High-Energy Hadron-Nucleus Collisions. I. Protons

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(Received 22 December 1967)

A semiclassical theory is developed to describe high-energy ($\gtrsim$10 BeV) reactions of hadrons with large nuclei. Coulomb phase effects are important, and they are accurately included in the formulas obtained. The theory is applied to data of Bellettini *et al.* at CERN for scattering of 19.3-BeV/c protons on large nuclei. Semiquantitative agreement of theory and experiment is obtained. However, for the case of a lead target, variation of the parameters in the theory did not produce a theoretical curve which agreed with the experimental p-Pb differential cross section over the entire range of angles measured. To obtain agreement at the smallest angles (before the first diffraction dip), it seems necessary to assume a much more diffuse nuclear surface than is indicated by electron scattering data. This is the regime in which the theory should be most reliable. To reproduce the oscillations in the data at larger angles, one seems forced to assume a fairly sharply defined surface, only slightly more diffuse than electron scattering suggests. Reasons are given why the effective nucleon density may be more diffuse for proton scattering than for electron scattering. Further experiments are suggested.

I. INTRODUCTION

OBSERVATION of hadron interactions with nuclei is a staple of the nuclear physicist. However, theoretical analysis of such observations is simplified enormously when the incident hadron has a very high energy. The main reason for this is that the hadron travels faster than the characteristic excitations of the nuclear medium. It arrives at any point in the nucleus before any other signal of its presence resulting from previous collisions with target nucleons. The hadron takes a "snapshot" of the nuclear ground state. Thus the full complexity of the many-body problem can be avoided, and one can use the incident hadron to probe nature's solution to the A-body problem for a nucleus of mass number A without solving the corresponding $(A+1)$ problem.

The large accelerators now available make it feasible to carry out such experiments. Quantitative theories with only a few fundamental parameters related to hadron-nucleon interactions and to the structure of the nucleus can thus be tested.

In this paper, we present a crude beginning of such theoretical efforts. In Sec. II, we give a semiclassical theory of high-energy hadron-nucleus interactions. The main advantage of our discussion over previous ones[1] is that it permits the easiest possible transfer of classical intuition to a quantum-mechanical problem. In Sec. III, we analyze some data of Bellettini *et al.*[2] on proton-nucleus scattering at 19.3 GeV/c. A future paper will contain an analysis of experimental data on "A_1" production in nuclei.

II. EIKONAL (HIGH-ENERGY) OPTICAL MODEL

In this section, we expound the conceptual basis of the high-energy optical model. More formal discussions may be found in the references.[1]

A. Eikonal Wave Function

(i) Basic assumptions. Consider a particle of very high wave number $\mathbf{k}=k\hat{z}$, incident on a many-body system. At any point $\mathbf{r}$, the system is nearly uniform on the scale of the incident wavelength. If it were completely uniform, we could deduce from translational invariance that the wave function would be

$$\psi(\mathbf{r})=e^{ikz}\varphi(\mathbf{r}),\qquad \varphi(\mathbf{r})=e^{iKz},\tag{2.1}$$

and $k'=k+K$ would be the wave number in the medium. Since the medium is not completely uniform, we take K as a slowly varying function of r and write the eikonal wave function

$$\varphi(\mathbf{r})=\exp\left[i\int_{-\infty}^{z}dz'K(x,y,z')\right].\tag{2.2}$$

To determine K, we assume that the medium is made up of scatterers which produce scattered waves $f(\theta)e^{ikr}/r$ when a wave e^{ikz} is incident upon them. Only the waves scattered in the forward direction combine coherently

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[1] R. Glauber, *Lectures in Theoretical Physics* (Interscience Publishers, Inc., New York, 1959), Vol. I, p. 315; M. Goldberger and K. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), Chap. 11. References to some earlier work are found in A. Goldhaber and M. Goldhaber, in *Preludes in Theoretical Physics*, edited by A. de-Shalit *et al.* (North-Holland Publishing Co., Amsterdam, 1966), p. 313.

[2] G. Bellettini, G. Cocconi, A. Diddens, E. Lillethun, G. Matthiae, J. Scanlon, and A. Wetherell, Nucl. Phys. **79**, 609 (1966).

to modify the incident wave. The modification $d\varphi$ from a slab of thickness dz is then

$$d\varphi = i\xi\rho(r) f^0 \varphi dz. \tag{2.3}$$

Here ρ is the density of scatterers and f^0 is the forward scattering amplitude on a single nucleon. The geometrical constant ξ is easily obtained. The density of the incoming wave obeys the equation

$$d|\varphi|^2/dz = -2\rho\,\mathrm{Im}(f^0\xi)|\varphi|^2, \tag{2.4}$$

but this must also be given by the classical attenuation formula

$$d|\varphi|^2 dz = -\sigma_T\rho|\varphi|^2, \tag{2.5}$$

where σ_T is the total cross section for interaction with a single scatterer. Using the optical theorem

$$\sigma_T = (4\pi/k)\,\mathrm{Im} f^0 = 2\lambda\,\mathrm{Im} f^0, \tag{2.6}$$

we find $\xi = \lambda$, the de Broglie wavelength, and

$$\psi(\mathbf{r}) = e^{ikz}\varphi(\mathbf{r}) = e^{ikz}\exp\left[i\int_{-\infty}^{z} dz'\lambda f^0 \rho(x,y,z')\right]. \tag{2.7}$$

All the calculations in this paper depend on such an approximation to describe the projectile wave in the region of the target system.

For later purposes, it is useful to note the relation between the wave-number shift $K(\mathbf{r})$ and the optical potential $V(\mathbf{r})$:

$$K(\mathbf{r}) = \delta k = (dk/dE)\delta E = (dk/dE)(-V) = -V/v. \tag{2.8}$$

We have used the relations $K/k \ll 1$ and $E+V=\mathrm{const}$; v is the velocity of the incident particle and E is its kinetic energy.

What we have found so far is this: Provided that the nucleus may be treated as a gas of slowly moving free nucleons and that the incident hadron propagates freely through the nucleus between collisions with these nucleons, then at high energies the *wave* function φ develops in the same way as the classical *intensity* distribution I_c for unscattered beam particles, with the substitution $-\sigma_T \to if\lambda = -\frac{1}{2}\sigma_T + i(\mathrm{Re} f)\lambda$. Since the quantum-mechanical intensity I_Q is simply $|\varphi|^2$, I_c and I_Q are the same if the target particles are uncorrelated, but we shall see that differences arise when correlations are present. These differences occur because the quantum-mechanical wave packet is coherent over regions large compared with internucleon spacing, or, for that matter, to the size of the nucleus. In the practically impossible case that the packet has dimensions of 1 F or less, one would use I_c instead of I_Q to describe the beam. Of course, in this limit there would be no diffraction scattering by the nucleus.

Our derivation applies only for a dilute gas of nucleons, i.e., one in which the separation of target particles is large compared with their size (the range of force between target and projectile). Nevertheless, (2.7) is correct even for a dense gas, provided that the density varies little over a target-particle diameter and that the nucleons are uncorrelated.[3]

(ii) Spin effects. To order A^{-1}, where A is the mass number, there are as many nucleons with spin up as with spin down at any point in a nucleus. Therefore the quantity f^0 which appears in (2.7) must be averaged over the target-nucleon spin $\frac{1}{2}\sigma$. For the same reason, f^0 is averaged over the target isospin $\frac{1}{2}\boldsymbol{\tau}$, with a weight factor $1-(N-Z)/A\,\boldsymbol{\tau}_z$. This spin and isospin averaging makes the nuclear scattering sensitive to a different amplitude from that which describes forward scattering on a free target nucleon. Thus comparison of results with nucleons and nuclei as targets may permit the isolation of spin-dependent effects. This could be useful, for example, in high-energy p-p scattering, where the forward scattering amplitude may be written $a + b\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + c\sigma_{1z}\sigma_{2z}$. Direct determination of a, b, and c separately would require a double polarization experiment, which is impractical at present.

(iii) Impulse approximation. There are several factors which may lead to deviations from (2.7). We have assumed that the scattering amplitude $f(\theta)$ is the same in the medium as for a free target particle. The validity of this assumption, which constitutes the "impulse approximation," is discussed in Appendix A. Suffice it to say here that the error associated with this approximation is expected to be small ($\lesssim 1\%$) for the applications which concern us.

