Field-Theoretic Formulation of the Bootstrap Hypothesis*

P. KAUS
University of California, Riverside, California 92501

AND

F. ZACHARIASEN
California Institute of Technology, Pasadena, California 91109

(Received 7 March 1968)

A precise statement of the bootstrap theory in the language of conventional Lagrangian field theory is given. Under no further assumptions than the Mandelstam representation, it is shown that this closed-form theory is equivalent to the more usual S-matrix formulation in terms of Regge trajectories. The assumptions of the bootstrap theory are stated in terms of the vanishing of well-defined renormalization constants Z. All standard results and approximation methods in the bootstrap theory are shown to follow directly from the $Z=0$ conditions.

1. INTRODUCTION

It is our purpose in this paper to present, to the extent we can at present, a complete and systematic development of the bootstrap theory of strongly interacting particles, couched in the language of conventional Lagrangian field theory. In doing this, we do not mean to imply that this is in any way a different bootstrap theory from that usually expressed in the language of S-matrix theory and Regge poles[1]; to the contrary, we are convinced the theories are the same. Nevertheless, we feel it of some value, in the light of the limited practical success of bootstrap calculations phrased in S-matrix language, to try to make available different, albeit equivalent, ways to say the same thing, in the hope that more useful approximate methods, for some bootstrap problems at least, will suggest themselves. In addition, there may be some virtue in using the language of field theory to formulate the bootstrap idea, because the formulation in terms of S-matrix theory is at present incomplete. The assumptions of S-matrix theory can still not be stated precisely; one is still required to make assumptions like "all S-matrix elements are as analytic as possible consistent with unitarity," which while useful intuitively and meaningful for simple cases are still not completely satisfying. Lagrangian field theory, on the other hand, has the aura of being well-defined, even if in fact there are many points of dubious mathematical validity in it.

Our plan is, or rather should be, the following. We first define, in any Lagrangian field theory, the wave function and vertex renormalization constants Z associated with any given elementary particle. We then show that if the (physical) mass and coupling constant of this particle are made to vary until the renormalization constants vanish (assuming that this can be done), then the particle will lie on a Regge trajectory; it has thus been made composite according to the criterion of compositeness assumed in the usual S-matrix bootstrap approach.[2] Having established that the vanishing of the renormalization constants is an acceptable definition of compositeness, we next systematically state a set of rules defining the bootstrap theory.

Briefly stated, these rules are:

(1) Assume a trial world—this means to assume the existence of a given set of elementary particles, with specified masses and spins.

(2) Write down a Lagrangian field theory for these particles—this means we assume particular interactions among the particles, with certain coupling constants.

(3) Solve, by whatever methods, this field theory. That is, calculate all S-matrix elements, including all bound or resonance states which may occur, all Regge trajectories, and anything else of interest, in terms of the parameters describing the field theory—that is, in terms of the masses and various coupling constants. In particular, calculate the renormalization constants of the elementary particles.

(4) Finally, choose the values of the various parameters so that all the renormalization constants vanish, if this can be done. If this cannot be done, the assumed world cannot bootstrap itself, and it must be rejected. If it can be done, then the assumed elementary particles have all been moved onto trajectories, have become dynamical, and all the parameters—masses and coupling constants —have been determined. We now have a bootstrapped world.

It is, of course, assumed in the above outline, that there are worlds for which a bootstrap solution of this type exists; in other words, that solutions to the equations $Z=0$ do exist. On the other hand, it is also assumed that the equations $Z=0$ are not identities in the various parameters. We can, as of now, prove

* Work supported in part by the U. S. Atomic Energy Commission. Prepared under Contract Nos. AT(11-1)—68 and AT(11-1)—34 for the San Francisco Operations Office, U. S. Atomic Energy Commission.

[1] G. F. Chew and S. Frautschi, Phys. Rev. Letters **7**, 394 (1961).

[2] The relevance of renormalization constants to compositeness and bootstraps was first discussed by A. Salam, Nuovo Cimento **25**, 224 (1962).

neither of these two statements, though it is of course to be hoped that proofs will be found sometime in the future.

We next turn to a discussion of uniqueness of the bootstrap solution, assuming that it exists. What we are concerned with here is not so much the question of how many bootstrapped worlds exist (for this question we have no answer), but rather the question of which sets of particles can we choose as the "elementary" particles of our trial world, and what kind of interaction Lagrangians can we write down? Suppose, for example, that we try some set of particles and some interactions. Then suppose that in step (3) we discover the existence of several bound states made up of these particles. Then suppose we took a new trial world which included among its "elementary" particles some particles with the same properties as the bound states of the first world. When we apply step (4) to both worlds, do we or do we not get to the same bootstrapped world? Intuitively, it is our belief that we should. It should not matter in which order the Z's of various particles are made to vanish; thus, if in the second trial world we first set the Z's of the added particles equal to zero, we expect to obtain the first trial world. Then, if the remaining Z's vanish, we have the bootstrap limit of the first trial world. But, if we set *all* Z's equal to zero at once in the second trial world, then we have the bootstrap limit of the second world. The two bootstrap limits should clearly be identical.

Therefore we have some freedom in guessing worlds. If we want to look for a world with a given set of particles, we do not have to take *all* of these particles as "elementary" input particles in our trial world; any subset of them having the remainder as bound (or resonant) states will do.

We have a further freedom in the choice of interactions in the Lagrangian of our trial world. Many different interactions, of the Yukawa and more complicated types, with or without derivative couplings, are consistent with Lorentz invariance and the space-time properties of our trial set of "elementary" particles. Again, it is our feeling that the final bootstrap limit should not depend on which form of interaction we assume. (Though, of course, if we allow non-Yukawa couplings, such as $\lambda\pi^4$ for π mesons, for example, then some additional conditions besides $Z=0$ will have to be imposed to reach the bootstrap limit. That is, exactly what the dynamical conditions are may depend on the form of the interaction.) Once all particles have been made composite, then the various interactions among them are effectively all prescribed, so that after the dynamical conditions have been imposed, the resulting theory will be the same regardless of which couplings we assumed among the prebootstrapped elementary particles.

Even though all this freedom is, we believe, in principle available to us, we shall at this stage in our understanding, nevertheless, have to impose certain restrictions on the particles we choose as "elementary" and on the form of the interaction. These restrictions result from the fact that, at present, we do not know how to handle nonrenormalizable field theories. (We do not know what "Z" means for a high-spin particle, for example.) Therefore we shall confine our choices of elementary particles to those of spin 0 or $\frac{1}{2}$, and we shall confine our choices of interactions to Yukawa couplings without gradients.

We hasten to emphasize that these restrictions do not prevent us from writing down trial bootstrap theories for any conceivable world. Any high-spin particles can be considered to be a bound system made of various spin-0 and spin-$\frac{1}{2}$ particles; therefore, a bootstrapped world containing high spins can be made from a trial world containing "elementary" low-spin particles, in the limit where these low-spin particles are made composite through $Z=0$.

It is undeniable that this selection for special treatment of spin 0 and spin $\frac{1}{2}$ is aesthetically unappealing, even though in the final bootstrap limit all particles, of whatever spin, do presumably appear on an equal footing. However, we reiterate our belief that this is not a basic requirement of the Lagrangian field theory $Z=0$ approach to the bootstrap; it is rather a reflection of our present ignorance of how to deal with high-spin elementary particles. Once we learn how to define Z for a high-spin particle in a nonrenormalizable field theory, then in the limit $Z=0$ the high-spin particle becomes composite, and when this happens the renormalization difficulties with the field theory disappear also.[3] Thus we may, with some confidence, look forward to a day when the restriction of our trial elementary particles to spin 0 and spin $\frac{1}{2}$ may be lifted. Similar remarks apply to the kinds of couplings we can use.

The outline of the remainder of the paper is the following: In Sec. 2 we define the various quantities, including the Z's, which will be of interest later. These definitions are made both in the field-theory language and, so far as possible, in the S-matrix language. Section 3 is devoted to demonstrating, for both spin-0 and spin-$\frac{1}{2}$ particles, that setting the Z's to zero moves the particle onto a Regge trajectory, so that by the usually accepted definition it becomes composite. Section 4 repeats the discussion of Sec. 3 for multiplets of particles with the same quantum numbers. In Sec. 5 the bootstrap rules are stated, and finally, in Sec. 6, we discuss several simple applications of the $Z=0$ conditions to practical bootstrap problems.

2. DEFINITIONS

In the Lagrangian field theory of spin-0 and/or spin-$\frac{1}{2}$ particles, the concepts of the renormalized proper vertex function and renormalized propagator are well defined.

[3] In fact, that renormalization difficulties must disappear when $Z\to 0$ is a convenient criterion to use in searching for an acceptible definition of Z.

Since these functions involve the behavior of off-mass-shell particles, they do not need to be defined in S-matrix theory, and indeed some proponents of S-matrix theory might say they have no meaning. Nevertheless, we believe they can be defined even in the language of S-matrix theory. Let us illustrate this for the case of spinless particles; the spin-$\frac{1}{2}$ case does not differ in any essential way.

Suppose we have two kinds of spinless particles, which we may call π and σ, and a Lagrangian field theory of these particles with the interaction

$$\mathcal{L}_I = -g_0^{\sigma\pi\pi}[\hat{\pi}(x)]^2\hat{\sigma}(x)\,, \tag{2.1}$$

where $\hat{\pi}$ and $\hat{\sigma}$ are unrenormalized Heisenberg fields for the two particles. It is conventional in the field theory to define the unrenormalized propagator for the σ particle by

$$\Delta_\sigma'(s) = -i\int \langle 0|(\hat{\sigma}(x),\hat{\sigma}(0))_+|0\rangle e^{iq\cdot x}d^4x\,, \tag{2.2}$$

where $s=q^2$. This function has a pole at $s=m_\sigma^2$, the physical mass of the σ, and we *define* the wave function renormalization constant Z_σ as the residue at this pole. We can also show that as $s\to\infty$, $s\Delta_\sigma'(s)\to 1$.

The renormalized Heisenberg field is now defined by

$$\sigma(x) = Z_\sigma^{-1/2}\hat{\sigma}(x) \tag{2.3}$$

and the renormalized propagator by

$$\Delta_\sigma(s) = -i\int \langle 0|(\sigma(x),\sigma(0))_+|0\rangle e^{iq\cdot x}d^4x = Z_\sigma^{-1}\Delta_\sigma'(s)\,. \tag{2.4}$$

Thus $\Delta_\sigma(s)$ has a pole at $s=m_\sigma^2$ with residue unity, and $s\Delta_\sigma(s)\to Z_\sigma^{-1}$ as $s\to\infty$.

Analogous definitions may be made for the π.

The vertex function may be defined in various ways; for example, as the sum of all proper vertex diagrams. A perhaps more convenient definition for our purposes involves first defining the form factor by

$$F_{\sigma\pi\pi}(s,m_\pi^2,m_\pi^2) = (4\omega'\omega)^{1/2}\langle \mathbf{p}'|(\Box+m_\sigma^2)\sigma(x)|\mathbf{p}\rangle|_{x=0}\,, \tag{2.5}$$

where $s=(p'-p)^2$ and $\omega=(\mathbf{p}^2+m_\pi^2)^{1/2}$. This is the form factor for two on-shell pions and an off-shell σ. We can also define form factors for an off-shell π coupled to an on-shell π and to an on-shell σ by

$$F_{\sigma\pi\pi}(m_\sigma^2,s,m_\pi^2) = (4E\omega')^{1/2}\langle \mathbf{p}'|(\Box+m_\pi^2)\pi(x)|\mathbf{q}\rangle|_{x=0}\,, \tag{2.6}$$

where $s=(p'-q)^2$ and $E=\mathbf{q}^2+m_\sigma^2$. One can also take more than one of the particles off-shell. The renormalized coupling constant $g_{\sigma\pi\pi}$ is defined to be

$$g_{\sigma\pi\pi} = F_{\sigma\pi\pi}(m_\sigma^2,m_\pi^2,m_\pi^2)\,. \tag{2.7}$$

The usual definition of the vertex renormalization constant is now

$$(g_{\sigma\pi\pi}/g_0^{\sigma\pi\pi}) = Z_\pi Z_\sigma^{1/2}/Z_{\sigma\pi\pi}\,. \tag{2.8}$$

However, we prefer another definition, which is generally believed to be equivalent to Eq. (2.8), but which has not been proved to be equivalent.[4] This is that

$$F_{\sigma\pi\pi}(s,m_\pi^2,m_\pi^2) \to g_{\sigma\pi\pi}Z_{\sigma\pi\pi}/Z_\sigma \tag{2.9}$$

as $s\to\infty$. Several things are implicit in this definition. First, that F does indeed approach a constant as $s\to\infty$. Second, that $Z_\sigma F_{\sigma\pi\pi}(s,m_\pi^2,m_\pi^2)$, $Z_\pi F_{\sigma\pi\pi}(m_\sigma^2,s,m_\pi^2)$, and $Z_\pi F_{\sigma\pi\pi}(m_\sigma^2,m_\pi^2,s)$ all approach the *same* constant as $s\to\infty$.[5] We shall accept all these things as true, and also accept the identity of the two definitions of $Z_{\sigma\pi\pi}$.

