Variational Principles for Crossing-Symmetric Off-Shell Equations*

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Variational principles for the vertex function and the two-body scattering amplitude are constructed, based on crossing-symmetric dynamical equations. Our expressions are stationary under variations of all the trial functions that enter. All relevant channels are treated in a manifestly symmetrical manner.

I. INTRODUCTION

ALTHOUGH the Bethe-Salpeter (BS) equation is an exact equation for the scattering amplitude, most recent studies of the equation have involved approximations that badly violate crossing.[1] The ladder approximation, for example, in which the interaction kernel consists of the one-particle-exchange graph, produces an amplitude satisfying elastic unitarity in one channel only. We shall consider here integral equations for the interaction kernel that will guarantee crossing symmetry, and thus yield an amplitude that satisfies elastic unitarity in all three channels.[2] The equations are highly nonlinear, and certainly intractable using conventional methods. However, we shall show that it is possible to write a variational principle that yields a stationary expression for the crossing-symmetric T matrix, which should provide a sounder basis for a study of approximate solutions to the equations.

We have considered a particular crossing-symmetric equation for the proper vertex function[3] and have found an associated variational principle for this equation also. The stationary expressions for these two problems are very similar in form but since there are fewer particle lines for the vertex problem, its variational principle is considerably more transparent. Since the essential features are present for the vertex function, it will be presented first to serve as a motivation for the form of the stationary expression for the T matrix.

II. VERTEX FUNCTION

A. Dynamical Equations

Given a trilinear scalar interaction $\gamma\varphi_1\varphi_2\varphi_3$, we should like to write a linear integral equation for the proper vertex function $\Gamma(k,P)$. The vector P is the momentum of the left leg in Fig. 1, and k is the relative momentum in the other two legs. Let us base our discussion on the BS equation for the T matrix illustrated in Fig. 1(a);

$$T(k',k;P)=K(k',k;P)+\int d^4q\, T(k',q;P)G(q,P)K(q,k;P). \quad (2.1)$$

The interation kernel $K(q,k;P)$ is the usual two-body irreducible kernel minus the direct-channel pole graph in the variable P^2, i.e., it contains all graphs that do not have a one- or two-particle intermediate state. The Green's function $G(p;P)$ is the product of two single-particle propagators:

$$G(q;P)=\Delta_1'((q-\tfrac{1}{2}P)^2)\Delta_2'((q+\tfrac{1}{2}P)^2). \quad (2.2)$$

We define $\Gamma(k,P)$ to be the sum of all three-line graphs that do not have a one-particle intermediate state [Fig. 1(b)]:

$$\Gamma(k,P)=\gamma+\gamma\int d^4q\, G(q;P)T(q,k;P). \quad (2.3)$$

Inserting Eq. (2.3) in Eq. (2.1), we can easily arrive at

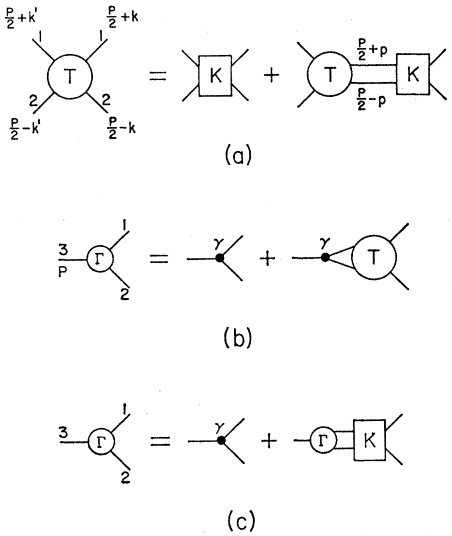

FIG. 1. (a) Bethe-Salpeter equation for the T matrix; (b) definition of the vertex function; (c) dynamical equation for the vertex function.

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[1] C. Schwartz and C. Zemach, Phys. Rev. **141**, 1454 (1966); M. J. Levine, J. Wright, and J. A. Tjon, *ibid.* **154**, 1433 (1967).