(iv) Correlation effects. We have assumed so far that the target particles are uncorrelated. We may see the effect of correlations by semiclassical considerations. Assume that there are only two-particle correlations. We wish to compute the probability δp that the trajectory of an incoming projectile will intercept exactly one target particle in an interval δz small compared with the mean free path but large compared with correlation distances. To second order in δz, this probability is

$$\begin{aligned}\delta p &= (\text{probability of at least one collision}) \\ &\quad - (\text{probability of two collisions}) \\ &= \int_0^{\delta z} dz\,\rho(z\hat{z})\sigma_T - \int_0^{\delta z} dz \int_z^{\delta z} dz' \rho^{(2)}(z\hat{z}, z'\hat{z})\sigma_T^2,\end{aligned} \tag{2.9}$$

with $\rho^{(2)}(\mathbf{r},\mathbf{r}') = \rho(\mathbf{r})\rho(\mathbf{r}')[1 + C(\mathbf{r},\mathbf{r}')]$, and the correlation function C depends strongly on the difference of its arguments $\mathbf{s} = \mathbf{r} - \mathbf{r}'$, but weakly on the average $\mathbf{R} = \frac{1}{2}(\mathbf{r}+\mathbf{r}')$. The result is

$$\delta p = \rho\sigma_T\delta z - \tfrac{1}{2}(\rho\sigma_T\delta z)^2 - \rho\sigma_T\delta z(\rho\sigma_T\mathcal{R}_c), \tag{2.10}$$

with

$$\mathcal{R}_c(\mathbf{r}) = \int_0^\infty ds\, C(\mathbf{r}, \mathbf{r} + s_z).$$

The approximation of integrating C to infinite $s = z' - z$ is justified because δz is large compared with correlation distances.

[3] Glauber (Ref. 1), p. 390.

The second term in (2.10) is simply the second term in the expansion of $\delta p=1-e^{-\rho\sigma_T\delta z}$, the probability of interaction in the absence of correlations. The third term, linear in δz, implies a correction to the mean free path to account for correlations:

$$\rho\sigma_T\rightarrow\rho\sigma_T(1-\mathcal{R}_c\rho\sigma_T). \tag{2.11}$$

One may express the implications of (2.11) quite simply. When target particles clump together, it becomes more likely that a projectile which misses one will miss another, and the mean free path is increased. The argument is equally valid with all signs reversed.

An analysis in terms of probability amplitudes instead of probability density ($-\sigma_T\rightarrow if\lambda$) leads to the correlation correction to the wave-number shift K,

$$K_2(\mathbf{r})=\lambda\rho(\mathbf{r})f^0[1+i\lambda\rho(\mathbf{r})f^0\mathcal{R}_c(\mathbf{r})]. \tag{2.12}$$

The alert reader may notice that the correlation term in (2.12) is only half that suggested by applying the optical theorem to (2.11). This is a genuine quantum-mechanical effect.

Following Johnston and Watson,[4] we shall take $\mathcal{R}_c\approx-0.8$ F in saturated nuclear matter. This turns out to decrease by about 20% the mean free path of a proton in the interior of a nucleus. The negative $\mathcal{R}_c$ reflects the almost universal assumption, still untested experimentally, that nucleons in the nucleus do not interpenetrate each other.

In the outer region of the nucleus the density falls below its central, saturated value, and the nucleons may cluster in regions of higher than average density, leaving holes between them. Such clustering would imply positive values of $\mathcal{R}_c(\mathbf{r})$ at the nuclear surface. Alpha-particle clustering (which, of course, involves up to four-nucleon correlations) could lead to a reduction of K (no correlations) by, say, 25%. Such strong correlation effects are best computed, not in terms of $\mathcal{R}_c$, but by carrying out the analysis leading to (2.7) for a system of target α particles,

$$K\approx\lambda\rho_\alpha f_\alpha{}^0=\tfrac{1}{4}\lambda\rho f_\alpha{}^0, \tag{2.13}$$

where $f_\alpha{}^0$ is the forward hadron-α scattering amplitude. If the total p-α cross section is thrice the p-p cross section, we obtain the 25% figure mentioned above.[5] Furthermore, the α particle would be "blacker" than a nucleon, so that the real part of f^0 for an α could be proportionately smaller than that for a nucleon. Since the degree of surface clustering is a matter of considerable debate, we have left it adjustable when trying to fit experimental data.

B. Elastic Scattering Amplitude

(i) Huygen's principle and analogy with partial-wave expansion. Knowing the eikonal wave function $\psi(\mathbf{r})$, we are ready to compute the amplitude for elastic scattering. The wave just beyond the target system has suffered a complex phase shift 2δ relative to a wave in the absence of the target, with

$$2\delta(x,y)=\int_{-\infty}^{\infty}dz\,K(x,y,z). \tag{2.14}$$

We may compute the scattering amplitude by Huygen's principle from the difference between the phase-shifted wave and an unscattered wave,

$$F(\mathbf{q})=\frac{k}{2\pi i}\int dxdy\,e^{i(q_1x+q_2y)}(e^{2i\delta(x,y)}-1), \tag{2.15}$$

where $\mathbf{q}$ is the momentum transfer to the projectile. If the target is spherically symmetric, then $\delta(x,y)$ will be simply $\delta(b)$, with $b=(x^2+y^2)^{1/2}$, and F may be written

$$F(q)=\frac{k}{i}\int_0^{\infty}bd\,b\,J_0(qb)(e^{2i\delta(b)}-1), \tag{2.16}$$

using Parseval's integral representation of the Bessel function J_0.

For small $q=k\theta$, we may make the substitutions $kb\rightarrow l+\frac{1}{2}$, $J_0(qb)\rightarrow P_l(\cos\theta)$ to write the integral as a sum of partial-wave amplitudes:

$$F(q)=\frac{1}{2ik}\sum_l(2l+1)P_l(\cos\theta)(e^{2i\delta_l}-1). \tag{2.17}$$

Thus, to the extent that our approximations are valid we have obtained the partial-wave phase shifts $\delta_l=\delta((l+\frac{1}{2})/k)$.

(ii) Coulomb effects. In practice, we must take account of long-range Coulomb forces as well as short-range strong forces. We do this by recalling the effect of Coulomb forces on the usual partial-wave expansion[6]:

$$\begin{aligned}
&F(q)=F_C(\theta)+F_N(\theta),\\
&F_N(\theta)=(2ik)^{-1}\sum_l(2l+1)P_l(\cos\theta)e^{2i\sigma_l}(e^{2i\delta_l}-1),\\
&F_C(\theta)=-(\eta/2k\sin^2\tfrac{1}{2}\theta)\,e^{-2i\eta\ln\sin\frac{1}{2}\theta}e^{2i\sigma_0},\\
&\quad\eta=Ze^2/\hbar v,\quad\sigma_l=\arg\Gamma(l+1+i\eta).
\end{aligned} \tag{2.18}$$

Using approximations appropriate to the regime of the eikonal treatment, $l_{\max}\gg1$ and $\theta\ll1$, we write

$$F(q)=F_C(\theta)+\frac{k}{i}\int_0^{\infty}db\,bJ_0(qb)e^{2i\eta\ln(kb)}(e^{2i\delta(b)}-1). \tag{2.19}$$

There is a small additional subtlety here. Equation (2.19) is obtained by matching partial waves inside the nucleus to solutions in the presence of a point Coulomb potential outside. The actual Coulomb potential inside the nucleus is that of an extended charge distribution. Thus, in order to match to point Coulomb solutions

[4] R. Johnston and K. Watson, Nucl. Phys. **28**, 583 (1961).

[5] This is consistent with the ratio $2\sigma(p\text{-He})/(\sigma(p\text{-}p)+\sigma(p\text{-}n))$ obtained for 1-BeV protons by G. Igo, J. Friedes, H. Palevsky, R. Sutter, G. Bennett, W. Simpson, D. Corley, and R. Stearns (to be published).

[6] Goldberger and Watson (Ref. 1), Chap. 6. A direct derivation of Coulomb effects in the eikonal approximation is given by Glauber (Ref. 1).

outside, one must introduce inside the nucleus an effective potential

$$V_{\rm eff}(r)=V_{\rm Coul}(\text{extended})-V_{\rm Coul}(\text{point})$$
$$=-4\pi\eta v\int_r^\infty dr'(r')^{-2}\int_{r'}^\infty dr''(r'')^2\rho_{\rm ch}(r''),\qquad(2.20)$$

where $\rho_{\rm ch}(r)$ is the charge density distribution, normalized to unity. The final expression for the wave-number shift K is obtained with the help of (2.8):

$$K(r)=K_2(r)-(1/v)V_{\rm eff}(r).\qquad(2.21)$$

For high-energy scattering of protons from lead, the Coulomb amplitude F_C is at least comparable with the nuclear amplitude at all angles, and precise calculation of Coulomb effects is essential.