From now on, let us replace the notation

$$F_{\sigma\pi\pi}(s,m_\pi^2,m_\pi^2)$$

simply by $F_{\sigma\pi\pi}(s)$. If the occasion arises to talk about situations with one or more of the other particles off the mass shell, it will be made clear at the time.

We may now define the vertex function. We write

$$\Gamma_{\sigma\pi\pi}(s) = F_{\sigma\pi\pi}(s)/(s-m_\sigma^2)\Delta_\sigma(s)\,. \tag{2.10}$$

Thus we have $\Gamma_{\sigma\pi\pi}(m_\sigma^2)=g_{\sigma\pi\pi}$ and $\Gamma_{\sigma\pi\pi}(s)\to g_{\sigma\pi\pi}Z_{\sigma\pi\pi}$ as $s\to\infty$. A perturbation expansion of Γ defined in this way also shows that this is precisely the sum of all proper vertex diagrams, with the two π's on the mass shell and the σ off in this case.

One can show that the propagator satisfies a dispersion relation of the form

$$\Delta_\sigma(s) = \frac{1}{s-m_\sigma^2} + \frac{1}{\pi}\int \frac{\mathrm{Im}\Delta_\sigma(s')}{s'-s}ds'\,, \tag{2.11}$$

with a "unitarity relation"

$$\begin{aligned}\mathrm{Im}\Delta_\sigma(q^2) &= -\tfrac{1}{2}\sum_n (2\pi)^4\delta^4(p_n-q)|\langle n^{(-)}|\sigma(0)|0\rangle|^2 \\ &= -\tfrac{1}{2}(q^2-m_\sigma^2)^{-2}\sum_n (2\pi)^4\delta^4(p_n-q) \\ &\quad\times|\langle n^{(-)}|(\Box+m_\sigma^2)\sigma(x)|0\rangle|^2|_{x=0}\,.\end{aligned} \tag{2.12}$$

Similarly, it is reasonable to expect that F satisfies a dispersion relation

$$F_{\sigma\pi\pi}(s) = g_{\sigma\pi\pi} + \frac{s-m_\sigma^2}{\pi}\int \frac{\mathrm{Im}F_{\sigma\pi\pi}(s')}{(s'-m_\sigma^2)(s'-s)}ds'\,, \tag{2.13}$$

with a unitarity relation

$$\begin{aligned}\mathrm{Im}F_{\sigma\pi\pi}(q^2) &= -\tfrac{1}{2}\sum_n (2\pi)^4\delta^4(p_n-q)(2\omega')^{1/2} \\ &\quad\times\langle p'|(\Box+m_\pi^2)\pi(x)|_{x=0}|n^{(+)}\rangle \\ &\quad\times\langle n^{(+)}|(\Box+m_\sigma^2)\sigma(x)|_{x=0}|0\rangle\,.\end{aligned} \tag{2.14}$$

[4] G. Källén, Helv. Phys. Acta **25**, 417 (1952); M. Gell-Mann and F. Zachariasen, Phys. Rev. **126**, 2201 (1962) (Appendix A).

[5] This amounts to saying that the unrenormalized vertex function approaches unity whenever *any* virtual mass goes to infinity.

The contribution to (2.12) and (2.14) from 2π intermediate states alone is simply

$$\mathrm{Im}\Delta_\sigma(s)=\rho(s)|F_{\sigma\pi\pi}(s)|^2/(s-m_\sigma^2)^2 \qquad (2.15)$$

and

$$\mathrm{Im}F_{\sigma\pi\pi}(s)=\rho(s)t(s)F_{\sigma\pi\pi}^*(s)\,, \qquad (2.16)$$

where $t(s)$ is the s-wave $\pi\pi$ scattering amplitude. $t(s)$ is thus an on-shell quantity directly accessible to measurement. $\rho(s)$ is a phase-space factor:

$$\rho(s)=(1/16\pi)[(s-4m_\pi^2)/s]^{1/2}. \qquad (2.17)$$

More generally, $\langle p'|(\Box+m_\pi^2)\pi(x)|_{x=0}|n^{(+)}\rangle$ is an on-shell scattering amplitude, representing the process $n\to\pi\pi$, and $\langle n^{(-)}|(\Box+m_\sigma^2)\sigma(x)|_{x=0}|0\rangle$ is a generalization of the form factor itself which we may label $F_{\sigma n}/(\prod_n 2E_n)^{1/2}$. A unitarity relation for this more general form factor then takes the form

$$\mathrm{Im}F_{\sigma n}=\sum_m t_{nm}F_{\sigma m}^*. \qquad (2.18)$$

Dispersion relations for the $F_{\sigma n}$ thus have the form of linear integral equations, the kernels of which involve physically measurable S-matrix elements. All the $F_{\sigma n}$ can therefore, in principle, be constructed in the language of S-matrix theory, if one so chooses. In the same way, Eq. (2.12) says

$$\mathrm{Im}\Delta_\sigma=\sum_n|F_{\sigma n}|^2/(s-m_\sigma^2)^2\,, \qquad (2.19)$$

so that the propagator too can be constructed, entirely within the framework of S-matrix theory.

To summarize, we do not need to use field theory to define the renormalization constants. The generalized form factors are solutions of integral equations which require as input only on-shell information[6]; the propagator is an integral over the squares of such form factors, the vertex is the ratio of the form factor to the propagators, and the Z's, finally, are defined in terms of the asymptotic behavior of the vertex and the propagator. If all this seems a bit too symbolic, may we remind the reader that S-matrix theory itself, as a method of calculating on-shell amplitudes, consists of little more than the same sorts of words.

We may conclude this section by defining a mass operator, which is sometimes of some convenience, and by writing dispersion relations for it and for the vertex function itself. Define

$$M_\sigma(s)=s-1/\Delta_\sigma(s)\,. \qquad (2.20)$$

Then near $s=m_\sigma^2$, $M_\sigma(s)=m_\sigma^2+O((s-m_\sigma^2)^2)$, and $\mathrm{Im}M_\sigma(s)=\mathrm{Im}\Delta_\sigma(s)/|\Delta_\sigma(s)|^2$. Thus we have

$$M_\sigma(s)=m_\sigma^2+\frac{(s-m_\sigma^2)^2}{\pi}\int\frac{\mathrm{Im}M_\sigma(s)}{(s'-m_\sigma^2)^2(s'-s)}ds' \qquad (2.21)$$

[6] Continued below threshold in some cases.

and

$$\mathrm{Im}M_\sigma(q^2)=-\tfrac{1}{2}\sum_n(2\pi)^4\delta^4(p_n-q)\times\left|\frac{\langle n^{(-)}|(\Box+m_\sigma^2)\sigma(x)|_{x=0}|0\rangle}{(q^2-m_\sigma^2)\Delta_\sigma(q^2)}\right|^2. \qquad (2.22)$$

The contribution to $\mathrm{Im}M_\sigma$ from the 2π state alone is simply

$$\mathrm{Im}M_\sigma(s)=\rho(s)|\Gamma_{\sigma\pi\pi}(s)|^2. \qquad (2.23)$$

Next, we have

$$\Gamma_{\sigma\pi\pi}(s)=g_{\sigma\pi\pi}+\frac{s-m_\pi^2}{\pi}\int\frac{\mathrm{Im}\Gamma_{\sigma\pi\pi}(s')}{(s'-m_\sigma^2)(s'-s)}ds'. \qquad (2.24)$$

The unitarity relation for Γ can be derived directly from the definition of Γ, Eq. (2.10). We obtain, from the $\pi\pi$ intermediate state alone,

$$\mathrm{Im}\Gamma_{\sigma\pi\pi}(s)=\rho(s)[t(s)-\Gamma_{\sigma\pi\pi}(s)\Delta_\sigma(s)\Gamma_{\sigma\pi\pi}(s)]\Gamma_{\sigma\pi\pi}^*(s)\,. \qquad (2.25)$$

Finally, from Eqs. (2.21) and (2.24) we can obtain expressions for the Z's. We recall that $\Gamma_{\sigma\pi\pi}(s)\to g_{\sigma\pi\pi}Z_{\sigma\pi\pi}$, and note that $M_\sigma(s)\to s(1-Z_\sigma)$, as $s\to\infty$. Thus we have

$$Z_\sigma=1+\frac{1}{\pi}\int\frac{\mathrm{Im}M_\sigma(s')}{(s'-m_\sigma^2)^2}ds' \qquad (2.26)$$

and

$$Z_{\sigma\pi\pi}=1-\frac{1}{\pi}\int\frac{\mathrm{Im}\Gamma_{\sigma\pi\pi}(s')}{s'-m_\sigma^2}ds'. \qquad (2.27)$$

As we shall see in Sec. 6, when we use these equations to calculate the Z's in the elastic unitarity approximation, both M and Γ have poles when the Z's are sufficiently small. Thus $\mathrm{Im}M$ and $\mathrm{Im}\Gamma$ contain, in general, δ-function terms.

In writing Eqs. (2.26) and (2.27), an implicit assumption is made that the integrals exist. In fact, this assumption is essential to our entire approach, for without it, there are no Z's which we can set equal to zero. For the case explicitly under discussion here of completely spinless particles, the assumption is valid even in each order of perturbation theory. For the case where spin-$\frac{1}{2}$ particles are included (which, as we have said, goes through formally in essentially the same manner), the assumption is *not* true in perturbation theory. We can only trust that the actual Z's are finite in spite of this.

3. $Z=0$ AND COMPOSITENESS

The next step in our development is to show that, for a given particle, setting the wave function and vertex renormalization constants equal to zero moves the particle smoothly onto a Regge trajectory, and removes the Kronecker δ terms in the angular momentum from the scattering amplitude.[7] Let us again begin with the case of spinless particles[8]; the spin-$\frac{1}{2}$ situation will be described at the end of this section.

For the π-σ model described in Sec. 2, assume that we may write a Mandelstam representation for the $\pi\pi$ scattering amplitude. Judging by the perturbation expansion, no subtractions are required, so that

$$\begin{aligned}T(s,t)=&-\frac{g^2}{s-m_\sigma^2}-\frac{g^2}{t-m_\sigma^2}\\&+\frac{1}{\pi}\int\frac{f(s')}{s'-s}ds'+\frac{1}{\pi}\int\frac{f(t')}{t'-t}dt'\\&+\frac{1}{\pi^2}\int\int\frac{\rho(s',t')}{(s'-s)(t'-t)}ds'dt'\\&+\frac{1}{\pi^2}\int\int\frac{\bar\rho(t',u')}{(t'-t)(u'-u)}dt'du'\\&+\frac{1}{\pi^2}\int\int\frac{\bar\rho(u',s')}{(u'-u)(s'-s)}du'ds',\quad(3.1)\end{aligned}$$

where $T(s,t)$ is the $\pi\pi$ scattering amplitude. In using this unsubtracted form, we are explicitly confining ourselves to the Yukawa interaction of Eq. (2.1). If we permitted, in addition, a contact interaction of the type

$$\mathfrak{L}_I'=\tfrac{1}{4}\lambda_0[\hat\pi(x)]^4\quad(3.2)$$

for example, then the representation for $T(s,t)$ would require an over-all subtraction, and the subtraction constant $T(s_0,t_0)$ would be identified with the renormalized coupling constant λ. We choose to exclude this type of interaction and, as stated in the Introduction, confine ourselves entirely to Yukawa couplings.[9]

A partial-wave decomposition of $T(s,t)$ may be made in the usual way. The pole and single integral in t as well as the double integral terms all yield partial-wave amplitudes which have analytic continuations into the

[7] P. Kaus and F. Zachariasen, Phys. Rev. **138**, B1304 (1965); T. Saito, *ibid.* **145**, 1302 (1966). The first of these papers uses the condition $Z_1/Z_3\rightarrow 0$, $Z_3\rightarrow 0$. The second, using the same method, shows that $Z_1\rightarrow 0$, $Z_3\rightarrow 0$ is sufficient. Contrary to some authors, and as will be clear from the text and the example in the Appendix, the final solution is always the same regardless of the order in which the Z's are made to vanish.

[8] For the case of spinless particles, the use of $Z=0$ as a compositeness condition has been studied in the context of a number of models and approximations. For a recent well-referenced review of the subject, we refer the reader to K. Hayashi, M. Hirayama, T. Muta, N. Seto, and T. Shirafuji, Fortschr. Physik **15**, 126 (1967). What is new in the present paper is the derivation of the conditions with no restriction other than the Mandelstam analyticity, as well as a treatment of the spin-$\frac{1}{2}$ case and the complete treatment of several particles with the same quantum numbers, in a way that makes a formulation of the bootstrap problem possible.

[9] Later in this section, when we introduce spin-$\frac{1}{2}$ particles, an interaction of the type (3.2) will be necessary for the renormalizability of the theory. We then need an extra condition besides $Z=0$, which will be set s_0, $t_0\rightarrow\infty$ and $\lambda\rightarrow 0$. This reduces the subtracted Mandelstam representation to the unsubtracted form, so that the argument given here may be used unchanged.