[2] Historical precedents using various approaches to satisfy unitarity and crossing will be found in K. Ter-Martirosyan, Phys. Rev. **111**, 948 (1958); W. Zimmerman, Nuovo Cimento **21**, 249 (1961); N. Khuri, *ibid.* **22**, 1023 (1961); J. G. Taylor, Suppl. *ibid.* **1**, 857 (1963); K. Symanzik, *Symposia on Theoretical Physics* (Plenum Press, Inc., New York, 1967), Vol. 3; R. E. Cutkosky, Phys. Rev. **154**, 1375 (1967); J. G. Cordes, *ibid.* **156**, 1707 (1967).

[3] This type of equation has also been studied by J. Harte, Phys. Rev. **165**, 1557 (1968).

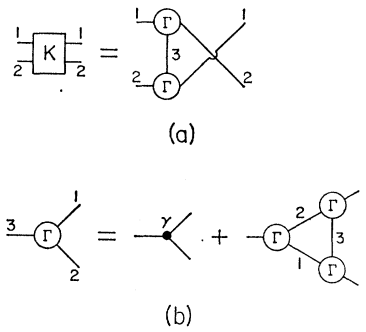

FIG. 2. (a) Approximate interaction kernel; (b) resulting crossing-symmetric equation for the vertex function.

an equation for $\Gamma(k,P)$ [Fig. 1(c)]:

$$\Gamma(k;P)=\gamma+\int d^4q\,\Gamma(q;P)G(q;P)K(q,k;P). \quad (2.4)$$

Equation (2.4) is an exact equation for the proper vertex function, and thus $\Gamma(k;P)$ will be crossing-symmetric. However, the approximation of $K(q,k;P)$ by a finite set of graphs (e.g., single-particle exchange) will clearly break crossing symmetry since the equation will iterate that set of graphs in one channel only. Rather than including a few graphs in K, we should like to introduce an approximate K that preserves crossing symmetry for Γ. Such a K is simple to find, as shown in Fig. 2(a):

$$K(q,k;P)=\Gamma(\tfrac{1}{2}(-P+k-k');-k-k')\times\Delta_3'((k-k')^2)\Gamma(\tfrac{1}{2}(-P+k-k');k'+k). \quad (2.5)$$

The equation for Γ now becomes

$$\Gamma(k;P)=\gamma+\int d^4q\,\Gamma(q;P)\Delta_1'((\tfrac{1}{2}P-q)^2)\times\Delta_2'((\tfrac{1}{2}P+q)^2)\Gamma(\tfrac{1}{2}(-P+k-k');-k-k')\times\Delta_3'((k-k')^2)\Gamma(\tfrac{1}{2}(P+k-k');k'+k). \quad (2.6)$$

This equation is obviously no longer linear in Γ. An iterative solution of this equation shows that it sums all nested triangle graphs, as one can see by iterating Fig. 2(b).

B. Variational Principles

Our goal is to write a stationary expression for Γ satisfying Eq. (2.6), but first let us write a variational principle based on the linear integral equation (2.4) and point out its drawbacks for yielding a crossing-symmetric vertex function.

The desired stationary expression can be written in operator form [Fig. 3(a)]:

$$[\Gamma]=\gamma+\Gamma GK+[-\Gamma+\gamma+\Gamma GK]GT. \quad (2.7)$$

Let us examine the first-order variation of $[\Gamma]$ with respect to Γ and T [Fig. 3(b)]:

$$\delta[\Gamma]=\delta\Gamma G[K-T+KGT]+[-\Gamma+\gamma+\Gamma GK]G\delta T. \quad (2.8)$$

Thus this expression is stationary with respect to first-order variations of Γ and T about the exact solution to Eqs. (2.4) and (2.1), respectively. However, we notice from Eq. (2.8) that T must satisfy the transpose of Eq. (2.1), but for symmetric potentials the two forms are equivalent. Now the error in $[\Gamma]$ will be of second order in the error of a trial Γ and a trial T inserted on the right. We have assumed that K is a known operator and is independent of Γ and T. This expression is *not* stationary with respect to variations in K and thus if K is not known exactly, an error in K will produce a first-order error in Γ.

The crossing-symmetric equation (2.6) has an unknown potential K, since K depends in turn on the unknown function Γ. This makes the variational principle (2.7) inadequate for our purposes. It is also clear that (2.7) does not treat the three legs of Γ symmetrically.

