(u)+1]\delta_i(u)\frac{P_{\gamma_i}(-hz-g)}{\sin\pi\gamma_i},\quad(29)$$

where $\gamma_i(u)$ are the positions and $\delta_i(u)$ are, to within constant factors, the residues of the poles. Unfortunately, since (11) is not valid for $u\approx0$, we cannot use theorems 1–3 in establishing the analytic properties or the asymptotic behavior of $b(u,l)$ for $u\approx0$ and $-\frac{1}{2}<\text{Re}l<N$.

A similar analysis of the term corresponding to the left-hand cut shows that we can find similar Sommerfeld-

Watson formulas, except for the fact that their domain of validity is even smaller, namely, $u>(M+\mu)^2$. The analysis of this term proceeds in a way similar to the one we have already discussed. We define new partial waves:

$$b^{\pm}(u,l)=\frac{1}{\pi}\int_{k_1}^{\infty} A_t(u,t')Q_l(hz'+g)hdz' \pm\frac{1}{\pi}\int_{-k_0}^{\infty} A_s(u,s')Q_l(hz'+g')hdz', \quad (30)$$

with one difference: $g'(u)$ is chosen so that in the vicinity of $u=0$, as z' varies from $-k_0$ to ∞, $hz'+g'$ goes from a positive value larger than 1 to infinity. We again define

$$B^+(u,z)=2\sum_{l=0}^{\infty}(2l+1)b^+(u,l)P_l(hz+g') \quad (31)$$

and similarly $B^-(u,z)$. Now the partial waves appearing in (16) are given by

$$b(u,l)=b^+(u,l)+b^-(u,l)\,, \quad (32)$$

and the term corresponding to the left-hand cut by

$$A^+(u,-z)-A^-(u,-z)=B^+(u,-z)-B^-(u,-z)\,. \quad (33)$$

Finally, we discuss how the new representation may be useful in other regions of the Mandelstam plane. The conventional Regge representation has been established for restricted regions of the Mandelstam plane by proving that the partial-wave amplitudes can be defined for noninteger as well as complex l, with $\mathrm{Re}l>N$. For the same regions of the Mandelstam plane and $-\frac{1}{2}\leq\mathrm{Re}l\leq N$, one must assume meromorphicity and certain asymptotic behavior. One should not overlook several attempts[7] to prove the properties for $-\frac{1}{2}\leq\mathrm{Re}l\leq N$ using elastic unitarity, but these should be considered only as an approximation. A different approach would be to define new partial waves which are well defined in a larger region of the Mandelstam plane whenever $\mathrm{Re}l>N$ and then assume that they have for $-\frac{1}{2}\leq\mathrm{Re}l\leq N$ the same properties which are assumed for the ordinary Froissart-Gribov waves. This is not much more restrictive than what has been assumed so far, because if we will ever succeed in proving the properties of $a^{\pm}(u,l)$ for $-\frac{1}{2}\leq\mathrm{Re}l\leq N$ from a set of basic assumptions, then it seems very plausible that the same procedure will give a proof of the desirable properties of $b(u,l)$ for $-\frac{1}{2}\leq\mathrm{Re}l\leq N$. If we look at the problem from this point of view, then representation (29) is valid not only for $u>(M+\mu)^2$ but also for $u>-\frac{1}{2}M^2+3\mu^2$.

IV. DISCUSSION AND CONCLUSIONS

Under the assumptions of the Mandelstam representation and of two "Regge-type" assumptions, we have shown that we can obtain a new Regge representation which is defined in restricted regions of the Mandelstam plane, and it satisfies Mandelstam analyticity and Regge asymptotic behavior. In the regions of the Mandelstam plane, where the ordinary Regge representation is valid, the new representation follows rigorously from the existence of the ordinary Regge representation. In addition, we have established nontrivial relations between Regge poles and the singularities appearing in the new representation.

This approach suggests that we look at the problem from a different point of view. Although the definition of a function in terms of an expansion can fail outside its circle of convergence, in many cases the function can still be analytically continued by expanding around a new point in terms of a new variable. This is the underlying idea in changing the argument of the Legendre functions. In fact, if we naively assume the validity of the ordinary Sommerfeld-Watson transform for $u<0$ and then replace $P_\alpha(-\cos\theta_u)$ by its asymptotic expansion, which is justified there, we obtain a leading term, which aside from inessential factors, goes like $\beta_\alpha(u)(su)^\alpha$. The next term of the expansion goes like $\beta_\alpha(u)(su)^{\alpha-1}$ and so on. The factor u^α, of course, makes the asymptotic expansion invalid at $u\approx0$. In the new representation the factor u^α is again present, but now it is contained in the residue and not in $P_\alpha(hz+g)$.[12] This follows from (26) and (29).

Two other groups have recently published papers dealing with the same subject.[3] It should be emphasized that the new Sommerfeld-Watson formulas give the correct Mandelstam analyticity in the $(hz+g)$ plane, as it was demanded by Jones and Goldberger. Furthermore, our approach accounts, *within the limitations which we have already emphasized*, for the following model-independent results:

(1) Regge asymptotic behavior near the backward direction.