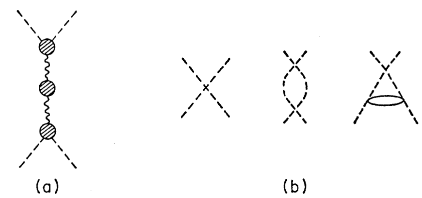

FIG. 1. (a) Feynman diagrams included in the s-channel pole and single integral. The dashed line represents π, the wavy line σ. (b) Additional diagrams present if a π^4 interaction exists.

l plane and contain no Kronecker δ's in l; the only s-channel poles appearing in these terms are Regge poles. The pole and single integral in s, in contrast, produces a δ_{l0} term in the partial-wave amplitude. Thus at this stage, the σ particle shows up as a fixed pole at $s=m_\sigma^2$, in a Kronecker δ-type term.

Now the pole and single-integral terms in Eq. (3.1) are nothing more than the Feynman diagrams shown in Fig. 1(a), so that we may write

$$\begin{aligned}T(s,t)=&-\Gamma_{\sigma\pi\pi}(s)\Delta_\sigma(s)\Gamma_{\sigma\pi\pi}(s)-\Gamma_{\sigma\pi\pi}(t)\Delta_\sigma(t)\Gamma_{\sigma\pi\pi}(t)\\&+(\text{double-integral terms}),\quad(3.3)\end{aligned}$$

where Γ and Δ are the functions defined in Sec. 2. Note that if there were subtractions, this identification would not in general be possible. For example, with the interaction of Eq. (3.2), diagrams of the sort shown in Fig. 1(b) would be included in the single-integral terms and these are not of the form written in Eq. (3.3).

Let us next try to write down forms for Γ and Δ incorporating the few facts we know about them, namely,

$$\begin{aligned}&\Gamma_{\sigma\pi\pi}(m_\sigma^2)=g\,,\quad \Gamma_{\sigma\pi\pi}(s)\xrightarrow[s\rightarrow\infty]{} Z_{\sigma\pi\pi}g\,,\\&(s-m_\sigma^2)\Delta_{\sigma\pi\pi}(s)|_{m_\sigma^2}=1\,,\quad s\Delta_{\sigma\pi\pi}(s)\xrightarrow[s\rightarrow\infty]{} Z_\sigma^{-1}.\end{aligned}\quad(3.4)$$

First, write

$$\Gamma_{\sigma\pi\pi}(s)=\frac{Z_{\sigma\pi\pi}g_{\sigma\pi\pi}}{[Z_{\sigma\pi\pi}+(s-m_\sigma^2)\alpha(s)]}.\quad(3.5)$$

Then we have

$$\alpha(s)=-\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\left(\frac{\mathrm{Im}\Gamma_{\sigma\pi\pi}(s')}{\Gamma_{\sigma\pi\pi}(\infty)}\right)\times\left|\frac{\Gamma_{\sigma\pi\pi}(\infty)}{\Gamma_{\sigma\pi\pi}(s)}\right|^2\frac{ds'}{(s'-m_\pi^2)(s'-s)}.\quad(3.6)$$

As $Z_{\sigma\pi\pi}\rightarrow 0$, i.e., as $\Gamma_{\sigma\pi\pi}(\infty)\rightarrow 0$, the scale of $\Gamma_{\sigma\pi\pi}(s)$ changes, but the scale change factors out in the ratio $\Gamma_{\sigma\pi\pi}(s)/\Gamma_{\sigma\pi\pi}(\infty)$. Hence there is no reason to expect any remarkable behavior of $\alpha(s)$ in this limit.

Second, write

$$\Delta_\sigma(s)=(1/Z_\sigma)[Z_\sigma/(s-m_\sigma^2)+\gamma(s)];\quad(3.7)$$

again nothing significant should be expected to happen to γ as $Z_\sigma\rightarrow 0$.

We can combine Eqs. (3.5) and (3.7) to write a representation for the form factor

$$F_{\sigma\pi\pi}(s)=\frac{Z_{\sigma\pi\pi}g_{\sigma\pi\pi}}{Z_{\sigma\pi\pi}+(s-m_\sigma^2)[Z_\sigma\beta_1(s)+Z_{\sigma\pi\pi}\beta_2(s)]}, \quad (3.8)$$

where

$$\beta_1(s)=\alpha(s)/[Z_\sigma+(s-m_\sigma^2)\gamma(s)] \quad (3.9)$$

and

$$\beta_2(s)=-\gamma(s)/[Z_\sigma+(s-m_\sigma^2)\gamma(s)]. \quad (3.10)$$

For convenience, we may also define

$$Z_\sigma\beta=Z_\sigma\beta_1+Z_{\sigma\pi\pi}\beta_2, \quad (3.11)$$

so that

$$F_{\sigma\pi\pi}(s)=\frac{Z_{\sigma\pi\pi}g_{\sigma\pi\pi}}{[Z_{\sigma\pi\pi}+(s-m_\sigma^2)Z_\sigma\beta(s)]}. \quad (3.12)$$

Because of the asymptotic behavior of $\Gamma_{\sigma\pi\pi}$ and $F_{\sigma\pi\pi}$, we will have $\alpha\to(1/s)(1-Z_{\sigma\sigma\pi})$, $\beta_1\to 1/s$, and $\beta_2\to -1/s$, so that $\beta\to(1/s)[1-(Z_{\sigma\pi\pi}/Z_\sigma)]$ as $s\to\infty$.

Now let us suppose $Z_{\sigma\pi\pi}$ is small. Then $\Gamma_{\sigma\pi\pi}(s)$ will develop a pole at s_0 near m_σ^2, where

$$s_0=m_\sigma^2-Z_{\sigma\pi\pi}/\alpha(s_0)\approx m_\sigma^2-Z_{\sigma\pi\pi}/\alpha(m_\sigma^2). \quad (3.13)$$

This pole will be called the vertex pole. $F_{\sigma\pi\pi}(s)$ may or may not develop a pole near m_σ^2 when $Z_{\sigma\pi\pi}$ becomes small, depending on the ratio $Z_{\sigma\pi\pi}/Z_\sigma$. However, when this ratio becomes small, then $F_{\sigma\pi\pi}(s)$ will have a pole at $\bar{s}_0$, where

$$\bar{s}_0=m_\sigma^2-Z_{\sigma\pi\pi}/Z_\sigma\beta(\bar{s}_0)=m_\sigma^2-Z_{\sigma\pi\pi}/Z_\sigma\beta(m_\sigma^2). \quad (3.14)$$

In the Appendix we solve a very simple model which displays all the quantities just discussed and serves as a convenient guide to see how the various limits work.

Regardless of whether $F_{\sigma\pi\pi}(s)$ has a pole near m_σ^2 or not, it in any case does not have a pole at s_0. Therefore the vertex pole at s_0 does not represent a physical particle and is not a pole of $T(s,t)$; hence the double-integral terms in T must also contain a pole at s_0 which just cancels the pole in the single integral term. This pole in the double integral, of course, must lie on a Regge trajectory. We shall refer to this pole as the "compensating pole."

Let us now write the single-integral terms for small $Z_{\sigma\pi\pi}$ near $s=m_\sigma^2$:

$$-\Gamma_{\sigma\pi\pi}(s)\frac{1}{s-m_\sigma^2}F_{\sigma\pi\pi}(s)\approx-\frac{g_{\sigma\pi\pi}Z_{\sigma\pi\pi}/\alpha(m_\sigma^2)}{s-s_0}\frac{1}{s-m_\sigma^2}\frac{g_{\sigma\pi\pi}Z_{\sigma\pi\pi}/Z_\sigma\beta(m_\sigma^2)}{s-m_\sigma^2+Z_{\sigma\pi\pi}/Z_\sigma\beta(m_\sigma^2)}$$

$$=-\frac{g_{\sigma\pi\pi}^2}{s-m_\sigma^2}+\frac{1}{s-s_0}\left[\frac{g_{\sigma\pi\pi}^2\alpha(m_\sigma^2)}{\alpha(m_\sigma^2)-Z_\sigma\beta(m_\sigma^2)}\right]+\frac{g_{\sigma\pi\pi}^2}{(s-m_\sigma^2)Z_\sigma\beta(m_\sigma^2)+Z_{\sigma\pi\pi}}\left[\frac{Z_\sigma^2\beta^2(m_\sigma^2)}{Z_\sigma\beta(m_\sigma^2)-\alpha(m_\sigma^2)}\right]. \quad (3.15)$$

Now let $Z_{\sigma\pi\pi}\to 0$ and $Z_\sigma\to 0$ in any order. First, note that the third term, corresponding to the possibly existing pole at $\bar{s}_0$ of $F_{\sigma\pi\pi}(s)$, will simply vanish. To realize this, it is helpful to notice that while $\beta(s)$ contains terms proportional to $(Z_{\sigma\pi\pi}/Z_\sigma)$, and hence will become infinite if Z_σ vanishes first, $Z_\sigma\beta(m_\sigma^2)$ will remain finite. We recall from Eq. (3.11) that $Z_\sigma\beta$ is of the form $Z_\sigma\beta_1(m_\sigma^2)+Z_{\sigma\pi\pi}\beta_2(m_\sigma^2)$.

The vertex pole at s_0 goes to m_σ^2 in the limit, and its residue approaches $+g^2$; thus the first two terms in Eq. (3.15) cancel, and the whole single-integral term vanishes. The compensating pole in the double-integral term, however, which must cancel the second term in Eq. (3.15), has moved to $s_0=m_\sigma^2$ during the limiting process, and its residue has become $-g^2$. The σ particle has thus been transferred to a Regge trajectory and has become dynamical.

It is also obvious from Eq. (3.15) that while there is no difference in the end result, there is a difference in the behavior of the third term (which corresponds to an actual pole of the physical amplitude at $\bar{s}_0$ when such a pole exists), depending on whether $Z_{\sigma\pi\pi}$ goes to zero first and then Z_σ, or vice versa.

Since $Z_\sigma\beta(m_\sigma^2)=Z_\sigma\beta_1(m_\sigma^2)+Z_{\sigma\pi\pi}\beta_2(m_\sigma^2)$, we see that the third term in Eq. (3.15) has a pole at

$$\bar{s}_0=m_\sigma^2-Z_{\sigma\pi\pi}/(Z_\sigma\beta_1+Z_{\sigma\pi\pi}\beta_2), \quad (3.16)$$

with a residue

$$\bar{R}_0=g_{\sigma\pi\pi}^2(Z_\sigma\beta_1+Z_{\sigma\pi\pi}\beta_2)/(Z_\sigma\beta_1+Z_{\sigma\pi\pi}\beta_2-\alpha). \quad (3.17)$$

If $Z_{\sigma\pi\pi}/Z_\sigma$ vanishes, this pole moves in with the other two poles at m_σ^2 with a residue

$$\bar{R}_0\to g^2Z_\sigma\beta_1/(Z_\sigma\beta_1-\alpha). \quad (3.18)$$

This residue then vanishes with Z_σ. If $Z_\sigma/Z_{\sigma\pi\pi}$ goes to zero, however, this pole moves to

$$\bar{s}_0=m_\sigma^2-1/\beta_1, \quad (3.19)$$

with a residue

$$\bar{R}_0\to g^2Z_{\sigma\pi\pi}\beta_2/(Z_{\sigma\pi\pi}\beta_2-\alpha) \quad (3.20)$$

which vanishes with $Z_{\sigma\pi\pi}$.

In either case, the third term in Eq. (3.15) represents a bound state of the system at $\bar{s}_0$, which becomes an "extinct" bound state in the limit $Z_{\sigma\pi\pi}\to 0$, $Z_\sigma\to 0$, but not necessarily at m_σ^2, depending on the value of the ratio $Z_{\sigma\pi\pi}/Z_\sigma$.

Let us now specialize this general discussion to the elastic unitarity approximation. This means that we confine ourselves to $\pi\pi$ intermediate states only in the s channel. The elastic approximation has the virtue that the various quantities introduced above can be explicitly represented in terms of various D functions, whose properties are well known. Thus we can assure ourselves, at least within this model, that nothing of importance has been overlooked in the general discussion just completed, and that the assorted functions such as α and β introduced there do in fact have the properties assumed.