(iii) Total cross sections. By analogy with the usual partial-wave discussion, we may deduce the total inelastic cross section:

$$\sigma_{\rm in}=2\pi\int_0^\infty b\,db[1-e^{-4\,{\rm Im}\delta(b)}].\qquad(2.22)$$

To the extent that Coulomb phases may be ignored, we may obtain the "total nuclear cross section" from

$$\sigma_{\rm in}+\sigma_{{\rm el}N}=\sigma_T-\sigma_{\rm Coul}=4\pi\int_0^\infty b\,db[1-\cos 2\,{\rm Re}\delta e^{-2\,{\rm Im}\delta}],$$
$$\sigma_{{\rm el}N}\equiv\int d\Omega|F_N(\theta)|^2.\qquad(2.23)$$

However, when Coulomb phases are big, interference between F_C and F_N plays an important part and (2.23) has no direct physical significance. The quantities which can be compared with experiment are (2.22) and $d\sigma/d\Omega=|F_N+F_C|^2$.

C. Corrections to Elastic Scattering Cross Section

There are several reasons for possible deviation from the prediction (2.19) of the elastic scattering from a nucleus. First, let us backtrack a bit to formula (2.8), relating $K(\mathbf{r})$ to an optical potential $V(\mathbf{r})$ acting on the projectile. If V were a true potential appearing in a Dirac or Klein-Gordon equation, then (2.19) would be the eikonal approximation to the exact scattering amplitude. Saxon and Schiff[7] have estimated the relative error resulting from the eikonal approximation. For our case, this takes the form

$$\delta F_N/F_N=O(q^2R/k),\qquad(2.24)$$

where R is the radius of the target. For 20-BeV p-Pb scattering at $q=1$ F^{-1}, this is $O(7\%)$.

We have taken a smoothly varying optical potential. In fact, one might expect nonlocal effects over distances of the order of a nucleon size ($\sim$1 F) or a nucleon correlation length (also $\sim$1 F). Thus we expect deviations from our calculation for $q\gtrsim 1$ F^{-1}.

Finally, one might say that the whole approach taken here has dubious validity because it entails taking seriously an ordinary space-time picture of events on a scale of 10^{-13} cm. Since we do not know how to proceed otherwise, we feel justified in ignoring this objection until and unless experiment forces us to face it.

D. Inelastic Reactions

(i) Single-interaction effects. If an experiment is not sensitive to excitation of the nuclear target, then scattering which excites or fractures the nucleus will not be distinguished from elastic scattering. To compare our results with data from such experiments, we must estimate the inelastic scattering. If the particles in the nucleus are uncorrelated, we may estimate the differential cross section for exciting the nucleus semiclassically. First, suppose that the excitation occurs in a collision with a single-target nucleon; then $d\sigma/d\Omega$ is given by

$$\frac{d\sigma}{d\Omega_{\rm in}}=\frac{d\sigma}{d\Omega_f}[1-P_G(\mathbf{q})]\sum_n p_n.\qquad(2.25)$$

Here $d\sigma/d\Omega_f$ is the differential cross section for elastic scattering on a single-target nucleon, P_G is the probability that the nucleus remains in its ground state, and we expect $P_G\ll 1$ for $qR\gg 1$.[8] The factor p_n is the probability that a projectile passing through the position of target nucleon n will not be absorbed while passing through the nucleus. The "effective number" of free nucleon scatterers is

$$\mathcal{A}(A)=\sum_n p_n\approx\int d^3r\,\rho(\mathbf{r})\times\exp\left[-\sigma_T\int_{-\infty}^\infty dz'\rho(x,y,z')\right].\qquad(2.26)$$

In (2.26), we neglect the fact that the path of the projectile is bent slightly by the small-angle scattering at the point $\mathbf{r}$.

Equation (2.25) actually gives a lower limit to $d\sigma/d\Omega_{\rm in}$, since the possibility of exciting the nucleus by exciting two or more target nucleons is excluded. The ratio of double scattering (without particle production) to single scattering is $d\sigma_2/d\sigma_1\approx\sigma_{\rm el}/\sigma_T$, since two *elastic* scatterings must occur. For incident protons $\sigma_{\rm el}/\sigma_T$ is about $\frac{1}{4}$,[9] and therefore the order of magnitude of the inelastic scattering is correctly given by (2.25) if correlations are unimportant.

Explicit computations confirm the intuitive expectation that most of the contributions to the effective

[7] D. Saxon and L. Schiff, Nuovo Cimento **6**, 614 (1957).

[8] See Appendix B for a more careful discussion on this point.

[9] K. Foley, S. Lindenbaum, W. Love, S. Ozaki, J. Russell, and L. Yuan, Phys. Rev. Letters **11**, 425 (1963); K. Foley, R. Jones, S. Lindenbaum, W. Love, S. Ozaki, E. Platner, C. Quarles, and E. Willen, *ibid.* **19**, 837 (1967).

number of target nucleons $\mathcal{A}(A)$ originate in the surface of the nucleus. Thus strong surface correlations could appreciably alter $\mathcal{A}(A)$. To obtain an upper limit to such an effect, again suppose that the surface nucleons are clustered tightly into α particles. The scattering amplitude on an α is certainly less than four times that on a single nucleon, probably about three.[5] The cross section on an α particle would then be about 9/4 that on four uncorrelated nucleons, dropping to the uncorrelated value for momentum transfers large compared with an inverse α-particle radius. Thus $\mathcal{A}(A)$ calculated from (2.26) might be doubled or tripled at small momentum transfers ($<1\ \mathrm{F}^{-1}$) by surface correlation effects, but should be very nearly correct as it stands at larger momentum transfers.

(ii) Deformation effects. We have discussed inelastic scattering involving single-nucleon or α-particle cluster excitation. There is another class of corrections associated with the possibility that the ground state of the nucleus is deformed. Let us treat these effects in the adiabatic approximation, that is, let us neglect the motion of the nucleus during the passage of the fast ($v\sim c$) particle. This should be a very good approximation for nonspherical nuclei, which have characteristic rotation frequencies of tens of keV, such as U^{238}.[10] For nuclei which have only oscillating deformations about a spherical equilibrium, with characteristic frequencies of MeV, such as (perhaps) Pb^{208},[11] the passage time of the fast particle is about 10% of an oscillation period. Thus the adiabatic approximation should give a slight overestimate of the oscillation effect, which is small in any case.

When the adiabatic approximation holds, we may calculate the scattering for each orientation $\boldsymbol{\alpha}$ of the nucleus. The eikonal calculation proceeds as before, with a wave-number shift

$$K_\alpha(\mathbf{r})=\lambda f\rho_\alpha(r)\,, \tag{2.27}$$

where $\rho_\alpha(\mathbf{r})$ is the density distribution of nucleons for orientation α. For example, a prolate nucleus might have a density distribution

$$\begin{aligned}\rho_\alpha(\mathbf{r})&=\rho_0/\{1+\exp[(r-r_0(\theta,\varphi))/a]\}\,,\\ r_0(\theta,\varphi)&=R[1+\epsilon P_2(\cos\Theta)]\,,\end{aligned} \tag{2.28}$$

where $\cos\Theta$ is the projection of $\hat{r}$ onto the long axis of the nucleus, specified by $\boldsymbol{\alpha}=(\bar{\theta},\bar{\varphi})$:

$$\cos\Theta=\cos\theta\,\cos\bar{\theta}+\sin\theta\,\sin\bar{\theta}\,\cos(\varphi-\bar{\varphi})\,.$$

This ρ_α gives rise to an elastic scattering amplitude $f(\theta,\varphi,\boldsymbol{\alpha})$. Since the value of $\boldsymbol{\alpha}$ could be measured, at least in principle, after the incident projectile had passed, the amplitudes for different $\boldsymbol{\alpha}$ are incoherent. Therefore the proper way to compute the differential cross section is not to superpose amplitudes, but rather to average the cross section over $\boldsymbol{\alpha}$:

$$\frac{d\sigma}{d\Omega}(\theta)=(4\pi)^{-1}\int d\Omega_\alpha|f(\theta,\varphi,\boldsymbol{\alpha})|^2\,. \tag{2.29}$$