(2) It follows from (26) and (27) that every Regge pole will generate a sequence of satellite poles. The satellite trajectories are obtained by decreasing the real part of the Regge trajectory by units of one. A Regge pole of given signature generates trajectories with alternating signatures.

(3) The leading asymptotic term is

$$[-(s-M^2-\mu^2)/2s_0+(M^2-3\mu^2)/s_0]^{\alpha(u)}\,,$$

where $\alpha(u)$ is the conventional leading Regge pole.

(4) All our results are compatible with the cancellation mechanism of "daughter" trajectories conjectured by Freedman and Wang. It follows from (26) and (27) that the residue of the leading pole $\delta(u)$ has the following u dependence as $u\to0$:

$$\delta(u)\to \mathrm{const}\beta_\alpha(u)/h^\alpha(u)\to \mathrm{const}\beta_\alpha(u)u^{\alpha(u)}\,,$$

where $\beta_\alpha(u)$ is the residue of the leading conventional Regge pole. The analytic properties of $\delta(u)$ depend cri-

[12] I would like to thank Dr. J. Boccio for discussions concerning this point.

tically on those of $\beta_\alpha(u)$. Freedman and Wang[13] argued that $\beta_\alpha(u)u^{\alpha(u)}$ is analytic in the vicinity of $u=0$ and then conjectured a cancellation mechanism for the singularities of the remaining satellite poles. Such a mechanism can also be true in our case. But it should be pointed out that this conjecture could be misleading, because the analytic continuation of the Khuri amplitudes to $u=0$ is arbitrary.

(5) Our method can trivially be extended to cases with spin by replacing the Legendre functions by the d functions of a modified argument. Consequences and applications of this approach will be discussed elsewhere.

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APPENDIX: ESTIMATES OF INTEGRALS

In estimating the integrals we need asymptotic estimates of the Legendre functions. This estimate is obtained using

$$P_l(z)=F(1+l,\,-l,\,1;\,\tfrac{1}{2}-\tfrac{1}{2}z) \tag{A1}$$

and the asymptotic estimate of the hypergeometric function $F(a,b,c;\,z)$ given in Bateman's manuscript, Vol. I [p. 77, Eq. (17)][14,15]:

$$|P_l(z)|\le[f(z)/l^{1/2}]e^{|\mathrm{Re}\,l\,\mathrm{Re}\,\xi-\mathrm{Im}\,l\,\mathrm{Im}\,\xi|}\quad\text{as}\quad |l|\to\infty\,, \tag{A2}$$

where

$$z=\cosh\xi\,,$$

and, if $\xi=\eta+i\zeta$, where η, ζ are real, it will be supposed that $\eta\ge 0$, $-\pi\le\zeta\le\pi$. This formula holds for z in the exterior of an arbitrary closed curve which encloses the segment $[-1,\,1]$, l large and such that $-\frac{1}{2}\pi-\delta<\arg l<\frac{1}{2}\pi+\delta$, with $\delta>0$. For more details see Ref. 15.

We want to estimate

$$I(x\pm i\epsilon)=\int_C dl(2l+1)\frac{P_l(-x\mp i\epsilon)}{\sin\pi l}\times[a^+(u,l)+a^-(u,l)]\,, \tag{A3}$$

C being the semicircle at infinity shown in Fig. 2. Consider $I(x-i\epsilon)$ and use the upper bound given in (A2), and contour in Fig. 4.

$$I(x-i\epsilon)\le\text{const}\int_C dl\, l^{1/2}[k_1+(k_1^2-1)^{1/2}]^{-|\mathrm{Re}\,l|}\times|-x+i\epsilon+[(-x+i\epsilon)^2-1]^{1/2}|^{|\mathrm{Re}\,l|}\frac{e^{|\mathrm{Im}\,\xi||\mathrm{Im}\,l|}}{e^{\pi|\mathrm{Im}\,l|}}\,. \tag{A4}$$

For $\mathrm{Im}\,l\neq 0$ the integral goes to zero as $|l|\to\infty$ because of the fraction at the right-hand side. The only complication can arise when $\epsilon\to 0$, but since we first take the limit of $|l|\to\infty$ and then $\epsilon\to 0$ the integral is still zero. For $\mathrm{Im}\,l=0$ and $x<k_1$ the integral again goes to zero because of the remaining exponentials. The same argument can be repeated for $I(x+i\epsilon)$.

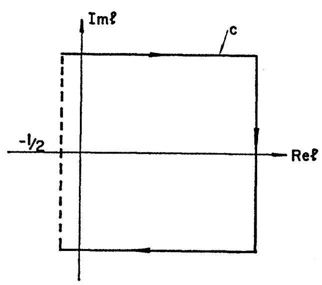

FIG. 4. Rectangle at infinity.

[13] D. Z. Freedman and J.-M. Wang, Phys. Rev. 153, 1596 (1967).

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