In the elastic unitarity approximation, we can represent the partial wave amplitude near $l=0$ as[10]

$$T(s,l)=\frac{\bar{N}(s,l)}{\bar{D}(s,l)}=\frac{N(s)}{D(s)}-g_{\sigma\pi\pi}{}^2\frac{D(m_\sigma{}^2)\bar{D}(m_\sigma{}^2)}{D(s)(s-m_\sigma{}^2)\bar{D}(s)}\delta_{l0}. \quad (3.21)$$

The first term on the right of Eq. (3.21) represents the contribution of the double-integral term and is unitary as well as analytic in l. We can now identify

$$\begin{aligned}&\Gamma_{\sigma\pi\pi}(s)=g_{\sigma\pi\pi}D(m_\sigma{}^2)/D(s)\,,\\ \text{therefore}\quad &D(m_\sigma{}^2)=Z_{\sigma\pi\pi};\\ &F_{\sigma\pi\pi}(s)=g_{\sigma\pi\pi}\bar{D}(m_\sigma{}^2)/\bar{D}(s)\,,\\ \text{therefore}\quad &\bar{D}(m_\sigma{}^2)=Z_{\sigma\pi\pi}/Z_\sigma;\end{aligned} \quad (3.22)$$

and

$$\Delta_\sigma(s)=\left(\frac{1}{s-m_\sigma{}^2}\right)\frac{\bar{D}(m_\sigma{}^2)}{\bar{D}(s)}\frac{D(s)}{D(m_\sigma{}^2)}=\left(\frac{1}{s-m_\sigma{}^2}\right)\frac{D(s)}{Z_\sigma\bar{D}(s)}.$$

We may now expand $D(s)$ and $\bar{D}(s)$ around $m_\sigma{}^2$ and find

$$\begin{aligned}\Gamma_{\sigma\pi\pi}(s)&=\frac{Z_{\sigma\pi\pi}g_{\sigma\pi\pi}}{Z_{\sigma\pi\pi}+(s-m_\sigma{}^2)D'(m_\sigma{}^2)+\cdots}\,,\\ F_{\sigma\pi\pi}(s)&=\frac{(Z_{\sigma\pi\pi}/Z_\sigma)g_{\sigma\pi\pi}}{Z_{\sigma\pi\pi}/Z_\sigma+(s-m_\sigma{}^2)\bar{D}'(m_\sigma{}^2)+\cdots}.\end{aligned} \quad (3.23)$$

Again, we may say that the fact that $\bar{D}(s)$ is a function which has no poles in s on the physical sheet, but is equal to $Z_{\sigma\pi\pi}/Z_\sigma$ at $s=m_\sigma{}^2$ and approaches 1 as $s\to\infty$, assures us that we can write $\bar{D}(s)$ as

$$\bar{D}(s)=\bar{d}_1(s)+Z_{\sigma\pi\pi}/Z_\sigma\bar{d}_2(s)\,, \quad (3.24)$$

where

$$\begin{aligned}\bar{d}_1(s)&=0, \quad s=m_\sigma{}^2\\ &=1, \quad s=\infty\end{aligned}$$

and

$$\begin{aligned}\bar{d}_2(s)&=1, \quad s=m_\sigma{}^2\\ &=\frac{ic}{s}, \quad s=\infty\,.\end{aligned}$$

It follows that $\bar{D}'(m_\sigma{}^2)=\bar{d}_1'(m_\sigma{}^2)+Z_{\sigma\pi\pi}/Z_\sigma\bar{d}_2'(m_\sigma{}^2)$, so that

$$F_{\sigma\pi\pi}(s)=\frac{Z_{\sigma\pi\pi}g_{\sigma\pi\pi}}{Z_\sigma+(s-m_\sigma{}^2)[Z_\sigma\bar{d}_1'(m_\sigma{}^2)+Z_{\sigma\pi\pi}\bar{d}_2'(m_\sigma{}^2)]+\cdots} \quad (3.25)$$

[10] We use the notation of Ref. 7.

near $s=m_\sigma{}^2$. These quantities are displayed explicitly in the model of the Appendix.

The propagator may be written

$$\begin{aligned}\Delta_\sigma(s)&=(s-m_\sigma{}^2)^{-1}[F_{\sigma\pi\pi}(s)/\Gamma_{\sigma\pi\pi}(s)]\\ &=1/s-m_\sigma{}^2\frac{Z_{\sigma\pi\pi}+(s-m_\sigma{}^2)\alpha(s)}{Z_{\sigma\pi\pi}+(s-m_\sigma{}^2)Z_\sigma\beta(s)}\,,\end{aligned} \quad (3.26)$$

where $\alpha(s)=[D(s)-Z_{\sigma\pi\pi}]/(s-m_\sigma{}^2)$ and

$$Z_\sigma\beta(s)=\{Z_\sigma\bar{d}_1(s)+Z_{\sigma\pi\pi}[\bar{d}_2(s)-1]\}/(s-m_\sigma{}^2)\,.$$

Thus we see that the renormalized propagator always blows up as $Z_{\sigma\pi\pi}\to 0$ and $Z_\sigma\to 0$. In addition, in the limit, the entire vertex function $\Gamma_{\sigma\pi\pi}(s)$ vanishes, the form factor $F_{\sigma\pi\pi}(s)$ either vanishes if $(Z_{\sigma\pi\pi}/Z_\sigma)\to 0$ or remains finite and goes asymptotically to $Z_{\sigma\pi\pi}/Z_\sigma$ when $Z_{\sigma\pi\pi}/Z_\sigma$ remains a constant, or remains finite but goes asymptotically as $s^{1/2}$ when $Z_{\sigma\pi\pi}/Z_\sigma\to\infty$. This can be seen most easily from the model in the Appendix; however, we expect this behavior to be true quite generally, independent either of the explicit model or the elastic approximation.

The *unrenormalized* vertex and propagator, defined by

$$\begin{aligned}\Gamma_{\sigma\pi\pi}'(s)&=(1/Z_{\sigma\pi\pi})\Gamma_{\sigma\pi\pi}(s)\,,\\ \Delta_\sigma'(s)&=Z_\sigma\Delta_\sigma(s)\,,\end{aligned} \quad (3.27)$$

remain finite with $\Delta_\sigma'(s)$, however, vanishing when $Z_{\sigma\pi\pi}/Z_\sigma\to\infty$. In the limit, the pole of $\Gamma_{\sigma\pi\pi}'(s)$ moves to $s=m_\sigma{}^2$, and $\Delta_\sigma'(s)$ either loses its pole at $m_\sigma{}^2$ when $Z_{\sigma\pi\pi}/Z_\sigma$ becomes constant or keeps it when $Z_{\sigma\pi\pi}/Z_\sigma\to 0$.

The final amplitude is always the same in the limit $Z_{\sigma\pi\pi}\to 0$ and $Z_\sigma\to 0$, even though the functions $\bar{D}(s)$ and $\bar{N}(s)$ differ, depending on $Z_{\sigma\pi\pi}/Z_\sigma$. This ratio affects the position of $\bar{s}_0$, which is a zero of $\bar{D}(s)$. In the limit, however, a zero of $N(s)$ always moves to the position $\bar{s}_0$, whether $\bar{s}_0=s_0$ $(Z_{\sigma\pi\pi}/Z_\sigma\to 0)$ or not. In $\bar{N}/\bar{D}=T$, the pole of $\bar{s}_0$ becomes extinct in any case, leaving only a dynamic pole at $s_0=m_\sigma{}^2$.

Finally, we see that the bare coupling constant, defined by

$$g_0{}^{\sigma\pi\pi}=g_{\sigma\pi\pi}(Z_{\sigma\pi\pi}/Z_\sigma{}^{1/2}Z_\pi)\,, \quad (3.28)$$

behaves in an unknown way depending on the ratio $Z_{\sigma\pi\pi}/Z_\sigma{}^{1/2}$.

This completes our discussion of the spinless case. We have argued quite generally that in the limit $Z_{\sigma\pi\pi}\to 0$ and $Z_\sigma\to 0$ the σ particle ceases to be a fixed pole associated with a Kronecker δ in l, and becomes a Regge pole in an amplitude entirely free of Kronecker δ's. Thus it becomes dynamical. The crucial point in the derivation was that the Mandelstam representation for the scattering amplitude which contained the elementary σ particle pole required no subtractions. When we proceed to the case of spin, there are, in general, several invariant amplitudes for a given scattering process. If a particular scattering process has a pole corresponding to some elementary particle, it will normally appear in more than one of the invariant amplitudes. If any one of these amplitudes in which the pole appears requires no subtractions, then the single integral and pole terms may be identified with $\Gamma\Delta\Gamma$, where Δ is the propagator for the elementary particle, and Γ is the vertex function coupling the elementary particle to the scattering particles. The derivation used in the spinless case may be carried through essentially unchanged, and the dynamical situation is reached when the wave-function renormalization constant vanishes and the vertex renormalization constant, defined in terms of Γ at infinite energy, vanishes.

As we remarked in the Introduction, the case of high spin, that is of a nonrenormalizable field theory, requires an infinite number of subtractions in the Mandelstam representation, so there are unlikely to be any unsubtracted invariant amplitudes. Let us therefore make the above discussion explicit for the case of spin $\frac{1}{2}$.

To proceed with the spin-$\frac{1}{2}$ case, let us introduce a new particle in addition to the π and σ of our previous discussion, which we may call a "nucleon" and label N. Let us take the π and σ to be pseudoscalar and scalar, respectively, and assume an interaction of them with N of the form[11]

$$\mathcal{L}_1 = g_0^{\pi NN}\hat{N}(x)\gamma_5\hat{\pi}(x)\hat{N}(x) + g_0^{\sigma\pi\pi}\hat{N}(x)\hat{\sigma}(x)\hat{N}(x) \quad (3.29)$$

added to our earlier interaction.

Let us first look at the nucleon pole in π-N scattering. There are two invariant amplitudes $A(s,t)$ and $B(s,t)$ and, as usual, the entire amplitude is

$$T = (\bar{u}_{p'}|A(s,t) + \tfrac{1}{2}B(s,t)(q_1'+q_1)|u_p). \quad (3.30)$$

Now, judging by the perturbation expansion with the interaction of Eq. (3.29), the A amplitude requires a subtraction in s and one in u, but the B amplitude requires no subtractions, and the nucleon pole occurs only in the B amplitude. The form of the single integral and pole terms in B may consequently be calculated in terms of the nucleon vertex. Let us write the π-N vertex, for an off-shell N, as

$$\Gamma = \Gamma_1\gamma_5 + \Gamma_2\gamma_5(P-M).$$

Here P is the four-momentum of the off-shell N; Γ_1 and Γ_2 are functions of $s=P^2$. As $s\to\infty$, we have $\Gamma_1 \to Z_{\pi NN}$ and $\Gamma_2 \to 0$ and of course $\Gamma_1(M^2) = g_{\pi NN}$. The propagator for the nucleon may be broken up in a similar way:

$$S = S_1(P+M) + S_2.$$

Thus S_1 contains a pole at $s=M^2$, with residue 1, and S_2 does not.

In terms of these quantities, we may express the pole and single integral in B as

$$|\Gamma_1|^2S_1 + (\Gamma_1^*\Gamma_2 + \Gamma_2^*\Gamma_1)S_2 + |\Gamma_2|^2(S_1(s-M^2) - 2MS_2).$$

[11] See Ref. 9.

The nucleon pole is contained in the first term here, which is of the same form as in the spinless case. As in the spinless case, in the limit $Z_{\pi NN}\to 0$, $Z_N \to 0$, a compensating pole develops, and will end up at $s=M^2$ with a residue $g_{\pi NN}^2$. However, it is not obvious at first glance that this pole is on a Regge trajectory, because we apparently still have the Kronecker δ terms associated with Γ_2 and S_2 present in B, as well as the other terms analytic in the angular-momentum plane. We must therefore ask how these terms behave as $Z_{\pi NN}\to 0$.

The most direct argument is that, as in the spinless case, it is the unrenormalized vertex $\Gamma' = \Gamma_1'\gamma_5 + \Gamma_2'\gamma_5(P-M)$ which remains finite. We may, as a result, say that both Γ_1 and Γ_2 vanish, since $\Gamma_1 = Z_1\Gamma_1'$ and $\Gamma_2 = Z_2\Gamma_2'$. This can be confirmed if we look at the elastic unitarity approximation. The D function for the ${}^2P_{1/2}$ channel, normalized to 1 at infinity, may be related to the nucleon vertex as in the spinless case:

$$\Gamma_1(s) + (W-M)\Gamma_2(s) = g_{\pi NN}D(M)/D(W).$$

Again, we have $Z_{\pi NN} = D(M)$. Now, as $Z_{\pi NN}\to 0$, it is clear that not only Γ_1 but also Γ_2 vanishes.

In the limit, then, the Kronecker δ terms other than $\Gamma_1^*S_1\Gamma_1$ in the B amplitude vanish. The argument given in Sec. 3 for the spinless case thus may be carried over unchanged to the B amplitude; in the limit $Z_{\pi NN}\to 0$ and $Z_N\to 0$, the nucleon moves into a Regge trajectory and B loses all Kronecker δ's.

The single integrals in the A amplitude, while they do not contain a particle, nevertheless contribute Kronecker δ's to the partial-wave amplitude. However, those single integrals exist not only because of the $\Gamma S\Gamma$ term but also because the double integrals in A required, as far as we could see from perturbation theory, subtractions. Now that the nucleon has been made dynamical, the perturbation theory no longer has meaning, and things are presumably more convergent than before. It is then quite possible that in the $Z_{\pi NN}\to 0$, $Z_N\to 0$ limit, the double integrals in A will need no subtractions as well, so the single integrals will vanish in the limit. (On the other hand, since the pion is still elementary, perhaps we should not expect *all* Kronecker δ's in the amplitude to have vanished.)

In the same vein, while the nucleon is made dynamical by $Z_{\pi NN}\to 0$ and $Z_N\to 0$, in that it is moved onto a Regge trajectory, the introduction of the coupling in Eq. (3.29) apparently undoes the argument we made earlier to show that the σ became dynamical when $Z_{\sigma\pi\pi}\to 0$ and $Z_\sigma\to 0$. This is because, with this coupling, the $\pi\pi$ scattering amplitude now requires an overall subtraction; in other words, the theory is no longer renormalizable without an interaction of the form of Eq. (3.2). Thus the Kronecker δ's do not arise only from diagrams included in $\Gamma\Delta\Gamma$ and hence making $\Gamma\Delta\Gamma$, vanish does not eliminate all Kronecker δ's. Therefore, while it remains true that in the limit the σ particle has become the "compensating pole," this pole is not now manifestly a Regge pole. The compensating pole occurs in the sum of the double integral and t-channel single integral, which are smooth in l, *plus* the remainder of the s-channel single integrals not included in $\Gamma\Delta\Gamma$, and this last is *not* smooth in l. Thus the compensating pole could, for all we know, still be a Kronecker δ, unless the additional condition discussed in Ref. 8 is imposed.