The qualitative effect of deformation is clearly to blur radius effects, making maxima and minima in $d\sigma/d\Omega(\theta)$ less conspicuous. This effect is obvious in a comparison of the diffraction patterns of 19.3-BeV protons on Pb and U targets.[2] The Pb nucleus is thought to be nearly spherical, while U is quite deformed ($\epsilon\sim\frac{1}{4}$). The amplitude $f(\theta,\varphi,\boldsymbol{\alpha})$ is diagonal by construction for a set of nuclear basis states labeled by $\boldsymbol{\alpha}$. Thus, in the $\boldsymbol{\alpha}$ basis, we have computed the elastic scattering from a deformed nucleus with a specific orientation. However, if we revert to the angular-momentum basis $|J,m\rangle$, where $\mathbf{J}$ is the generator of rotations on $\boldsymbol{\alpha}$, the amplitude $\langle J'm'|f(\theta,\varphi)|J,m\rangle$ is not diagonal. Drozdov[12] and Blair[13] used the unitary transformation from $\boldsymbol{\alpha}$ to $\mathbf{J}$ to estimate matrix elements for excitation of collective states with angular momentum J' from the ground state $|J_0\rangle$. For our purposes here, the individual excited states of the nucleus and the ground state are indistinguishable, so that we must sum $(2J_0+1)^{-1}\times|\langle J'm'|f|J_0m\rangle|^2$ over all J', m', and m. This sum is precisely equivalent to the angular average (2.29), assuming that the energy differences $E_{J'}-E_J$ are insignificant for computing f, as is implied by the adiabatic approximation.

Note that the contribution of a particular inelastic state J' to $d\sigma/d\Omega(\theta)$ can be greater than the total deformation effect at angle θ. This is possible because the amplitude $\langle J_0|f(\theta)|J_0\rangle$ for scattering with no excitation may be reduced by the deformation, thus more or less compensating for the appearance of collective inelastic scattering.

E. Production Reactions and Other "Single-Scattering" Effects

(i) Production. Eikonal wave functions of the form (2.2) may also be used to calculate differential cross sections for production reactions. To orient ourselves here, we first consider a case in which the incident projectile has such a small interaction with each target nucleon that it is unlikely to interact more than once while passing through the nucleus. Call a the amplitude for production of some final particle on a single nucleon, e.g., $\pi+p\rightarrow A_1+p$, and A the corresponding amplitude for the whole nucleus. In the weak interaction (wi) limit we simply add the amplitudes from each nucleon, with a relative phase depending on momentum transfer and position:

$$A_{\mathrm{wi}}(q)=\sum_{i=1}^{N+z}a(q)e^{i\mathbf{q}\cdot\mathbf{r}_i}=a(q)\int d^3r\,\rho(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}\,. \tag{2.30}$$

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[10] J. Davidson, Rev. Mod. Phys. **37**, 105 (1965).

[11] A. Lane and E. Pendlebury, Nucl. Phys. **15**, 39 (1960).

[12] S. Drozdov, Zh. Eksperim. i Teor. Fiz. **28**, 734, 736 (1955) [English transl.: Soviet Phys.—JETP **1**, 591, 588 (1955)].

[13] J. Blair, Phys. Rev. **115**, 928 (1959).

This is just the first Born approximation with a potential proportional to $a(q)\rho(\mathbf{r})$.

To account for the strong interaction of the projectile with the nucleus, we must modify our treatment. Let us now make the reasonable assumption that the production process involves only a single nucleon, while the waves before and after production are modified by wave-number shifts K and $\bar{K}$, respectively. For small-angle production, we neglect deviation of the path through the nucleus from a straight line, and write

$$A(q)=a(q)\int d^3r \exp\left[i\int_{-\infty}^{z} dz'\, K(x,y,z')\right] e^{i\mathbf{q}\cdot\mathbf{r}}(\mathbf{r}) \times\exp\left[i\int_{z}^{\infty} dz'\, \bar{K}(x,y,z')\right] e^{2i\sigma(b)}, \quad b=(x^2+y^2)^{1/2}. \tag{2.31}$$

This is an eikonal distorted-wave Born approximation for a production potential proportional to $a\rho(\mathbf{r})$. The factor $e^{2i\sigma(b)}$ accounts for long-range Coulomb effects on the in- and out-going waves.

The absorption effects summarized in (2.31) alter a commonly stated conclusion about thresholds for coherent production of a resonance on a nucleus. The minimum momentum transfer to a target T in the production reaction $X+T\rightarrow Y+T$ is found when Y comes out parallel to the incident X. This longitudinal momentum transfer $q_{||}$ is given at high energies by

$$q_{||}\approx (m_Y^2-m_X^2)/2p_X, \tag{2.32}$$

where the notation is obvious. In the absence of absorption, coherence of the production reaction over the whole nucleus would require $q_{||}R\ll 1$, where R is the nuclear radius. However, as a result of the absorption of the coherent wave, coherence in the direction parallel to $\mathbf{p}_X$ is only maintained over dimensions of order Λ, where Λ is the mean free path in the nucleus. Thus we have

$$q_{||}\Lambda\ll 1, \tag{2.33}$$

and there should not be a strong A dependence of the coherent-production threshold. In particular, we would *not* expect coherent production at zero degrees to drop as A increases, but rather it should rise, since the production takes place mainly in a ring of radius $R\propto A^{1/3}$, and the ring gets bigger.

(ii) Single-scattering effects in elastic reactions. Formula (2.31) may also be applied to elastic scattering amplitudes for which the single-scattering assumption is valid. For example, in elastic scattering of protons, the amplitude for a single-target nucleon may be written $f(q)+s(q)\boldsymbol{\sigma}\cdot\hat{k}_i\times\hat{k}_f$; we ignore the spin of the target nucleon, which is unimportant for coherent processes with a target nucleus. If $s(q)|\hat{k}_i\times\hat{k}_f|$ is small compared with f^0, then we may assume that $s(q)$ acts only once on passage through the nucleus, and write the amplitude for spin-orbit coupling with the nucleus using (2.31), with $a(q)\rightarrow s(q)\hat{k}_i\times\hat{k}_f\cdot\boldsymbol{\sigma}$.

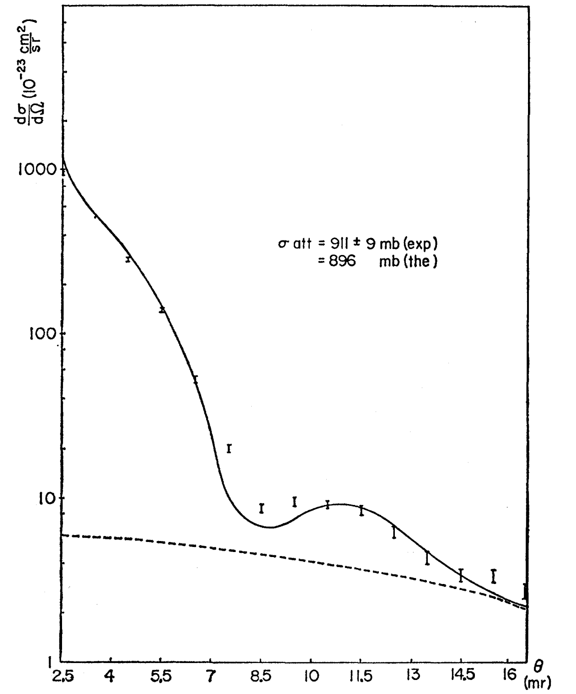

FIG. 1. Laboratory differential cross section for proton scattering on copper at 19.3 BeV/c in units of 10^{-23} cm^2/sr. The experimental points come from Ref. 2. The theoretical curve uses a Woods-Saxon nuclear density, with $R=4.3$ F, $a=0.55$ F (Ref. 18), to compute the elastic-scattering cross section. The only inelastic scattering included in the theory is quasielastic scattering from six equivalent free nucleons. The solid curve is the sum of elastic and inelastic, the dashed curve, inelastic alone.

The analogous procedure may be applied to estimate corrections to elastic scattering due to the finite size of target nucleons. Taking $d(q)=f(q)-f^0$, we may substitute d for a in (2.31) to determine the amplitude D which should be added to F in (2.18), to account for the fact that $f(q)$ varies somewhat in the range of momentum transfers to which we apply (2.18). For high-energy protons at $q=1$ F^{-1}$[|f(q)|-|f^0|]/|f^0|$ is about -0.2. If that ratio is unchanged by removing absolute value signs, then the single-scattering assumption is justified for computing $D(q)$, $q\lesssim 1$ F^{-1}.