It is possible to invent a form that overcomes both of these difficulties. Consider the expression

$$[\Gamma]=\gamma+\Gamma GK+T_3G_3[-\Gamma+\gamma+\Gamma GK], \quad (2.9)$$

where T_3 and G_3 are the three-body T matrix and free Green's function, respectively, and the products have a meaning as defined in Fig. 4(a). Let us consider variations of the unknown functions on the right. The result of carrying out the variation can be seen most easily in pictures as shown in Fig. 4(b). The variation δT_3 gives Eq. (2.6) for Γ, and the variation $\delta\Gamma$ gives a linear inte-

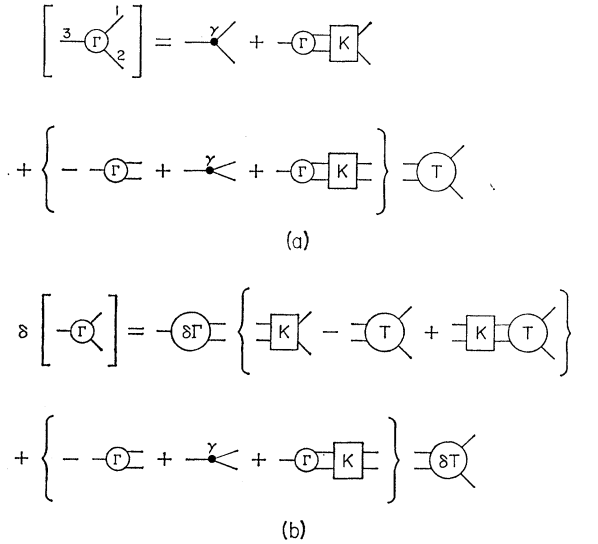

FIG. 3. (a) Stationary expression for the vertex function based on the linear equations of Fig. 1; (b) first-order variation of the above expression with respect to the trial functions. The brackets vanish if T and Γ satisfy Eqs. (2.1) and (2.4), respectively.

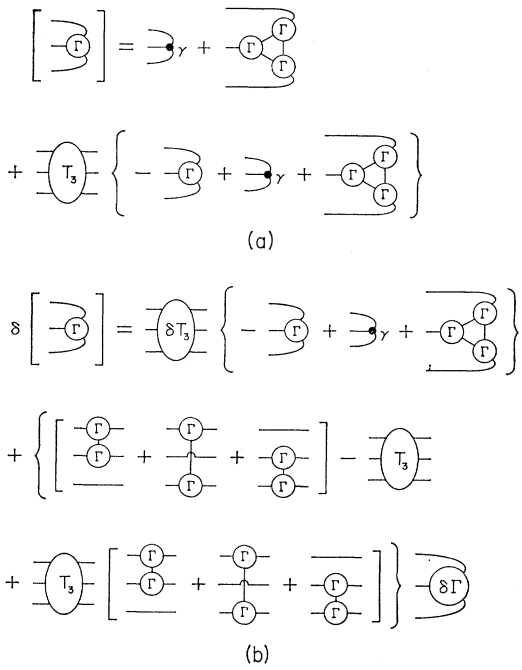

FIG. 4. (a) Crossing-symmetric variational principle based on Fig. 2(b); (b) first-order variation of the vertex function. The coefficient of δT_3 vanishes by construction. The vanishing of the coefficient of $\delta\Gamma$ yields a linear three-body equation for T_3.

gral equation for the three-body T matrix with the symmetric sum of two-body potentials.

It appears necessary to introduce the full three-body amplitude in order to achieve a crossing-symmetric variational principle; it should be noted, however, that the three-body amplitude is needed only for the total three-body energy equal to zero. The three-body equation has five nontrivial integrations for $E\neq 0$, but only three at $E=0$. This also suggests the conjecture that an n-particle linear equation is associated with an n-line crossing-symmetric amplitude in constructing a crossing-symmetric variational principle. This is, in fact, true for $n=4$ as we will show in the following sections.

III. CROSSING-SYMMETRIC T-MATRIX EQUATIONS

Let us now consider the elastic scattering of two self-conjugate spin-zero particles. For simplicity, we shall demand that the process be elastic in *all three* channels in order to avoid inelastic channel sums, hence the condition of self-conjugacy. The basic interaction can be left open for the time being, except that we will not allow a φ^3 coupling with all three masses equal. The presence of this coupling causes complications that will be discussed later. We shall consider the particles to be pseudoscalar, so that a three-field interaction is not allowed. For definiteness, we shall assume a φ^4 interaction in the following discussion, but our final results are true under much more general types of interactions.