The final questions to be asked, to complete the $J^P=0^+,\ 0^-,\ \frac{1}{2}^+$ world, concern the N pole in σN scattering and the π and σ poles in $N\bar{N}$ scattering.

As to the first, let us look at the σN scattering process. This is again described in terms of two scalar amplitudes A and B, as in Eq. (3.30). As in the πN case, A requires subtractions while B does not. The only difference is that the nucleon pole now occurs in both A and B instead of only in B. Nevertheless, the previous argument goes through unchanged. Look at B alone. Since it requires no subtractions, the Kronecker δ term contains a pole and will vanish as $Z_{\sigma NN}\to 0$. Thus the compensating pole is necessarily again a Regge pole, and hence in the limit the nucleon moves to a trajectory.

Exactly the same remarks apply to $N\bar{N}$ scattering; the Kronecker δ term in the unsubtracted amplitudes (of which there are three of the five that exist altogether) is again entirely expressible in terms of $\Gamma\Delta\Gamma$, but now $\Gamma=\Gamma_1\gamma_5$ or $\Gamma=\Gamma_1 1$ only; for an off-shell π or σ rather than nucleon there is just one vertex function, and as $s\to\infty$, $\Gamma_1\to Z_{\pi NN}$, or $Z_{\sigma NN}$, according to which particles we are discussing.

To conclude, in a world containing σ, π, and N, with a Lagrangian given by Eq. (3.29), we find that in all channels in which it occurs N is made composite (i.e., made to lie on a Regge trajectory) by the conditions $Z_N=0$, $Z_{\sigma NN}=0$, $Z_{\pi NN}=0$; similarly, π is made composite by $Z_\pi=0$, $Z_{\pi NN}=0$, $Z_{\sigma\pi\pi}=0$, and, finally, σ is composite if $Z_\sigma=0$, $Z_{\sigma NN}=0$, and $Z_{\sigma\pi\pi}=0$. Presumably, then, *all* are made composite if all six separate conditions hold.

4. MULTIPLETS WITH IDENTICAL QUANTUM NUMBERS

The $Z=0$ conditions can be generalized directly to the case of multiplets with the same quantum numbers. Consider n π-particles described by fields π_i, $i=1\cdots n$ and N σ-particles described by fields σ_α, $\alpha=1\cdots N$, with the interaction term of the Lagrangian given by

$$\mathcal{L}_I(x)=-\sum_{i,j=1}^{n}\sum_{\alpha=1}^{N}(g_0)_{ij}{}^{\alpha}\pi_i(x)\pi_j(x)\sigma_\alpha(x). \quad (4.1)$$

Let us call the mass of the σ_α particles μ_α instead of $m_{\sigma\alpha}$ to reduce the number of indices and avoid confusion, and the mass of the π_i particles m_i.

We write the renormalized propagator as

$$\Delta_{\alpha\beta}(q^2)=-i\int d^4x\, e^{iq\cdot x}\langle 0|(\sigma_\alpha(x),\sigma_\beta(0))_+|0\rangle$$

$$=\frac{\delta_{\alpha\beta}}{s-\mu_\alpha^2}+\frac{1}{\pi}\int\frac{\mathrm{Im}\Delta_{\alpha\beta}(s')}{s'-q^2}ds'. \quad (4.2)$$

Then $s\Delta_{\alpha\beta}(s)\to Z^{-1}{}_{\alpha\beta}$ as $s\to\infty$, and $Z_{\alpha\beta}$ is a real symmetric matrix. We define the form factor by

$$F_{ij}{}^{\alpha}(s)=(4\omega_j'\omega_i)^{1/2}\langle \mathbf{p}',j|(\Box+\mu_\alpha^2)\sigma_\alpha(x)|_{x=0}|\mathbf{p},i\rangle, \quad (4.3)$$

where

$$s=(p'-p)^2,\quad \omega_i=(\mathbf{p}^2+m_i^2)^{1/2},$$

and

$$\omega_j'=(\mathbf{p}'^2+m_j^2)^{1/2}.$$

These definitions are equivalent to the commutation rule

$$[\sigma_\alpha(\mathbf{x},t),\sigma_\beta(\mathbf{y},t)]=iZ^{-1}{}_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y}) \quad (4.4)$$

and the relation

$$\sigma_\alpha(x)=\sum_\beta Z^{-1}{}_{\alpha\beta}\hat{\sigma}_\beta(x). \quad (4.5)$$

The vertex renormalization constant is defined by

$$F_{ij}{}^{\alpha}(s)\xrightarrow[s\to\infty]{}\sum_\beta Z^{-1}{}_{\alpha\beta}Z_{ij}{}^{\beta}g_{ij}{}^{\beta}. \quad (4.6)$$

The vertex function itself is defined by the equation

$$F_{ij}{}^{\alpha}(s)=\sum_\beta(s-\mu_\alpha^2)\Delta_{\alpha\beta}(s)\Gamma_{ij}{}^{\beta}(s), \quad (4.7)$$

from which it follows that, as $s\to\infty$,

$$\Gamma_{ij}{}^{\alpha}(s)\to Z_{ij}{}^{\alpha}g_{ij}{}^{\alpha}. \quad (4.8)$$

The single-integral term in the Mandelstam representation for the $\pi_i\pi_j\to\pi_k\pi_l$ amplitude $T_{ij,kl}(s,t)$ is written

$$-\sum_{\alpha,\beta}\Gamma_1(s)_{ij}{}^{\alpha}\Delta_1(s)_{\alpha\beta}\Gamma_1(s)_{kl}{}^{\beta}, \quad (4.9)$$

and since

$$\sum_\beta\Delta_1(s)_{\alpha\beta}\Gamma_1(s)_{kl}{}^{\beta}=(s-\mu_\alpha^2)^{-1}F(s)_{kl}{}^{\alpha}, \quad (4.10)$$

the single-integral term is

$$-\sum_\alpha\Gamma_1(s)_{ij}{}^{\alpha}(s-\mu_\alpha^2)^{-1}F(s)_{kl}{}^{\alpha}. \quad (4.11)$$

The vertex function $\Gamma_1(\mu_\alpha^2)_{ij}{}^{\alpha}$ must be equal to $g_{ij}{}^{\alpha}$, and $F_{kl}{}^{\alpha}(\mu_\alpha^2)=g_{kl}{}^{\alpha}$. This permits, in the neighborhood of $s=\mu_\alpha^2$, the representations analogous to (3.5) and (3.8) for $\Gamma_1(s)_{ij}{}^{\alpha}$ and $F(s)_{kl}{}^{\alpha}$:

$$\Gamma_1(s)_{ij}{}^{\alpha}=Z_{ij}{}^{\alpha}g_{ij}{}^{\alpha}/[Z_{ij}{}^{\alpha}+(s-\mu_\alpha^2)\alpha(s)_{ij}{}^{\alpha}] \quad (4.12)$$

and

$$F(s)_{kl}{}^{\alpha}=\frac{Z_{kl}{}^{\alpha}g_{kl}{}^{\alpha}}{Z_{kl}{}^{\alpha}+(s-\mu_\alpha^2)Z_{kl}{}^{\alpha}[\beta_1(s)_{ij}{}^{\alpha}/\sum_\beta Z^{-1}{}_{\alpha\beta}Z_{kl}{}^{\beta}g_{kl}{}^{\beta}+\beta_2(s)_{ij}{}^{\alpha}]}. \quad (4.13)$$

Arguments very similar to the case discussed in Sec. 3 shows that all the σ particles become dynamical, provided we require

$$Z_{ij}{}^{\alpha}g_{ij}{}^{\alpha}=0, \quad \text{for all } i,\ j,\ \alpha \tag{4.14}$$

and

$$g_{ij}{}^{\alpha}Z_{ij}{}^{\alpha}/\sum_{\beta} Z^{-1}{}_{\alpha\beta}g_{ij}{}^{\beta}Z_{ij}{}^{\beta}=0\,, \quad \text{for all } i,\ j,\ \alpha. \tag{4.15}$$

The second condition can also be written

$$\frac{\det Z_{\alpha\beta}}{\sum_{\beta} Z_{\alpha\beta}{}^{\text{cof}}(g_{ij}{}^{\beta}Z_{ij}{}^{\beta}/g_{ij}{}^{\alpha}Z_{ij}{}^{\alpha})}=0\,, \tag{4.16}$$

where $Z_{\alpha\beta}{}^{\text{cof}}$ is the cofactor matrix. From this we see that $\det Z_{\alpha\beta}=0$, in general, satisfies the second condition. However, while $\det Z_{\alpha\beta}=0$ is sufficient to rid the $\pi\pi$ scattering amplitudes of Kronecker δ's, it is not in fact sufficient to remove all the elementary σ particles; to accomplish this, we must require that all eigenvalues of the matrix $Z_{\alpha\beta}$ vanish, that is, that the entire matrix $Z_{\alpha\beta}$ vanishes.

To see this, let us simplify the problem to a single channel by restricting ourselves to a single pion, i.e., we drop the indices i, j, etc. The bootstrap conditions, Eqs. (4.14) and (4.15), become

$$Z^{\alpha}=0, \quad \text{for all } \alpha \tag{4.17}$$

and

$$X^{\alpha}=0, \quad \text{for all } \alpha\,, \tag{4.18}$$

where we define

$$X^{\alpha}=g^{\alpha}Z^{\alpha}/\sum_{\beta} Z^{-1}{}_{\alpha\beta}g^{\beta}Z^{\beta}. \tag{4.19}$$

From Eq. (4.19) it follows that

$$\sum_{\beta} Z_{\alpha\beta}V_{\beta}=X^{\alpha}V_{\alpha}\,, \tag{4.20}$$

where

$$V_{\alpha}=g^{\alpha}Z^{\alpha}/X^{\alpha}=F^{\alpha}(\infty)\,. \tag{4.21}$$

Since the form factors for different indices α are all the same up to a scale, we have

$$F^{\alpha}(s)/F^{\beta}(s)=F^{\alpha}(\infty)/F^{\beta}(\infty) \tag{4.22}$$

for all s; thus Eq. (4.20) may also be written

$$\sum_{\beta} Z_{\alpha\beta}F^{\beta}(s)=X^{\alpha}F^{\alpha}(s)\,. \tag{4.23}$$

Now the dynamical limit, as expressed by Eq. (4.18), leads to the result

$$\sum_{\beta} Z_{\alpha\beta}F^{\beta}(s)=0; \tag{4.24}$$

and this, as we shall see next, requires that all $Z_{\alpha\beta}$ vanish if we are to remove all the elementary σ particles. To exhibit the physical reason involved, it is convenient to look for a moment at the case of no mixing, where $Z_{\alpha\beta}$ is diagonal, so that $Z_{\alpha\beta}=Z_{\alpha}\delta_{\alpha\beta}$. In this case, the dynamical condition is simply

$$Z_{\alpha}F^{\alpha}(s)=0, \quad \text{for all } \alpha\,, \tag{4.25}$$

from which we conclude either that $Z_{\alpha}=0$ or $F^{\alpha}(s)=0$. But if for some α the second of these equations holds, then the particle σ_{α} has never coupled to the $\pi\pi$ system in the first place; thus this condition achieves the removal of Kronecker δ's from the $\pi\pi$ amplitude, but it does so by uncoupling the αth σ particle, not by making it dynamical. Thus this particle still exists in the Lagrangian; it merely does not couple to the pions. Therefore, if we wish to make the σ particle dynamical, we must choose the first alternative, namely $Z_{\alpha}=0$.

It is easy to see that the same physical situation occurs in the case with mixing unless all $Z_{\alpha\beta}$ vanish. The manipulations are as follows:

From Eq. (4.24), together with the definition of the form factor through Eq. (4.3), we have

$$\sum_{\beta} Z_{\alpha\beta}(\Box+\mu_{\beta}{}^{2})\sigma_{\beta}(x)=0 \tag{4.26}$$

as our condition. Now define

$$Z_{\alpha\beta}=\sum_{\gamma} U_{\alpha\gamma}{}^{-1}Z_{\gamma}U_{\gamma\beta}. \tag{4.27}$$

Z_{γ} are the eigenvalues of $Z_{\alpha\beta}$ and U is an orthogonal matrix since $Z_{\alpha\beta}$ is a real symmetric matrix. Define

$$\psi_{\alpha}(x)=\sum_{\beta} U_{\alpha\beta}\sigma_{\beta}(x)\,. \tag{4.28}$$

Using these definitions, we may rewrite Eq. (4.26) as

$$Z_{\gamma}(\Box+\mu^{2})\psi_{\gamma}(x)=Z_{\gamma}\sum_{\delta} C_{\gamma\delta}\psi_{\delta}(x)\,, \quad \text{for all } \gamma\,, \tag{4.29}$$

where

$$C_{\gamma\delta}=\sum_{\beta} U_{\gamma\beta}(\mu^{2}-\mu_{\beta}{}^{2})U_{\beta\delta}{}^{-1}; \tag{4.30}$$

thus $C_{\gamma\delta}$ is simply a number.