F. Summary of Theory

In this section, we have seen how to compute cross sections for reactions of fast elementary particles with large nuclei. The basis for the whole development is the use of an approximate wave function for a projectile inside the nucleus,

$$\psi(\mathbf{r})=e^{ikz}\exp\left[i\int_{-\infty}^{z} dz'\, K(x,y,z')\right],$$

where the complex wave-number shift K depends on the interaction of the projectile with individual nucleons and on the distribution of nucleons in the target. The main results are contained in (2.19),

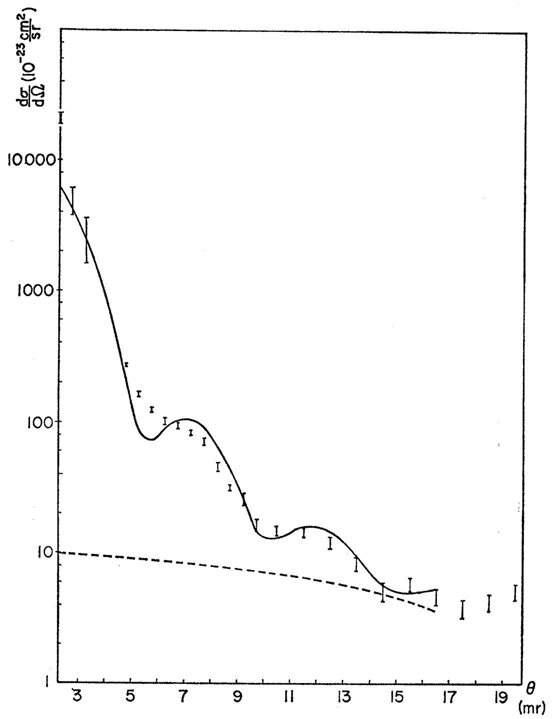

FIG. 2. Proton scattering on uranium at 19.3 BeV/c. See caption of Fig. 1. Here $R=6.8$, $a=0.5$ F (Ref. 18). We assume 10 equivalent free nucleons.

the eikonal approximation for elastic scattering, and (2.31), which gives single-scattering effects, including particle production.

III. SCATTERING OF 19.3-BeV/c PROTONS

Bellettini *et al.*[2] have observed "elastic" scattering of protons on a variety of nuclei: Li^6, Li^7, Be^9, C^{12}, Al^{27}, and natural Cu, Pb, and U. By "elastic" we indicate that the energy resolution was not sufficient to distinguish cases in which the target nucleus was excited from cases of true elastic scattering, although hadron production could be excluded. These experiments have already been the subject of several simplified analyses,[14] perhaps the most thorough being that of Frahn and Wiechers (FW).[14] These authors, following earlier work,[15] describe the scattering from a complex nucleus by a simple parametrization of the phase-shift function $e^{2i\delta_l}-1$. The fits which they obtain for the heavy nuclei (Cu, Pb, U) are similar in appearance to those in Figs. 1–4, described below. They also fit the lighter nuclei (Li^6–Al^{27}), but we did not.

Using the eikonal approximation, one may infer from the phase-shift function the effective proton-nucleus

[14] Bellettini *et al.* (Ref. 2); O. Benestad and H. Olsen, Phys. Rev. Letters **17**, 1031 (1966); A. Dar and S. Varma, *ibid.* **16**, 1003 (1966); W. Frahn and G. Wiechers, *ibid.* **16**, 810 (1966); Ann. Phys. (N. Y.) **41**, 442 (1967).

[15] W. Frahn and R. Venter, Ann. Phys. (N. Y.) **24**, 243 (1963); R. Venter, *ibid.* **25**, 405 (1963).

TABLE I. Comparison of the integrated optical potential of FW with the first-order high-energy model.

Target	$I(W)$[a] (F^2)	$-A\sigma_T(p\text{-}p)/2$[b] (F^2)	$I(V)$ (F^2)	$-\frac{1}{2}\alpha A\sigma_T(p\text{-}p)$ (F^2)
Li^6	−12.8	−11.7	4.9	2.9
Li^7	−14.4	−13.6	5.8	3.4
Be^9	−16.4	−17.5	7.3	4.4
C^{12}	−22.6	−23.2	7.1	5.9
Al^{27}	−55.5	−52.7	13.5	13.2
$Cu^{63.6}$	−161	−248	29.6	31.0
$Pb^{207.2}$	−935	−404	127	101
U^{238}	−935	−454	140	116

[a] Frahn and Wiechers (Ref. 14).
[b] Reference 17.

potential $(V+iW)(r)$.[16] This is not a reliable procedure for the interior of heavy nuclei, since they are nearly opaque at small impact parameters. It would hardly affect the computed differential cross section to set $e^{2i\delta(b)}=0$ at small b, but this would imply $W(r)=\infty$ at small r. However, for the light nuclei, which are semi-transparent even at the center, inference of the potential from the phase shift is meaningful. FW compute $I(W)$ and $I(V)$, the integrals over the nuclear volume of the imaginary and real parts of the potential. From (2.8) we deduce

$$I(V)+iI(W)=-vA\lambda f^0, \qquad (3.1)$$

where A is the mass number of the target nucleus. Table

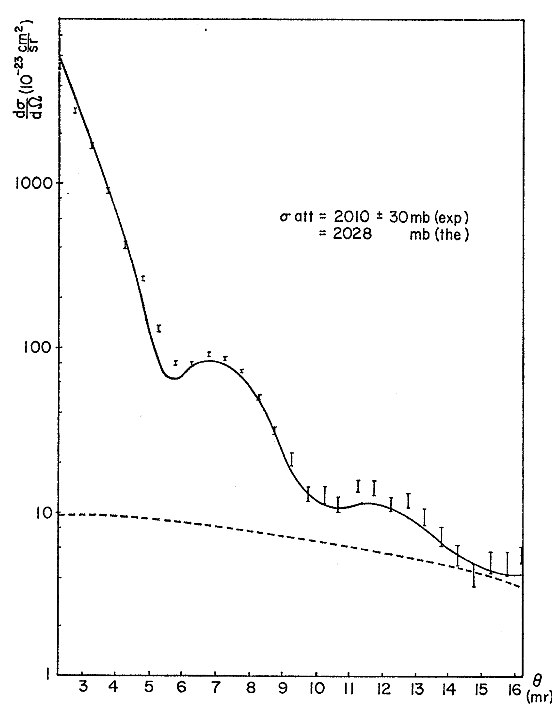

FIG. 3. Proton scattering on lead at 19.3 BeV/c. See caption of Fig. 1. Here $R=6.5$, $a=0.5$ F (Ref. 18). We assume 10 equivalent free nucleons.

[16] W. Frahn, Phys. Letters **24B**, 216 (1967), discusses this subject with a somewhat different emphasis.

TABLE II. Total cross sections of light nuclei.

Target	σ_T[a] (mb)	$A\sigma_T(p\text{-}p)$[b] (mb)
Li^6	232± 5	234
Li^7	250± 5	253
Be^9	278± 4	350
C^{12}	335± 5	468
Al^{27}	687±10	1007

[a] Reference 2.
[b] Reference 17.

I shows the comparison of the right and left sides of (3.1), using the values of $I(V)$ and $I(W)$ deduced by FW.

The first two columns of Table I agree very well with each other, except, as expected, for the heavy nuclei. In terms of the theory presented in Sec. II, the agreement is perhaps too good, since the absorption in the bulk of the nucleus should be about 25% bigger than the first approximation used here because of negative correlations in saturated nuclear matter. The answer may be that positive correlations in the nuclear surface cancel the negative correlation effect in $I(W)$.

The remaining two columns of the table present a cloudy picture. Consistency requires us to disregard the heavy nuclei, since V is not well determined in the nuclear interior. This may be less serious for $I(V)$ than for $I(W)$, since the V of FW does not grow enormously as $r\rightarrow 0$, but it precludes quantitative reliability. As it happens, even the values for light nuclei scatter a good deal. The general tendency suggests a larger value of $-\alpha$ than that determined by p-p scattering.[17]

Note that these nuclei are *not* so transparent that the first Born approximation holds. In Table II, we compare total cross sections of the light nuclei with the Born approximation $A\sigma_T(pp)$. We conclude that, for the semitransparent light nuclei, the unforced fits of FW give strong support to the ideas underlying the present work.