We shall rely heavily on pictures for this discussion, and it will be convenient to define a direction to correspond to the three possible scattering channels. The s channel will be horizontal, that t channel vertical, and the u channel into the paper.

First, consider the derivation of the standard BS equation in the ith channel ($i=s, t, u$). The sum of all graphs that cannot be cut into two separate graphs by cutting only two propagators in the ith direction will be denoted by K_i. Thus K_i is the two-particle irreducible kernel of the ith channel. Since all graphs are either in this class or have a two-particle cut, this serves as a complete classification of diagrams. All the graphs that contribute to the T matrix can be classified by starting at one end of the diagram and moving to the other in any one of the three channels.

If no two-particle cut is found moving in the ith direction, then that diagram belongs to K_i. The sum of all graphs that do have a two-particle cut are of a simple form. To the left of the first two-particle cut that can be made in the s channel, for example, there are no two-particle singularities, therefore the sum of all such contributions is K_s. To the right of the cut, one has all possible contributions, which is the T matrix itself. Thus one has the equations

$$T=K_i+K_iG_iT, \tag{3.1}$$

where G_i is the product of two Feymnan propagators in the ith channel. Figure 5 shows diagrammatic representation of these equations. The T matrix and the K's are represented by a circle and squares, respectively; the lines inside the squares serve to tell at a glance which direction has no two-particle state. For example, in the s channel [Fig. 5(a)] the lines are meant to indicate that there is no two-particle state for the process $1+2\rightarrow\bar{3}+\bar{4}$ but there may be two-particle states for the processes $1+3\rightarrow\bar{2}+\bar{4}$ and $1+4\rightarrow\bar{2}+\bar{3}$.

If we find a T that is a solution of all three of these equations it will be guaranteed to be unitary in all three

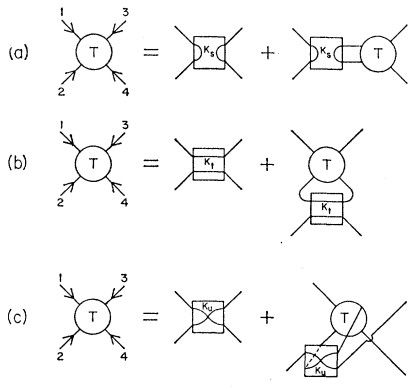

FIG. 5. Bethe-Salpeter equations for the s, t, and u channels. The lines inside the interaction-kernel squares indicate which channel has no two-particle state, e.g., the lines in K_s connect $1+3$ to $\bar{2}+\bar{4}$, and $1+4$ to $\bar{2}+\bar{3}$, but do not connect $1+2$ to $\bar{3}+\bar{4}$, the channel with no two-particle state.

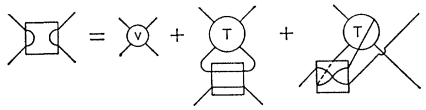

FIG. 6. Equation (3.2) for the interaction kernel K_s.

channels. If we approximate G by two bare propagators, then unitarity will be ensured only in the two-particle sector. The usual approximations to the irreducible kernel by a finite set of graphs will clearly lead to an inconsistency between the three BS equations.

The next step is to find equations for the K's. Consider K_s, for example, all contributing graphs can be classified into four groups: (i) those with no two-particle t or u cuts, (ii) those with a t cut, (iii) those with a u cut, and (iv) those with both cuts. Taking our example of φ^4 coupling, it can easily be seen that there are no diagrams of type (iv). The sum of type-(i) diagrams we shall denote by V, which we shall term *completely* irreducible. The sum of diagrams of type (ii) takes an already familiar form, K_tG_tT, and similarly, type (iii) takes the form K_uG_uT. So that we finally have the equation illustrated in Fig. 6:

$$K_s=V+K_tG_tT+K_uG_uT. \tag{3.2}$$

It can easily be checked that K_tG_tT has only a t-channel two-particle cut, and K_uG_uT is only a u-channel two-particle cut, so that this is indeed a unique classification of diagrams. There are similar equations for K_t and K_u, giving, finally:

$$K_i=V+\sum_{j\neq i} K_jG_jT. \tag{3.3}$$

The six equations [(3.1) and (3.3)] are not independent. It can be easily checked that the set of K equations, together with the BS equation in one channel, imply the BS equations in the other channels. Note also that there is one algebraic relation among the functions:

$$T=\tfrac{1}{2}(\textstyle\sum K_i-V). \tag{3.4}$$

We can now define a crossing-symmetric model by restricting the input V to a small number of diagrams. The only information about the specific interaction appears in V. For a φ^4 theory, the basic coupling itself [Fig. 7(a)] would serve as a suitable model V. For a three-field coupling, the sum of pole terms shown in Fig. 7(b) is appropriate provided that the internal particle has a mass μ which is larger than the mass of the external particles. Then a unitary T matrix in the two-particle sectors up to the 2μ particle-production threshold will be produced by the equations. Pole terms in which the internal particle is the same as the external particles are not allowed. The T matrix and all the K's contain V to lowest order. Looking then at the K_s equation for example, (Fig. 6), the K_tG_tT term will contain a box diagram giving a two-particle state in the *s channel* which is not allowed. In fact, the equations will generate the three different box diagrams with corrected vertices and propagators that must be subtracted out of V. This subtraction not only avoids double counting, but also restores the two-particle irreducibility of the K matrices.

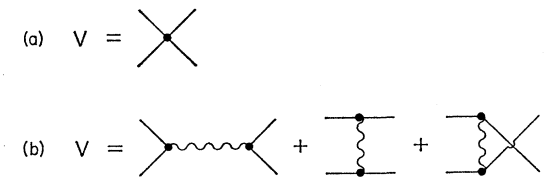

FIG. 7. Examples of approximations to the "completely irreducible" kernel V.

IV. VARIATIONAL PRINCIPLE FOR THE CROSSING-SYMMETRIC T MATRIX

A. Statement of the Stationary Expression

Our goal is to write an expression for the T matrix that is stationary with respect to first-order variations of all unknown functions about their true solutions. Our approach is to write an equation for T and add to it terms that go to zero if all functions are exact solutions but chosen so as to cancel the first-order error in T.

Our first step is to choose four independent equations out of the set (3.1) and (3.3) in a symmetrical way;

$$T=V+\textstyle\sum K_iG_iT, \tag{4.1a}$$

$$T=K_i+K_iG_iT. \tag{4.1b}$$

We can then write our result in operator form by defining

$$L\equiv V+\sum_i (K_iG_iT+TG_iK_i -K_iG_iK_i-K_iG_iTG_iK_i). \tag{4.2}$$

Our variational principle is then

$$[T]=L+T_4G_4\{L-T\}, \tag{4.3}$$

where T_4 is a four-particle T matrix and G_4 is the four-particle free Green's function. Some discussion is in order about the meaning of the operator products. The meaning and structure of this stationary form is conveniently expressed in terms of pictures.

Half the battle is won by drawing all external lines for each term to the left. The definition of L is shown pictorially in Fig. 8. The terms in the sum for $i=s$ are shown. The first term, K_sG_sT, is the same diagram as the product in Fig. 5(a) except that lines 3 and 4 are crossed back to the left. The terms K_tG_tT and K_uG_uT are shown explicitly, and it will be left to the reader to complete the other t and u terms. This is a straightforward task if it is remembered that G_i propagates two free particles in the ith channel. Having defined L, the variational principle takes the simple form shown in Fig. 9.

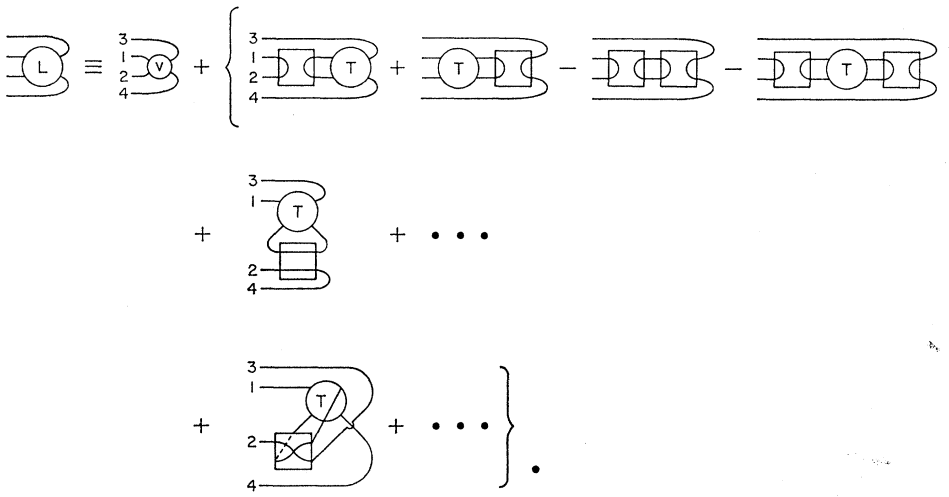

FIG. 8. Definition of the function L.