Equation (4.29) may be satisfied either by

$$Z_{\gamma}=0 \tag{4.31}$$

or by

$$(\Box+\mu^{2})\psi_{\gamma}(x)=\sum_{\gamma} C_{\gamma\delta}\psi_{\delta}(x)\,. \tag{4.32}$$

But the second of these is just the set of field equations for a collection of elementary particles, linear combinations of the σ particles, of mass μ, and interacting through the trivial Lagrangian

$$\mathfrak{L}_{1}=\sum_{\gamma\delta} \psi_{\delta}x(\psi)C_{\gamma\delta}\psi_{\delta}(x)\,. \tag{4.33}$$

Thus if Eq. (4.32) is the way our dynamical condition is satisfied, then we have not in fact removed all elementary σ particles; we have simply uncoupled them from the $\pi\pi$ amplitude. We therefore make the first choice, that is, $Z_\gamma=0$, all γ.

To conclude this section, then, in the multiplet case the dynamical conditions are simply

$$Z_{ij}{}^{\alpha}=0, \quad \text{for all } i,\ j,\ \alpha \tag{4.34}$$

and

$$Z_{\alpha\beta}=0, \quad \text{for all } \alpha,\ \beta. \tag{4.35}$$

We may note that there are precisely the right number of conditions: There is one $Z_{ij}{}^{\alpha}=0$ condition for each $g_{ij}{}^{\alpha}$ and the conditions $Z_{\alpha\beta}=0$ are actually only N conditions, namely, the conditions that all N eigenvalues of the $N\times N$ matrix $Z_{\alpha\beta}$ vanish.

5. $Z=0$ BOOTSTRAP HYPOTHESIS

We are now in a position to state the bootstrap assumptions for several possible cases. In general, the sequence of steps should be as outlined in the Introduction; however, as we have seen, we are as yet unable to handle more than a few simple cases. There are

(i) World containing spinless "elementary" particles: Assume the existence of a set of scalar particles σ_α and a set of pseudoscalar ones π_i. Choose as the interaction Lagrangian

$$\mathfrak{L}_1=\sum_{\alpha\beta\nu}(g_0{}^{\sigma\sigma\sigma})_{\alpha\beta\nu}\hat\sigma_\alpha(x)\hat\sigma_\beta(x)\hat\sigma_\nu(x) +\sum_{\alpha ij}(g_0{}^{\sigma\pi\pi})_{ij}{}^{\alpha}\hat\sigma_\alpha(x)\hat\pi_i(x)\hat\pi_j(x). \tag{5.1}$$

Calculate $Z_{\alpha\beta\nu}$, $Z_{ij}{}^{\alpha}$, $Z_{\alpha\beta}$, and Z_{ij}, and set these all equal to zero.

(ii) World containing pseudoscalar and spinor "elementary" particles: Assume the existence of the set of pseudoscalars π_α and the set of spinors N_i. Choose as the interaction

$$\mathfrak{L}_1=\sum_{\alpha ij}(g_0{}^{\pi NN})_{ij}{}^{\alpha}\hat{\bar N}_i(x)\hat\pi_\alpha(x)\gamma_5\hat N_j(x). \tag{5.2}$$

Calculate $Z_{ij}{}^{\alpha}$, $Z_{\alpha\beta}$, and Z_{ij} and set these all equal to zero.

(iii) World containing scalar, pseudoscalar, and spinor "elementary" particles: We add to (ii) the set σ_ν and write for the interaction Lagrangian

$$\begin{aligned}\mathfrak{L}_1=&\sum_{\nu\mu\lambda}(g_0{}^{\sigma\sigma\sigma})_{\nu\mu\lambda}\hat\sigma_\nu(x)\hat\sigma_\mu(x)\hat\sigma_\lambda(x)\\ &+\sum_{\nu\alpha\beta}(g_0{}^{\sigma\pi\pi})_{\alpha\beta}{}^{\nu}\hat\sigma_\nu(x)\hat\pi_\alpha(x)\hat\pi_\beta(x)\\ &+\sum_{\nu ij}(g_0{}^{\sigma NN})_{ij}{}^{\nu}\hat{\bar N}_i(x)\hat\sigma_\nu(x)\hat N_j(x)\\ &+\sum_{\alpha ij}(g_0{}^{\pi NN})_{ij}{}^{\alpha}\hat{\bar N}_i(x)\hat\pi_\alpha(x)\gamma_5\hat N_j(x).\end{aligned} \tag{5.3}$$

Calculate $Z_{\nu\mu\lambda}$, $Z_{\alpha\beta}{}^{\nu}$, $Z_{ij}{}^{\nu}$, $Z_{ij}{}^{\alpha}$, $Z_{\nu\mu}$, $Z_{\alpha\beta}$, and Z_{ij} and set these all equal to zero.

We cannot as yet write down worlds more complicated than these. Even the second and third examples mentioned are not strictly complete as stated, in that for renormalizability there should be a π^4 coupling included in $\mathfrak{L}_1$, and extra constraints have to be imposed to determine the parameters associated with this coupling. Fortunately, as explained in Ref. 8, we know what these are.

While the worlds described above are very restrictive, they still include many possible situations including, in particular, the one we believe to occur in nature. All the known strongly interacting particles can, in principle, be made of, for example, K meson and nucleons, or pions and sigmas, or all the pseudoscalar and baryon octets, and all these are worlds of type (ii).

If we so desire, the trial worlds can be chosen to have some internal symmetry such as isospin or $SU(3)$. This merely requires grouping the particles into degenerate multiplets and writing $\mathfrak{L}_1$ in an invariant way. However, it is perhaps more in the bootstrap spirit *not* to impose an internal symmetry, but to let the bootstrap produce it; in this case, the more general type of couplings we have written down in Eqs. (5.1)–(5.3) are more appropriate.

We mentioned earlier that the number of constraints $Z=0$ exactly equals the number of coupling constants plus the number of masses. However, this means that the number of conditions we impose is one greater than the number of parameters to be determined, because only dimensionless mass ratios can be calculated (or measured for the matter). The result is one redundant condition, and this has always been embarassing to any method of formulating the bootstrap theory. The redundancy is very basic, arising from the fact that all our formalisms for dealing with elementary particles are expressed in terms of separate particle masses and not mass ratios. We have no resolution to offer here; hopefully, either the extra equation will turn out to be an identity or else there is some as yet unknown parameter which exists in all these theories and which can be determined by it.[12]

The assumptions defining bootstrapped worlds, at least of the types mentioned above, are all quite explicit and well defined (assuming that Lagrangian field theory is well defined). We start with a completely specified Lagrangian for a specified set of particles, then impose well-defined constraints on all the parameters in the Lagrangian by means of explicit equations (assuming that the Z's exist and are finite). Thus no ambiguity exists in principle.

The practical situation is, however, another matter. It is all very well to have an explicit Lagrangian, but no one has ever been able to calculate accurately with any strong interaction Lagrangians. Therefore it will

[12] F. Zachariasen, in *Proceedings of the International Conference on Particles and Fields, Rochester, 1967* (Interscience Publishers, Inc., New York, 1967). The parameter in question may be a universal cutoff.

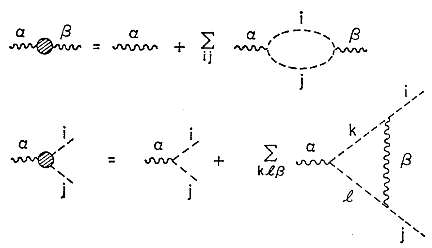

FIG. 2. Pictorial representation of the mass operator and vertex function in perturbation theory.

not be easy to calculate the Z's so it will be difficult to see if bootstrap solutions in fact exist. We will probably not be able to *accurately* calculate mass ratios and coupling constants. There is also the question of what other particles exist in the theory, besides the postulated "elementary" ones. These are bound states or resonances, even before the $Z=0$ conditions are imposed, but we will not find it an easy job to predict their existence, let alone their positions, couplings, or widths, and so forth. Thus the $Z=0$ approach is hardly a panacea for all the difficulties which have been encountered in attempts to implement the bootstrap idea.

Nevertheless, a few applications of $Z=0$ can easily be made, and if these do not lead to any results which have not already been obtained by other means, they are still of interest. We turn to these in Sec. 6.

6. APPLICATIONS

A. Cutkosky Relations

Cutkosky, some years ago, derived from the bootstrap idea the result that if a set of n vector mesons of equal mass bootstrapped themselves, then their mutual coupling constants were the structure factors of a Lie group.[13] The result was later generalized by several other authors.[14]

As a first application of the $Z=0$ conditions, let us rederive Cutkosky's equations.

Suppose we have n pseudoscalar mesons π_α, $\alpha=1\cdots n$, of equal mass m_π and N nucleons N_i, $i=1\cdots N$, also of equal mass m_N. To bootstrap the pions, for example, we must impose the conditions

$$Z_{\alpha\beta}=0,\quad \alpha,\beta=1\cdots n$$
$$Z_{ij}{}^{\alpha}=0,\quad \alpha=1\cdots n;\, i,j=1\cdots N$$

where $Z_{\alpha\beta}$ is the wave-function renormalization matrix for the pions and $Z_{ij}{}^{\alpha}$ is the vertex renormalization for the vertex coupling the ith nucleon to the jth nucleon and the αth pion.

The most trivial approximation is to calculate the Z's in perturbation theory in the pion-nucleon coupling constants $g_{ij}{}^{\alpha}$. As illustrated in Fig. 2, we evidently have

$$Z_{\alpha\beta}=\delta_{\alpha\beta}-\sum_{i,j} g_{ij}{}^{\alpha} g_{ij}{}^{\beta} I_1(m_\pi,m_N) \tag{6.1}$$

and

$$Z_{ij}{}^{\alpha} g_{ij}{}^{\alpha}=g_{ij}{}^{\alpha}-\sum_{k,l,\beta} g_{kl}{}^{\alpha} g_{ik}{}^{\beta} g_{jl}{}^{\beta} I_2(m_\pi,m_N)\,, \tag{6.2}$$

where, because of the assumed degeneracy in mass, I_1 and I_2 do not depend on any indices.

The bootstrap conditions now yield

$$\sum_{i,j} g_{ij}{}^{\alpha} g_{ij}{}^{\beta}=\delta_{\alpha\beta}/I_1 \tag{6.3}$$

and

$$\sum_{kl\beta} g_{kl}{}^{\alpha} g_{ik}{}^{\beta} g_{jl}{}^{\beta}=g_{ij}{}^{\alpha}/I_2\,, \tag{6.4}$$

which are precisely Cutkosky's equations.

The bootstrap is completed by requiring that

$$Z_{ij}=0,\quad i,j=1\cdots N$$

where Z_{ij} is the wave-function renormalization matrix of the nucleon. This, in the perturbation approximation, tells us that

$$\sum_{\alpha,k} g_{ik}{}^{\alpha} g_{jk}{}^{\alpha}=\delta_{ij}/I_3. \tag{6.5}$$

However, this equation does not contain any new information beyond that already encompassed by Eqs. (6.3) and (6.4).

The Cutkosky conditions are very restrictive. For example, if $n=3$, $N=2$, and we impose charge conservation and charge conjugation, then the relations among the couplings are those following from isotopic spin conservation.[13]

B. Reciprocal Bootstrap

Next, let us apply $Z=0$ to the model in which the nucleon is bootstrapped out of a pion and a nucleon, with both nucleon and N^* exchanges, in the static model.[15] We wish to calculate $Z_{\pi NN}$ and Z_N in the static nucleon limit and set them equal to zero. The most convenient way to do this is as follows.

We recall that in the elastic unitarity approximation

$$Z_{\pi NN}=D(0)\,, \tag{6.6}$$

where $D(\omega)$ is the D function, subtracted at infinity, for πN scattering in the $P_{1/2}$ channel. We take the nucleon mass at zero, as is usual in the static model, and ω is the pion energy.

Similarly, we can write

$$Z_{\pi NN}/Z_N=\bar{D}(0) \tag{6.7}$$

and we remind the reader that $\bar{D}$ is calculated with the elementary nucleon pole in the input while D does not include this in the input.

[13] R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963).

[14] Chan Hong-Mo, P. C. DeCelles, and J. E. Paton, Phys. Rev. Letters **11**, 521 (1963).

Now in lowest order we may write

$$D(0)=1-\frac{1}{\pi}\int_{mk}^{\infty}\frac{\rho(\omega)d\omega}{\omega}B(\omega) \tag{6.8}$$

and

$$\bar{D}(0)=1-\frac{1}{\pi}\int_{mk}^{\infty}\frac{\rho(\omega)d\omega}{\omega}[B(\omega)+B'(\omega)], \tag{6.9}$$

where $B(\omega)$ consists of the crossed poles and $B'(\omega)$ is the direct pole of the Born approximation.