Let us turn to our optical-model description of the proton scattering on Cu, Pb, and U. We have used the theory of Sec. II, taking f^0 from p-p scattering experiments.[17] The nuclear density distribution is given by the Woods-Saxon form

$$\rho_{WS}(r)=\rho_0/[1+e^{(r-R)/a}], \qquad (3.2)$$

where R and a are the parameters of the charge density distribution determined by electron scattering.[18] The scattering amplitude is

$$F_{scat}=F+D, \qquad (3.3)$$

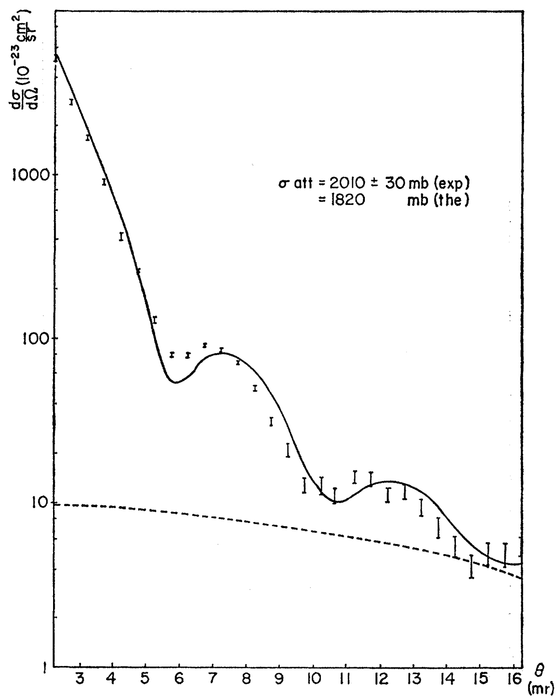

FIG. 4. Proton scattering on lead at 19.3 BeV/c. The only change from Fig. 3 is $a=0.7$ F.

with F given by Eq. (2.19) and D the amplitude described in Sec. II E (ii). The D term gives an approximate correction to F due to the finite range of the p-p interaction. It affects none of our conclusions.

There is another way to estimate the finite-range effect. In this method, we treat the nuclear charge density as a folding of the proton density and the charge distribution of the proton. We assume that the strong nuclear potential is given by folding the nuclear density with the proton-nucleon potential. Assuming Gaussian distributions for the proton charge distribution and the proton-nucleon potential, and taking the latter as the folding of the former with itself,[19] we find that the strong-interaction potential is slightly more extended than the charge distribution:

$$\frac{K(r)}{K(0)}\approx(2\pi)^{-3/2}\int d^3s\,[\rho_{ch}(s)/\rho_{ch}(0)]e^{-2(s-r)^2}, \qquad (3.4)$$

where all lengths are measured in fermis. For Pb^{208}, with charge distribution specified by $R=6.5$ and $a=0.5$, the modification (3.4) changes the distribution to one with $R\approx 6.7$, $a\approx 0.5$. This is a small effect, an increase in the half-density radius by 3% (which is less than $\frac{1}{10}$ of the surface thickness $t=4.4a$). We have *not* included the modification (3.4) in our *a priori* calculations of experimental quantities.

[17] We take $\sigma_T(p\text{-}p)=39$ mb, $\alpha=(\text{Re}f^0/\text{Im}f^0)=-0.25$, following K. Foley, R. Jones, S. Lindenbaum, W. Love, S. Ozaki, E. Platner, C. Quarles, and E. Willen, Phys. Rev. Letters **19**, 857 (1967). Our values of $-\alpha$ are somewhat larger than this reference suggests, but the difference does not affect any conclusions.

[18] L. Elton, *Nuclear Sizes* (Oxford University Press, London, 1961); H. Anderson, R. McKee, C. Hargrove, and E. Hincks, Phys. Rev. Letters **16**, 434 (1966).

[19] This simple assumption is quite adequate for our purposes. In fact, it has been exploited with remarkable success in relating p-p scattering to the charge form factor of the proton by T. Chou and C. Yang (to be published).

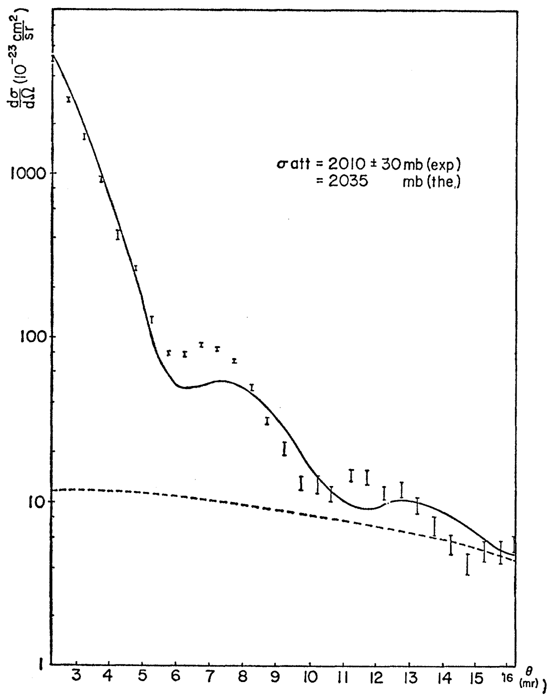

FIG. 5. Proton scattering on lead at 19.3 BeV/c. Here $R=4.0$, $a=1.1$ F. We use the computed 13.5 equivalent free nucleons for the inelastic scattering.

In calculating the phase shift $\delta(b)$, we use (2.21) and include the effect of repulsive correlations in the nuclear interior, but no positive correlations at the surface. The resulting *a priori* fits to the data are shown in Figs. 1–3. The only fitted parameter is $\mathcal{A}(A)$, the number of equivalent free nucleons to produce the incoherent large-angle inelastic scattering:

$$\frac{d\sigma}{d\Omega}=|F_{\rm scat}|^2+\mathcal{A}\frac{d\sigma}{d\Omega_{\rm el}}(p\text{-}p)\,. \qquad (3.5)$$

The fits to Cu and U data are impressive. Note the good agreement for Cu in the attenuation cross section:

$$\sigma_{\rm att}=\sigma_{\rm abs}+\sigma_{\rm el}(\theta>5 \text{ mrad}). \qquad (3.6)$$

Since both Cu and U are deformed nuclei, most of the discrepancies between theory and experiment may be attributed to the effect of averaging over orientations of the aspherical target. We have not included effects due to target deformation in our calculations.[20]

We come now to the case of Pb. Even a casual glance at Fig. 3 indicates two reasons to believe the lead target

TABLE III. Values of $\mathcal{A}(A)$ for $a=0.7$ and $R=1.2\,A^{1/3}$ F.

A	4	10	20	40	80	150	208
$\mathcal{A}$	2.8	4.7	6.3	7.8	9.3	10.5	10.8

[20] Estimates were made by G. Matthiae, Nucl. Phys. **87**, 809 (1967).

TABLE IV. Values of $\mathcal{A}(208)$ for $R=6.5$ F.

a	0.5	0.7	0.9	1.1	1.3
$\mathcal{A}$	6.6	9.8	13	17	21

is larger than we have assumed. First, $\sigma_{\rm att}$ as calculated is 10% below the experimental value. Secondly, the oscillations of the theoretical curve lag behind those given by experiment. There is a third reason. Using Eq. (2.26), we may compute $\mathcal{A}(A)$. It turns out that $\mathcal{A}(A)$ is quite sensitive to the diffuseness a, but fairly insensitive to the half-density radius R, as shown in Tables III and IV.[21]

For Cu, $\mathcal{A}(A)$ is about 5, with our *a priori* choice of parameters. This is in good agreement with the value $\mathcal{A}=6$ used in our fit. For Pb, our *a priori* choice would give $\mathcal{A}\approx 6.6$, which is not so close to the value $\mathcal{A}=10$ of our fit.

The arguments given above persuade us to change a from 0.5 to 0.7, yielding the curve in Fig. 4. Now a casual glance would leave one fairly happy with the agreement, but there is still a serious difficulty. At the first four small-angle points, where inelastic effects are negligible and the theory should be most reliable, the theoretical points are 10–12% above the experimental values. Close inspection of the curves of FW indicates a similar situation. Aside from the possibility of mechanical mistakes in theory or experiment, we see only two explanations. Either the theory is wrong in a deep way, perhaps having to do with Coulomb effects, or the effective nuclear density distribution is considerably different from the charge distribution.