We have chosen to write the variational principle in terms of T matrices but it could be translated into expressions involving wave functions or full Green's functions. This may not be the most efficient form for actual calculation since the number of integrals that must be performed depends on which form is chosen. Also, it may be desirable to introduce inverse propagators which are local operators in coordinate space rather than propagators themselves.

B. Demonstration of the Stationary Property

We shall consider the variation of the left-hand side of Eq. (4.3) due to variations in K_i, T_4, and T. The coefficients of the variations will be the Eqs. (4.1) and a defining equation for the new function T_4.

1. Variations of K_i

Note that K_i appears only in L. Thus we can write

$$\delta[T]=(1_4+T_4G_4)(\delta L/\delta K_i)\delta K_i, \tag{4.4}$$

where 1_4 just means the four-particle identity. The first factor is nonzero in general, so let us ignore it. The graphical meaning of the remainder of this expression is shown in Fig. 10, for $i=s$. Though the figure may look complicated, the rules for construction of the diagram are very simple: Compute the first-order variation of the expression, then pull the variational term to the right by deforming any necessary lines. It is clear that the first-order change vanishes if the s-channel BS equation is satisfied. Similarly, the variations of K_t and K_u will give the BS equations in the t and u channels, Eq. (4.1b).

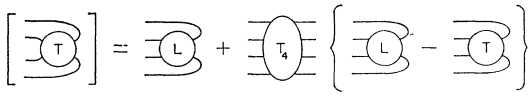

FIG. 9. Crossing-symmetric variational principle for the T matrix.

2. Variations of T_4

The variation of $[T]$ with respect to T_4 gives the equation $T=L$ (see Fig. 9). We can now use the three basic BS equations, and immediately deduce Eq. (4.1a).

3. Variations of T

The factor T appears linearly in the variational principle. Varying T and pulling δT to the right in each term yields

$$\delta[T]=\{-T_4G_4+\delta L/\delta T+T_4G_4\delta L/\delta T\}\delta T. \tag{4.5}$$

The expression in the brackets vanishes if T_4 satisfies the four-particle linear BS equation with a potential $(\delta L/\delta T)G_4^{-1}$, illustrated in Fig. 11.

We must still show that $(\delta L/\delta T)G_4^{-1}$ is indeed an acceptable potential for the four-body problem to justify calling T_4 a T matrix. Denoting $(\delta L/\delta T)G_4^{-1}$ by K_4, we define a notation for the various terms as follows:

$$K_4=K_4^s+K_4^t+K_4^u, \tag{4.6a}$$

where

$$K_4^s=K_sG_{\bar{s}}^{-1}+G_s^{-1}K_{\bar{s}}-K_sK_{\bar{s}}, \tag{4.6b}$$

and similar definitions hold for K_4^t and K_4^u. An s label

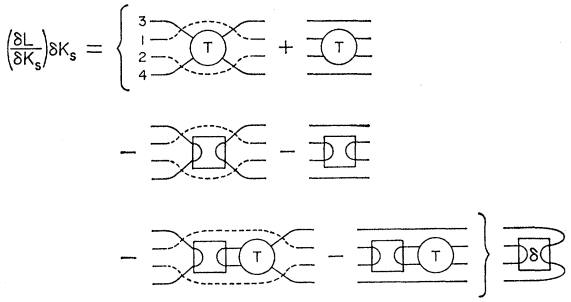

FIG. 10. Graphical demonstration that L is stationary with respect to variations of K_s. Similar results hold for variations of K_t and K_u.