To obtain the perturbation expansion of the Z's, we now substitute Eqs. (6.8) and (6.9) into (6.6) and (6.7) and expand (6.7). This results in

$$Z_{\pi NN}=1-\frac{1}{\pi}\int_{mk}^{\infty}\frac{\rho(\omega)d\omega}{\omega}B(\omega), \tag{6.10}$$

$$Z_N=1+\frac{1}{\pi}\int_{mk}^{\infty}\frac{\rho(\omega)d\omega}{\omega}B'(\omega). \tag{6.11}$$

Thus the wave-function renormalization constant is a dispersion integral over the direct Born-approximation pole, and the vertex renormalization constant is the same dispersion integral over the crossed Born-approximation pole.

Now, in the static model,

$$B'(\omega)=-(1/\omega)\gamma_{11}^2 \tag{6.12}$$

and

$$B(\omega)=(1/\omega)\sum_{IJ}C_{11,IJ}\gamma_{IJ}^2, \tag{6.13}$$

where γ_{IJ}^2 is the width for the decay of an isospin-I–spin-J resonance into πN and $C_{IJ,I'J'}$ is the crossing matrix.

Inserting these expressions into Eqs. (6.10) and (6.11) and setting the Z's to zero gives us

$$\gamma_{11}^2=\sum_{IJ}C_{11,IJ}\gamma_{IJ}^2. \tag{6.14}$$

The other equation merely determines the cutoff necessary in all the integrals.

If we also calculate πN scattering in the general IJ channel; that is, if we calculate the wave-function renormalization for the IJ resonance and vertex renormalization for πN coupled to the IJ resonance, we obviously get the usual reciprocal bootstrap equation[15]

$$\gamma_{IJ}^2=\sum_{I'J'}C_{IJ,I'J'}\gamma_{I'J'}^2. \tag{6.15}$$

Further generalizations are also possible, and permit one, for example, to reproduce the results of $SU(3)$ reciprocal bootstraps[16] and those obtained by Kumar.[17]

[15] G. F. Chew, Phys. Rev. Letters **9**, 233 (1963).
[16] R. F. Dashen, Phys. Letters **11**, 89 (1964).
[17] A. Kumar, Phys. Rev. **148**, 1347 (1966).

C. Elastic Unitarity Approximation

The reader will have observed that all the applications of $Z=0$ which we have discussed up to now have been based on perturbation theory for the Z's. This has the virtue of "vertex symmetry,"[18] in that all three particles interacting at a vertex are treated on an equal footing; nevertheless, it is of interest to ask if better approximations to the Z's can be usefully constructed, and to this end we shall next look at approximations based on elastic unitarity.

Let us, for convenience, confine ourselves to the π-σ model repeatedly discussed before. We wish to calculate Z_σ and $Z_{\sigma\pi\pi}$.

The most direct approach is obviously to use the "N-over-D" method to calculate $D(s)$ and $\bar{D}(s)$ for s-wave $\pi\pi$ scattering from the perturbation theory inputs

$$B(s)=(g_{\sigma\pi\pi}^2/2q^2)Q_0(1+m_\sigma^2/2q^2)$$

and

$$\bar{B}(s)=B(s)-g_{\sigma\pi\pi}^2/(s-m_\sigma^2),$$

respectively. Thus the Z's are obtained from $Z_{\sigma\pi\pi}=D(m_\sigma^2)$ and $Z_\sigma=D(m_\sigma^2)/\bar{D}(m_\sigma^2)$. This is almost indistinguishable from the usual N-over-D bootstrap calculation.

A less direct approach involves using two-particle unitarity in Eqs. (2.26) and (2.27) written down in Sec. 2, expressing the Z's in terms of the vertex function and the mass operator. In using these, however, one must recall that as the Z's become small the vertex function and mass operator will develop poles at the point we have called s_0, and this pole must be included. Thus Eq. (2.27) becomes, in the elastic approximation,

$$Z_{\sigma\pi\pi}g_{\sigma\pi\pi}=g_{\sigma\pi\pi}+\frac{R}{s_0-m_\sigma^2}-\frac{g_{\sigma\pi\pi}Z_{\sigma\pi\pi}}{\pi}\int\frac{\rho(s)N(s)}{|D(s)|^2}\frac{ds}{s-m_\sigma^2}. \tag{6.16}$$

We have used the fact that $\Gamma(s)=g_{\sigma\pi\pi}Z_{\sigma\pi\pi}/D(s)$, and from this fact it also follows that

$$R=g_{\sigma\pi\pi}Z_{\sigma\pi\pi}/D'(s_0). \tag{6.17}$$

Thus we have the result

$$Z_{\sigma\pi\pi}=\left(1-\frac{1}{D'(s_0)(s_0-m_\sigma^2)}+\frac{1}{\pi}\int\frac{\rho(s)N(s)}{|D(s)|^2}\frac{ds}{s-m_\sigma^2}\right)^{-1}. \tag{6.18}$$

In a perturbation expansion, $s_0-m_\sigma^2$ starts off as

[18] R. E. Cutkosky and M. Leon, Phys. Rev. **138**, B667 (1965).

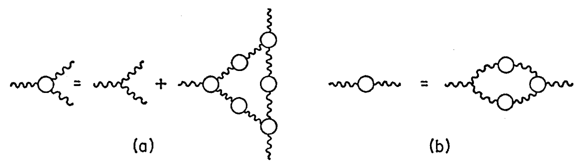

FIG. 3. Pictorial representations of integral equations for (a) the the vertex function and (b) the proper self-energy part.

$1/g_{\sigma\pi\pi}^2$, so that to lowest order Eq. (6.18) yields

$$s_0-m_\sigma^2=\left(1-\frac{1}{\pi}\int\frac{\rho(s)B(s)}{s-m_\sigma^2}ds\right)\Bigg/\frac{1}{\pi}\int\frac{\rho(s)B(s)}{(s-m_\sigma^2)^2}ds \quad (6.19)$$

from which, using the fact that

$$s_0-m_\sigma^2=-Z_{\sigma\pi\pi}/D'(m_\sigma^2)\,, \quad (6.20)$$

we find

$$Z_{\sigma\pi\pi}=1-\frac{1}{\pi}\int\frac{\rho(s)B(s)}{s-m_\sigma^2}ds\,, \quad (6.21)$$

as we should.

We can write similar expressions for Z_σ. From Eq. (2.26), we see that in the elastic approximation

$$M(s)=m_\sigma^2+\frac{(s-m_\sigma^2)^2}{\pi}\int\frac{\rho(s')|\Gamma(s')|^2}{(s'-m_\sigma^2)^2(s'-s)}ds' +\frac{(s-m_\sigma^2)^2}{s_0-m_\sigma}\frac{A}{s-s_0}. \quad (6.22)$$

The residue A can be calculated from the fact that

$$\Delta(s)=\frac{1}{Z_\sigma}\frac{1}{s-m_\sigma^2}\frac{D(s)}{\bar{D}(s)} \quad (6.23)$$

and we find

$$A=-Z_\sigma(s_0-m_\sigma^2)\bar{D}(s_0)/D'(s_0)\,. \quad (6.24)$$

Thus we get

$$Z_\sigma=\left(1-\frac{1}{\pi}g_{\sigma\pi\pi}^2Z_{\sigma\pi\pi}^2\int\frac{\rho(s)}{|D(s)|^2}\frac{ds}{(s-m_\sigma^2)^2}\right)\Bigg/\left(1-\frac{\bar{D}(s_0)}{D'(s_0)(s_0-m_\sigma^2)}\right). \quad (6.25)$$

Again, to lowest order, we replace D by 1 in the numerator and drop all but the one in the denominator. Thus

$$Z_\sigma=1-\frac{g_{\sigma\pi\pi}^2}{\pi}\int\frac{\rho(s)ds}{(s-m_\sigma^2)^2}\,, \quad (6.26)$$

which agrees with the perturbation expression for Z_σ calculated directly.

From these results, after some algebra, we can also obtain another representation for Z_σ, namely,

$$Z_\sigma=\frac{D'(s_0)}{\bar{D}'(s_0)}\left(1+g^2\frac{D'(s_0)}{N(s_0)}\right). \quad (6.27)$$

We may remark that the bootstrap equations of the conventional N-over-D method are precisely those obtained from setting $Z_\sigma=0$ in Eq. (6.27) and setting $Z_{\sigma\pi\pi}=0$ in the relation $Z_{\sigma\pi\pi}=D(m_\sigma^2)$.

Any of these equivalent equations expresses the Z's in terms of N, D, and $\bar{D}$. These functions can be calculated from the inputs B and $\bar{B}$ in any convenient manner, such as the complete N-over-D integral equation, or the determinantal method,[19] etc. The result is various approximate expressions for the Z's in the elastic approximation, or some variant of the elastic approximation. In principle, then, these Z's can be set equal to zero and the resulting equations used to calculate approximate values of $g_{\sigma\pi\pi}$, m_σ, and m_π.

Nevertheless, we feel that the basic inaccuracies imposed by the use of elastic unitarity and the choice of one-particle exchanges as the input force are so large that numerical results for bootstrap problems based on any of the approximations presented here are likely to be of dubious value. (This remark obviously applies to the usual N-over-D method as well.) Therefore, we have preferred to restrict most of our practical applications simply to the use of perturbation theory.

D. Approximations Based on the Dyson Equations

To conclude, let us describe an approximation method which includes more than the elastic unitarity approximation (to say nothing of perturbation theory), but which is nevertheless sufficiently simple to be susceptible to numerical solution. This is based on the Dyson equations for the various field theoretic quantities discussed earlier. To make the notation as trivial as possible, we shall illustrate the equations for the field theory composed only of one σ meson, with an interaction Lagrangian

$$\mathfrak{L}_1=g_0^{\sigma\sigma\sigma}[\sigma(x)]^3;$$

the extension to any other Yukawa couplings of any collection of 0^+, 0^-, or $\frac{1}{2}^+$ particles is obvious.

As illustrated in Fig. 3(a), we may write for the unrenormalized vertex function

$$\Gamma(p_1^2,p_2^2,p_3^2) =g_0^{\sigma\sigma\sigma}+i\int\frac{d^4p}{(2\pi)^4}\Gamma(p_1^2,\,(p+p_2)^2,\,(p+p_3)^2) \times\Delta((p+p_2)^2)\Gamma((p+p_2)^2,\,p_2^2,\,p^2)\Delta(p^2) \times\Gamma(p^2,\,p_3^2,\,(p+p_3)^2)\Delta((p+p_3)^2)\,, \quad (6.28)$$

where Δ is the unrenormalized propagator.

[19] F. Zachariasen and C. Zemach, Phys. Rev. 138, B441 (1965).

Now the renormalized quantities are defined by

$$\begin{aligned} g_{\sigma\sigma\sigma} &= Z_\sigma^{3/2}/Z_{\sigma\sigma\sigma} g_0^{\sigma\sigma\sigma}, \\ \Gamma_1 &= [g_{\sigma\sigma\sigma}/(g_0^{\sigma\sigma\sigma})] Z_{\sigma\sigma\sigma}\Gamma, \\ \Delta_1 &= Z_\sigma^{-1}\Delta. \end{aligned} \tag{6.29}$$

Thus, Eq. (6.28) becomes

$$\begin{aligned} \Gamma(p_1^2,p_2^2,p_3^2) &= g_{\sigma\sigma\sigma}Z_{\sigma\sigma\sigma} + i\int \frac{d^4p}{(2\pi)^4}\Gamma_1(p_1^2,\,(p+p_2)^2,\,(p+p_3)^2) \\ &\times \Delta_1((p+p_2)^2)\Gamma_1((p+p_2)^2,\,p_2^2,\,p^2)\Delta_1(p^2) \\ &\times \Gamma_1(p^2,\,p_3^2,\,(p+p_3)^2)\Delta_1((p+p_3)^2). \end{aligned} \tag{6.30}$$

Next, we may write an equation for the proper self-energy part, defined by

$$\Delta(p^2) = [p^2 - m_{\sigma 0}^2 - \Sigma(p^2)]^{-1}, \tag{6.31}$$

as illustrated in Fig. 3(b). We obtain

$$\Sigma(p^2) = \int \frac{d^4p'}{(2\pi)^3} g_0^{\sigma\sigma\sigma}\Delta(p'^2)\Delta((p-p')^2) \times \Gamma(p^2,\,(p-p')^2,\,p'^2). \tag{6.32}$$

Using Eqs. (6.29) and (6.31), we can now derive the following equation for Δ_1:

$$\frac{\partial}{\partial p^2}\left(\frac{1}{\Delta_1(p^2)}\right) = Z_\sigma - g_{\sigma\sigma\sigma}\int \frac{d^4p'}{(2\pi)^4}\Delta_1(p'^2)\frac{\partial}{\partial p^2} \times [\Delta_1((p-p')^2)\Gamma_1(p^2,\,(p-p')^2,\,p'^2)]. \tag{6.33}$$

These two equations, namely (6.30) and (6.33), now provide us with a closed system of equations from which we may calculate Γ_1, Δ_1, Z_σ, and $Z_{\sigma\sigma\sigma}$ in terms of m_σ and $g_{\sigma\sigma\sigma}$,[20] when we add the "boundary conditions"

$$\begin{aligned} &\Gamma_1(m_\sigma^2,m_\sigma^2,m_\sigma^2) = g_{\sigma\sigma\sigma}, \\ &\left.\frac{1}{\Delta_1(p^2)}\right|_{p^2=m_\sigma^2} = m_\sigma^2, \\ &\left.\frac{\partial}{\partial p^2}\frac{1}{\Delta_1(p^2)}\right|_{p^2=m_\sigma^2} = 1. \end{aligned} \tag{6.34}$$

The system of equations is obviously complicated, but it may not be beyond the reaches of possibility for large computers. In any case, precisely these equations have been studied by Cutkosky and Leon[18] in connection with their attempts to analyze the origin of symmetries on the basis of the bootstrap idea. Their equations, which they obtain from the Bethe-Salpeter equation, are precisely Eq. (6.30) evaluated for $p_1^2=p_2^2=p_3^2=m_\sigma^2$, and Eq. (6.33) evaluated at $p^2=m_\sigma^2$, and with $Z_{\sigma\sigma\sigma}=Z_\sigma=0$.