Let us follow the second alternative. If it were not for Coulomb effects, it would be clear that we need a more diffuse nuclear surface to get agreement with experiment. The slope of the theoretical curve is about right, but the absolute magnitude is high. If the scatterer had about the same size but lower surface density, one could then get agreement. This intuition turns out to apply also in the presence of Coulomb effects. Figure 5 shows the results for $R=4$, $a=1.1$. The agreement at small angles is excellent (see Table V); $\sigma_{\rm att}$ (the) is acceptable, and the main discrepancies have been shifted to larger angles.

Before discussing these, let us ask what such an effective density distribution would signify. First of all, because the opacity is so high at small impact parameter, we can say almost nothing about the central density. Thus one would expect equally good results from an effective density distribution which looked the same in the surface region, but had a central value identical to that for $R=6.5$, $a=0.7$. We have verified this. Thus we may assume that the central density has about the value indicated by many-body calculations. The novelty of the new distribution lies only in its description of the

[21] Similar results were found by R. Glauber and G. Matthiae, reported by R. Glauber, *High-Energy Physics and Nulcear Structure* (North-Holland Publishing Co., Amsterdam, 1967), p. 311.

TABLE V. Comparison of theory and experiment for the small-angle scattering on lead.

labθ (mrad)	2.25	2.75	3.25	3.75	4.25	4.75	5.25	5.75
$(d\sigma/d\Omega)$ (10^{-23} cm^2/sr)[a]	5110±100	2780±68	1630±48	910±39	423±27	262±5	131±4	80±2
$(d\sigma/d\Omega)$ (the)[b]	5300	3120	1870	1050	530	236	97	54
$(d\sigma/d\Omega)$ (the)[c]	5670	3230	1830	944	441	183	82	65
$(d\sigma/d\Omega)$ (the)[d]	5140	2840	1640	917	484	241	120	69
$(d\sigma/d\Omega)$ (the)[e]	5370	2970	1680	917	464	219	103	59
$(d\sigma/d\Omega)$ (the)[f]	5290	3050	1830	1060	570	274	124	65

[a] Reference 2.
[b] Parameters of Fig. 3.
[c] Parameters of Fig. 4.
[d] Parameters of Fig. 5.
[e] Parameters of Fig. 6.
[f] Reference 22.

nuclear surface.[22] At least two phenomena could alter the effective nuclear density distribution without affecting the determination of charge density by elastic electron scattering. First, there may be a neutron tail extending beyond the proton density distribution. Second, there could be strong positive correlations, exemplified by α-particle clusters, in the region just inside the neutron tail. These clusters would not show up in electron scattering because of the long range of the e-p interaction. To confirm the qualitative reasonableness of such a picture, we have obtained a fit to the p-Pb scattering with an effective density defined as follows. First the neutron tail is tacked on:

$$\begin{aligned} &\rho=\rho_{\mathrm{WS}}(r), \quad r\leq R+3a \\ &\rho=\rho_{\mathrm{WS}}(R+3a)e^{(R+3a-r)/2a}, \quad r>R+3a \\ &R=6.5, \quad a=0.7. \end{aligned} \tag{3.7}$$

Then negative correlations in the nuclear interior are accounted for as in Eq. (2.12);

$$\bar{\rho}(r)=\rho(r)[1+i\mathcal{R}_C\lambda f^0\rho(r)]. \tag{3.8}$$

We include positive correlations, as well as a reduction in neutron density corresponding to the shift of neutrons out to the tail by defining a reduced effective density just inside the tail:

$$\begin{aligned} &\rho_{\mathrm{eff}}(r)=u(r)\bar{\rho}(r)+[1-u(r)]\xi(r)\rho(r), \\ &u(r)=\rho(r)/\rho(0), \\ &\xi(r)=\tfrac{1}{2}, \quad r\leq R+2.5a \\ &\xi(r)=1-1/[2e^{(r-R-2.5a)/a}], \quad r>R+2.5a. \end{aligned} \tag{3.9}$$

The resulting differential and attenuation cross sections (Fig. 6) are similar to those of Fig. 5, though about 5% higher. Practically exact agreement with Fig. 5 is obtained by applying (3.9) for $R=6.0$, $a=0.7$ F.

There are still problems in explaining the large-angle data. The mean value of $d\sigma/d\Omega$ (i.e., averaged over oscillations) depends sensitively on the amount of α clustering. We may estimate the effect of clustering by extrapolating the 1-BeV p-α cross section,[23] assuming that $d\sigma(p\text{-}\alpha)/dt$ is independent of incident momentum, aside from a small reduction associated with the drop in p-p total cross section between 1.7 and 19.3 BeV/C:

$$\frac{d\sigma}{d\Omega}(p\text{-}\alpha,19.3)\approx\left(\frac{19.3}{1.7}\right)^2\left(\frac{2\sigma_T(p\text{-}p,19.3)}{\sigma_T(p\text{-}n,1.7)+\sigma_T(p\text{-}p,1.7)}\right)^2 \times\frac{d\sigma}{d\Omega}\left(p\text{-}\alpha,\frac{19.3}{1.7}\theta,1.7\right)=10^5e^{-\theta^2/100}\text{mb/sr}. \tag{3.10}$$

Clearly, the amount of α clustering has a significant

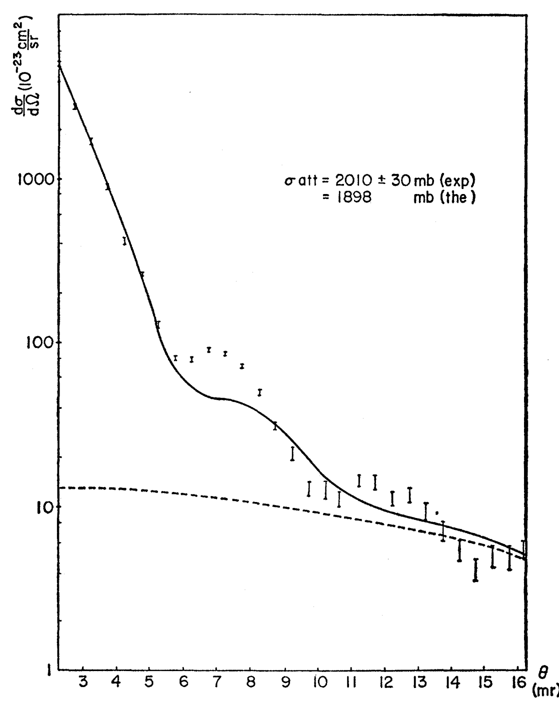

FIG. 6. Proton scattering on lead at 19.3 BeV/c. See text for assumed effective density distribution. We take 12 equivalent free nucleons for the inelastic scattering.

[22] One might ask whether a small change in f^0, the scattering amplitude on a single nucleon, could remove the small-angle discrepancy. The last row in Table V shows how row b would be modified if Imf^0 were multiplied by 0.7. At the same time, σ_{att} changes from 1.82 to 1.70 b. Clearly, reduction in f^0 is not the answer to the small-angle problem.

[23] H. Palevsky, J. Friedes, R. Sutter, G. Bennett, G. Igo, W. Simpson, G. Phillips. D. Corley, N. Wall, R. Stearns, and B. Gottschalk, Phys. Rev. Letters 18, 1200 (1967).

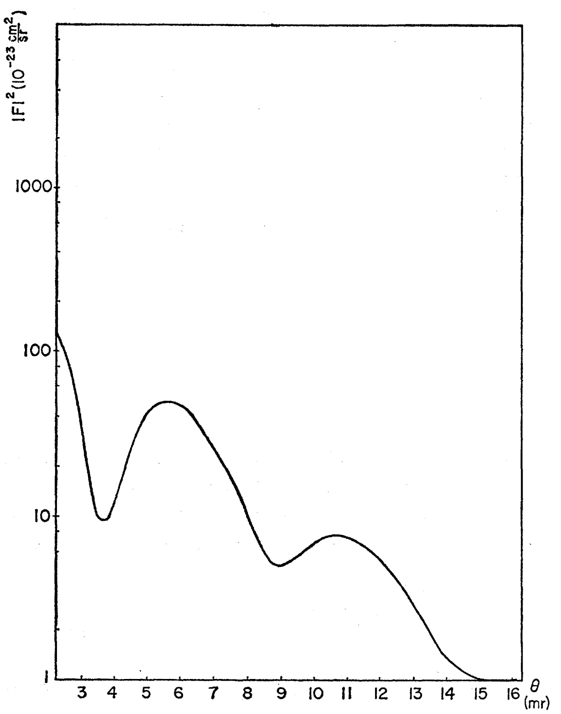

FIG. 7. The quantity $|\mathfrak{F}|^2$. If the differential cross section for a certain reaction on a single proton is $d\sigma/d\Omega = |\epsilon(\theta) f^0|^2$, then the single-scattering approximation gives the differential cross section on Pb, $d\sigma/d\Omega = |\epsilon(\theta)\mathfrak{F}(\theta)|^2$. We use the parameters of Fig. 5 in computing $\mathfrak{F}(\theta)$.

effect on $d\sigma/d\Omega$ at 5 mrad or more. Thus, in the diffuse-surface model, the magnitude of the large-angle cross section is not accurately predicted.