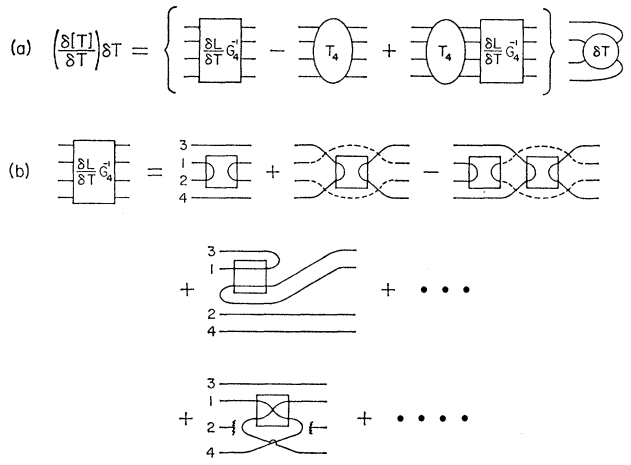

FIG. 11. (a) Variation of $[T]$ with respect to T; (b) four-body potential which yields T_4 and which therefore makes $[T]$ stationary with respect to variations of T.

indicates that the operator acts on particles 1 and 2, and $\bar{s}$ denotes 3 and 4. The inverse Green's functions are present to cancel the extra propagators in G_4 for those particles which do not interact.

First, note from Fig. 11(b) that the two-body K's are hooked up in the correct manner so that there is no direct two-particle state. This is clear from the structure of the diagrams because the lines inside the boxes do not connect an initial state to a final state. Second, we must justify the presence of the three terms in, for example, K_4^s (4.6b). The s and $\bar{s}$ labels refer to two distinct sets of particles, which do not interact if K_4^t and K_4^u are zero. The two sets of particles then propagate independently and thus the full four-particle Green's function must factor into two two-body Green's functions. The specific form of V_s ensures that this will happen. Consider

$$H_i = G_i + G_i K_i H_i, \quad i = s \text{ or } \bar{s} \tag{4.7a}$$

$$H_4 = G_4 + G_4 K_4^s H_4, \tag{4.7b}$$

where H_i is the full two-particle Green's functions for the set i, and H_4 in the full four-particle Green's function. The product equation is

$$H_s H_{\bar{s}} = G_s G_{\bar{s}} + G_s G_{\bar{s}} (K_s K_{\bar{s}} + H_s^{-1} K_{\bar{s}} + H_{\bar{s}}^{-1} K_s) H_s H_{\bar{s}}. \tag{4.8}$$

Since $H_i^{-1} = G_i^{-1} - K_i$ and $G_4 = G_s G_{\bar{s}}$, we immediately arrive at the form for K_4^s given in Eq. (4.6b).

V. DISCUSSION AND CONCLUSIONS

The stationary principles derived in this paper suffer from the defect of being quite complicated. However, the problem that one is trying to solve is extremely complicated also, and it is this property that is well hidden in the approximations (i.e., the ladder approximation, sum of bubbles, etc.) used so far in studies of the BS equation. Thus the crossing-symmetric equations and their associated stationary principles have the possibility of containing some of the physics of the problem.

Before proceeding to a general discussion, let us first outline a simple calculation which one might attempt with the stationary principle for the case of pion-pion scattering. The simplest choice for V is the four-field point interaction strength λ. Similarly, one can choose trial constants for the $K_i(=k)$ and $T(=t)$. For the four-particle T matrix T_4, one can choose K_4, where each two-particle K matrix is replaced by the trial constants k.

With the above choice for the trial functions, one finds that L can be expressed in terms of the constants λ, k, and t, and the bubble, diagrams in all three channels. Since these are infinite, one could renormalize by subtracting at the symmetry point and introducing renormalized values of $\bar{\lambda}$, $\bar{k}$, and $\bar{t}$. The variations of $\bar{k}$ and $\bar{t}$ are then carried out in such a way that the (renormalized) value of $[T]$ at the symmetry point is fixed at the value $\bar{\lambda}$. The general question of the renormalizability of these equations has not been studied. In any case, it is easy to see that the final form for $[T]$ involves only one-dimensional dispersion terms because of the simple choice that has been made for the trial functions. Thus it is a simple matter to find the stationary values of k and t. If these are not to depend on the energy variables in the three channels, then one must choose definite values of these variables in the region of primary interest and optimize for these values.