ACKNOWLEDGMENT

The authors are grateful for the hospitality of the Physics Division of the Aspen Institute for Humanistic Studies where some of this work was done.

APPENDIX: A SIMPLE MODEL

We want to present here the simplest possible model which is sufficiently interesting to have nonvanishing Z_1 and Z_3. By solving the corresponding ND^{-1} equations, we can display the functions $\Gamma(s)$, $F(s)$ and see that our representations for them in the text (3.5) and (3.8) are satisfied and how the various limits work.

We choose as a model the case where the part of the force $B(s,l)$ which is analytic in l is given by a simple pole on the imaginary q axis. To this is added a Kronecker-δ pole, also on the imaginary q axis, say at q_m with a given residue $-g^2$, where $s=q^2$.

We then have

$$\bar{B}(q,l) = \frac{g_1^2}{q^2-q_1^2} - \frac{g^2}{q^2-q_m^2}\delta_{l0}, \tag{A1}$$

where g_1 and q_1 vary smoothly with l.

This system meets all the conditions we need to solve for the relevant quantities. Near $l=0$,

$$T(q,l) = \frac{N(q)}{D(q)} - \frac{g^2 D(q_m)\bar{D}(q_m)}{D(q)(q^2-q_m^2)\bar{D}(q)}\delta_{l0} = \frac{\bar{N}(q)}{\bar{D}(q)}, \tag{A2}$$

where

$$\begin{aligned} N(q) &= g_1^2 D(q_1)/(q^2-q_1^2), \\ D(q) &= 1 - ig_1^2 D(q_1)/(q+q_1), \end{aligned} \tag{A3}$$

and therefore

$$D(q_1) = (1+ig_1^2/2q_1)^{-1}.$$

Similarly,

$$\begin{aligned} \bar{N}(q) &= g_1^2\bar{D}(q_1)/(q^2-q_1^2) - g^2\bar{D}(q_m)/(q^2-q_m^2), \\ \bar{D}(q) &= 1 - ig_1^2\bar{D}(q_1)/(q+q_1) + ig^2\bar{D}(q_m)/(q+q_m), \end{aligned} \tag{A4}$$

with

$$\begin{aligned} \bar{D}(q_1) &= \frac{D(q_1)\mathfrak{D}(q_1)}{1-[D(q_m)-1][\mathfrak{D}(q_1)-1]}, \\ \bar{D}(q_m) &= \frac{D(q_m)\mathfrak{D}(q_m)}{1-[D(q_m)-1][\mathfrak{D}(q_1)-1]}, \end{aligned} \tag{A5}$$

where[21]

$$\begin{aligned} \mathfrak{D}(q) &= 1 + ig^2\mathfrak{D}(q_m)/(q+q_m), \\ \mathfrak{D}(q_m) &= (1-ig^2/2q_m)^{-1}. \end{aligned}$$

---

[20] Actually, of course, only dimensionless quantities such as mass ratios may be calculated. This is the usual situation with bootstraps. This also assumes the existence without cutoffs of the relevant integrals.

[21] It is clear that $D(s)$ would be the D function if only the first ("force") pole (g_1^2,q_1) were present in $B(q,l)$; $\mathfrak{D}(s)$ if only the second (CDD) pole $(-g^2,\,q_m)$ were present; and $\bar{D}(s)$ is the actual D function with $\bar{B}(q,l)$, which contains both poles, as input.

We can now define the renormalization constants Z_1 and Z_3:

$$Z_1 \equiv D(q_m) = \frac{g_1^2(q_m - q_1) - 2iq_1(q_m + q_1)}{g_1^2(q_m + q_1) - 2iq_1(q_m + q_1)} \tag{A6}$$

and

$$Z_3 \equiv Z_1 \bar{D}^{-1}(q_m) = \mathfrak{D}^{-1}(q_m) - \mathfrak{D}^{-1}(q_m)[\mathfrak{D}(q_1) - 1][D(q_m) - 1] = \frac{(g^2 + 2iq_m)(g_1^2 - 2iq_1)(q_1 + q_m)^2 - 4q_1 q_m g_1^2 g^2}{2iq_m(q_1 + q_m)^2(g_1^2 - 2iq_1)}. \tag{A7}$$

Note that in spite of its definition Z_3 is *not* proportional to Z_1, because $\bar{D}(q_m)$ also has a factor Z_1 from (A5). The conditions $Z_1 \to 0$ and $Z_3 \to 0$ are therefore completely independent. Assuming g^2 and q_m as given, they are

$$\begin{aligned} Z_1 = 0: &\quad (ig_1^2/2g_1) = (q_1 + q_m)/(q_1 - q_m), \\ Z_3 = 0: &\quad (ig_1^2/2q_1) = -\frac{g^2(q_1 + q_m)^2 + 2iq_m(q_1 + q_m)^2}{g^2(q_1 - q_m)^2 + 2iq_m(q_1 + q_m)^2}. \end{aligned} \tag{A8}$$

This gives the unique solution for g_1^2 and q_1:

$$g_1^2 = g^2[(g^2 - 2iq_m)/(g^2 + 2iq_m)], \quad q_1 = q_m[(g^2 - 2iq_m)/(g^2 + 2iq_m)]. \tag{A9}$$

This solution can be approached along the line $Z_1 = 0$ or along the line $Z_3 = 0$, making Z_1/Z_3quite undefined. In the limit, the amplitude is dynamic and independent of the ratio Z_1/Z_3 although $\bar{N}$ and $\bar{D}$ separately are not.

We can also write explicit expressions for $\Gamma_1(q)$ and the form factor $F(q)$:

$$\Gamma_1(q) \equiv gD(q_m)/D(q) = g\frac{Z_1}{Z_1 + (q^2 - q_m^2)[(1 - Z_1)/(q + q_1)(q + q_m)]} \tag{A10}$$

and

$$F(q) \equiv g\bar{D}(q_m)/\bar{D}(q) = g\frac{Z_1}{Z_1 + (q^2 - q_m^2)\{Z_3 - Z_1[1 + (ig^2/2q_m)(q_1 - q_m)/(q + q_m)]\}/(q + q_1)(q + q_m)}, \tag{A11}$$

which gives for the functions $\alpha(q)$ and $\beta(q)$ from the text

$$\begin{aligned} \alpha(q) &= [(q + q_1)(q + q_m)]^{-1} - Z_1[(q + q_1)(q + q_m)]^{-1}, \\ Z_3\beta(q) &= \frac{Z_3}{(q + q_1)(q + q_m)} - Z_1\frac{1 + (ig^2/2q_m)(q_m - q_1)/(q + q_m)}{(q + q_1)(q + q_m)}. \end{aligned} \tag{A12}$$

Note that asymptotically as $q \to \infty$,

$$\Gamma_1(q) \to g\frac{Z_1}{1 - (1 - Z_1)(q_1 + q_m)/q} \tag{A13}$$

but

$$F(q) \to g\frac{Z_1}{Z_3 + (1/q)[(Z_1 - Z_3)(q_1 + q_m) - Z_1(ig^2/2q_m)(q_1 - q_m)]},$$

so that when Z_1 vanishes, $\Gamma_1(q)$ always vanishes with Z_1. However, the form factor $F(q)$, which usually vanishes with Z_1 and is asymptotic to gZ_1/Z_3, stays finite in the case $Z_3/Z_1 \to 0$, $Z_1 \to 0$ and goes asymptotically like

$$F(q) \to q\frac{g}{(q_1 + q_m) - (ig^2/2q_m)(q_1 - q_m)} = (q/q_m)[g + 2iq_m/g] \tag{A14}$$

from (A9).

Finally, the pole of $\Gamma_1(q)$ is at q_0 given by

$$q_0 = q_m - Z_1[q_m + q_1]$$

and therefore moves to q_m when $Z_1 \to 0$. The corresponding residue of $-\Gamma_1(q)(q^2-q_m{}^2)^{-1}F(q)$ in the amplitude T at q_0 is

$$R(q_0)=g^2(1-Z_1)/\{1-Z_3+Z_1(ig^2/2q_m)(q_1-q_m)/[2q_m-Z_1(q_m+q_1)]\}$$

and therefore, as $q_0 \to q_m$ with Z_1,

$$R(q_0) \to g^2/(1-Z_3)$$

and clearly then, when $Z_3 \to 0$, the residue approaches g^2 and cancels the elementary pole. This conclusion is again independent of any ratio Z_1/Z_3.

PHYSICAL REVIEW VOLUME 171, NUMBER 5 25 JULY 1968

Form Factors and Electromagnetic Masses*

TUNG-MOW YAN
Department of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 18 March 1968)

The π and ρ electromagnetic masses are calculated within the framework of chiral dynamics, including a ρ-γ form factor. "Anomalous-magnetic-moment"–type couplings of $\rho\pi\pi$ and $A_1\rho\pi$ are included, but it is found that their contribution to δm_π is very small. The dependence of the results on the form factor is discussed.

I. INTRODUCTION

SINCE the first successful chiral calculations of the pion electromagnetic mass splitting,[1,2] it has been discovered by several authors[3,4] that once the simplification $(m_\pi/m_\rho)^2=0$ is removed, the original procedure would lead to a logarithmically divergent result. In a recent paper[3,5] Schwinger has proposed that the low-energy description of a phenomenological local coupling of the photon to the neutral ρ meson must be eventually recognized as nonlocal and described by a form factor. With a suitable choice of the latter, the π as well as the ρ electromagnetic masses, among other things, are then calculable. In this paper we present these calculations within the framework of chiral dynamics.[6] Under the assumption that the form factor is dominated by a single mass value, we have found that the experimental value of the π mass splitting is reproduced if the mass value M that appeared in the form factor is about $5m_\rho$. This form factor is then used to calculate the ρ mass difference. For a wide range of M values, δm_ρ $(=m_{\rho^\pm}-m_{\rho^0})$ is found to be negative and with the magnitude about or less than 1 MeV. It is interesting that the negative sign of δm_ρ persists for such a wide range of M values. Experimentally, the situation for δm_ρ is not clear. However, new data seem to support the assertion that δm_ρ is negative.[7]

Since the same form factor also appears in the radiative corrections to weak processes, it would be desirable to apply the form factor determined here to study the quantitative correlations among these electromagnetic effects in strong and weak interactions. Another implication of the knowledge of the form factor lies in the following considerations. Electromagnetic and weak interactions are external disturbances to the strongly interacting particles. Thus the form factors connecting the photon and the (hypothetical) intermediate bosons to the strongly interacting particles are presumably alike, since they are a phenomenological way of summarizing the properties of strong-interaction effects. The low value of M ($\sim$4 BeV) is perhaps relevant in explaining the extremely small mass difference of $K_1{}^0$ and $K_2{}^0$ mesons. All these considerations will be deferred for another communication.

* Supported in part by the U. S. Air Force Office of Scientific Research.

[1] T. Das, G. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, Phys. Rev. Letters **18**, 759 (1967).

[2] Several derivations are given by J. Schwinger, Phys. Rev. **165**, 1714 (1968); **167**, 1546(E) (1968).

[3] J. Schwinger, Phys. Rev. Letters **19**, 1154 (1967).

[4] M. B. Halpern and G. Segrè, Phys. Rev. Letters **19**, 611 (1967); G. C. Wick and B. Zumino, Phys. Letters **25B**, 479 (1967); I. S. Gerstein, B. W. Lee, H. T. Nieh, and H. J. Schnitzer, Phys. Rev. Letters **19**, 1064 (1967).

[5] Recently T. D. Lee proposed that couplings between the photon and known particles (strongly and weakly interacting) are mediated by a hypothetical intermediate boson [T. D. Lee, Phys. Rev. **168**, 1714 (1968)]. There are differences between the two approaches, however. For instance, in Schwinger's proposal the form factor appears only in the couplings between the photon and strongly interacting particles.

[6] S. Weinberg, Phys. Rev. Letters **18**, 188 (1967); J. Schwinger, Phys. Letters **24B**, 475 (1967); J. A. Cronin, Phys. Rev. **161**, 1483 (1967); J. Wess and B. Zumino, *ibid.* **163**, 1727 (1967); L. S. Brown, *ibid.* **163**, 1802 (1967); P. Chang and F. Gürsey, *ibid.* **164**, 1752 (1967); B. W. Lee and H. T. Nieh, *ibid.* **166**, 1507 (1968).

[7] See the new compilation of A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968).