Another question that has not been settled completely is whether one can perform a Wick rotation in these equations. There does not seem to be any obvious difficulty in performing the rotation in all three channels simultaneously, but we have not been able to prove that it is always possible.

One possible approach to solving these equations is to examine the iterative solution, i.e., to find what low-order Feynman diagrams the equations generate. Denoting the sum of diagrams of order n by T_n, we have

$$T = \lambda f_1 T_1 + \lambda^2 f_2 T_2 + \cdots.$$

We can choose the numbers $f_n = 1$, which would yield the iterative solution again. But if we instead treat the f_n as variational parameters and insert this form in our variational principle along with similar forms for T_4 and K_i, and vary the parameters to find the extremum, we would hope to get an improved answer. This procedure can be carried out for a linear BS equation and the Schwinger variational principle; the result is that the best T is a Padé approximant[4] in λ. Put another way, given a truncated perturbation series in λ, Schwinger's principle tells one to construct a ratio of polynomials in λ and gives one the coefficients. Such a form has the possibility of generating poles in λ corresponding to bound states.

This procedure will generate something much more complicated than Padé approximants for our varia-

[4] J. Nuttall, Phys. Rev. 157, 1312 (1967).

tional principle. Because the equations are nonlinear we shall get branch points in the λ plane. Such branch points are expected because intermediate states containing two bound states are generated by the equation.

One final point is that the variational principle for the vertex function should be useful in dynamical calculations such as that performed by Cutkosky *et al.*[5] in their discussion of internal symmetry and symmetry breaking via the bootstrap hypothesis. However, there is some delicacy in using the vertex equation [Fig. 2(b)] for defining a bootstrap. If we set the inhomogeneous bare coupling term γ equal to zero and consider the remaining terms as an integral equation for Γ (i.e., $\Gamma=\Gamma GK$), we find $\Gamma\equiv 0$. The integral around the triangle loop has three nontrivial single integrals that can be chosen to be the invariant for each internal line. It is then necessary to carry the external invariants over all values to solve the integral equation. Now consider a BS equation with the potential [Fig. 2(a)]

$$T=K+KGT.$$

If this equation is closed off on the left to form

$$\Gamma GT=\Gamma GK+\Gamma GKGT,$$

then by using the integral equation $\Gamma=\Gamma GK$, we immediately find $\Gamma=0$ or $T=\infty$. If the happened for one value of s, we should be at a bound-state pole of T. But it is true for all s, hence $\Gamma\equiv 0$. It is clear that we can demand the relation $\Gamma=\Gamma GK$ only for isolated values of s, but then we no longer have an integral equation for Γ. This difficulty does not exist for $\gamma\neq 0$.

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Coupling of the Spacelike and Timelike Wave Functions at Infinite Momentum

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It is shown, using the $E(2)$ basis for the unitary representations of $SL(2,C)$, that the spacelike and timelike solutions to the Abers-Grodsky-Norton wave equation do not decouple at infinite momentum. It is concluded that this equation cannot be used to saturate the (isospin-factored) algebra of currents.

1. INTRODUCTION

IT has recently been suggested[1] that infinite-component wave equations could be used to saturate the algebra of currents with one-particle states at infinite momentum, at least for the case in which the isotopic spin is factored out. However, the possibility of such saturation depends critically on the assumption that either no spacelike solutions to the wave equation exist, or, if they exist, that they decouple from the timelike solutions at infinite momentum.

More recently, it has become clear[2] that for the Abers-Grodsky-Norton[3] (AGN) wave equation originally proposed, and more generally,[4] for any physically reasonable nontrivial infinite-component wave equation, spacelike solutions do indeed exist. The purpose of the present paper is to show that for the original (AGN) wave equation the spacelike solutions do *not* decouple at infinite momentum.

2. NORMALIZATION CONDITIONS

Let $\Psi_\sigma(\mathbf{p})$ and $\Psi_\tau(\mathbf{p})$ denote the spacelike and positive-frequency timelike solutions to the AGN wave equation, respectively, where Ψ is an infinite-component spinor which transforms according to a Dirac$\times$unitary irreducible representation (UIR) labeled (j_0,c) [5] of $SL(2,C)$. Here $\mathbf{p}$ denotes the 3-momentum; $\tau=(m,j_3)$, where m, which depends on j, is the mass of the particle; j is the spin; and j_3 is the third component of spin in the

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