The oscillations in the large-angle cross section can only be explained by an effect involving coherent scattering from a region of nuclear dimensions. Conceivably, making further adjustments with $\rho_{\text{eff}}(r)$ (perhaps putting in a kink, an abrupt change) would bring the theoretical curve into agreement with experiment. However, such adjustments would be artificial at our present state of understanding.

Two other possible sources of the oscillations are coherent excitation of a particular excited state and spin-flip scattering. The latter may be treated in the single-scattering approximation of Eq. (2.31). We may write the result as

$$\frac{d\sigma}{d\Omega_{\text{sf}}}(\theta) = \frac{d\sigma/d\Omega_{\text{sf}}(p\text{-}p,\theta)}{d\sigma/d\Omega_{\text{el}}(p\text{-}p,0)} |\mathfrak{F}|^2. \qquad (3.11)$$

The quantity $|\mathfrak{F}|^2$ is plotted in Fig. 7, and it is clear that spin-flip can only be important if spin-flip scattering on a single proton at 5 mrad is comparable with the elastic forward p-p cross section. At small angles $d\sigma/d\Omega_{\text{sf}}(p\text{-}p)$ vanishes as q^2. Thus a crude upper limit, sufficient for our purposes, can be obtained by taking the ratio multiplying $|\mathfrak{F}|^2$ in (3.11) as $bq^2 = \theta^2/(250\ \text{mrad}^2)$, where the elastic p-p cross section is approximated as $|f^0|^2 \exp(-bq^2)$.[24] This excludes spin-flip scattering as a significant contribution to p-Pb scattering. As to coherent excitation, it is hard to see why just one state should be excited; as in the deformation model of Sec. II, one expects the differential cross section to become smoother, not to develop oscillations, when inelastic scattering is included.

Thus we know of no simple and natural explanation for the oscillations at large angles if the small-angle data are taken seriously as a constraint on the effective density distribution.

One might ask if the diffuse effective density could be associated with octupole vibrations of the ground state, these being the strongest collective deformations hypothesized for Pb.[11] Using the parameters of Lane and Pendlebury,[11] we find that the root-mean-square variation in radius of the nuclear surface is less than 0.5 F, too little to explain the depression of the small-angle differential cross section. This deformation effect[20] is of the right order of magnitude to account for the discrepancy between theory and experiment in the first diffraction minimum of Fig. 4. However, already in the figure the diffraction dip is partly filled by constructive interference between the effects of the Coulomb force and of the repulsive real part of the p-N interaction. This is *not* true for Figs. 5 and 6. As a result, we would obtain a valuable test discriminating between the diffuse- and sharp-surface models by looking at $\pi^{\pm}$-Pb scattering. Since the $\pi^{\pm}$-nucleon amplitudes are essentially the same, while the Coulomb potential reverses sign between π^+ and π^-, we conclude that in the "conventional" model the first diffraction minimum would be much deeper for π^- than for π^+, but in the diffuse-surface model there would be little effect.

Finally, we find that the most straightforward application of a semiclassical picture gives excellent qualitative and fair quantitative agreement with experiment. If we take the model seriously in the domain where it is most reliable, the 10% discrepancy in small-angle p-Pb scattering forces us to resort to a model of the nucleus with a much more diffuse surface of the effective nucleon density distribution than that suggested by electron-scattering data. In this model, which is reasonable in terms of current ideas on nuclear phenomena at lower energies, the main difficulty is to explain the oscillations in the large-angle differential cross section. However, even if this effect should disappear, the fit of Fig. 4 already suggests strongly that the nuclear surface of Pb is more diffuse than the charge distribution. Thus any interpretation of the data of Belletini *et al.* suggests a diffuse surface. The questions remaining are, first, how diffuse is the surface, and second, how can the theory reconcile the small-angle and large angle data?

[24] Foley *et al.* (Ref. 9). This limit on spin-flip scattering was suggested by S. Lindenbaum.

IV. CONCLUSIONS

The results of Sec. IV demonstrate that high-energy scattering can provide useful data for the determination of nuclear structure. In particular, the data for lead imply a considerably more diffuse nuclear surface than suggested by electron scattering. In order to check this effect and to permit more complete analyses, many more experiments would be useful:

(1) Repetition of the original work with somewhat better statistics, using isotopically pure targets.

(2) The same experiment with $\pi^{\pm}$ beams. This would permit separation of hadron spin-flip effects from inelastic scattering. Also, at these energies the π^+ and π^- nucleon scattering amplitudes are nearly identical, so that comparison of the nuclear predictions with theory would give a useful check on the method of including Coulomb effects, and on the role of Coulomb-nuclear interference in filling the first diffraction minimum.

(3) Variation of the beam energy. Since the theory is essentially energy-independent, this would give a sensitive test.

(4) Scattering with polarized protons.

(5) Detection of recoil nuclear excitation. Experiments (4) and (5) are very hard, but their value is evident.

From the theoretical point of view, a more complete discussion of inelastic effects would complement future experiments. This would include the appropriate angular averaging for deformed nuclei, as well as multiple inelastic reactions.[25]

At this point, the promise of the subject is just as clear as the need for more work.

ACKNOWLEDGMENTS

The calculations of Sec. III were done with the CDC 6600 computers at the Lawrence Radiation Laboratory, Berkeley, Calif. We thank the computer staff for constant help. Many colleagues contributed helpful suggestions. In particular, we thank Henry J. Lubatti for useful criticism and assistance in some of the calculations. One of us (A.S.G.) acknowledges a pleasant stay at the Physics Division of the Aspen Institute, Aspen, Colorado, where he found an important mistake. Part of this work was supported for one of us (A.S.G.) by the Miller Institute for Scientific Research, University of California, Berkeley, Calif., and for the other (C.J.J.) by the Belgian American Educational Foundation. Part of this work was supported by the U. S. Atomic Energy Commission, and part by the U. S. Air Force, Office of Scientific Research. Finally, we thank G. Cocconi, who provided unpublished data requisite for our analysis.

[25] Multiple inelastic reactions, ignoring cluster effects, were estimated by Glauber (Ref. 21).

APPENDIX A: ACCURACY OF APPROXIMATIONS

(i) The Impulse approximation. We wish to estimate the error caused by assuming that the scattering amplitude on a target nucleon in the nucleus is the same as that for a free target nucleon. The standard estimate for the error of the impulse approximation is given, e.g., by Goldberger and Watson[1]:

$$|\Delta T/T| \sim O(|fkU/\epsilon|), \tag{A1}$$

where f is the forward scattering amplitude on a single free nucleon, k is the incident particle wave number, U is an energy characteristic of the binding of the system, and ϵ is the incident particle energy. For very high energies, (A1) becomes arbitrarily large, if the total cross section on a single nucleon approaches a constant;

$$\left|\frac{fkU}{\epsilon}\right| > \left|\frac{k^2\sigma_T}{4\pi}\frac{U}{\epsilon}\right| \sim \left|\frac{\sigma_T}{4\pi}kU\right|. \tag{A2}$$

However, (A1) was obtained on the assumption that $|f(\theta)|$ is of the same order of magnitude for all θ.

If one uses the same set of assumptions [embodied in Eq. (11.29) of Goldberger and Watson] without imposing the isotropy of $f(\theta)$, one obtains the more general expression

$$|\Delta T^0| \sim 2\pi^2 \left| U \int \frac{d\Omega}{4\pi} \frac{d}{d\epsilon_i}\left(\frac{k^2}{d\epsilon_i/dk} T_{fi} T_{if}\right)_{\epsilon_f=\epsilon_i}\right|, \tag{A3}$$
$$T^0 = -(2\pi^2\epsilon)^{-1} f^0,$$

which may be reduced to (A1) with the assumption of isotropy, provided that derivatives of T_{fi} and T_{if} with respect to ϵ are unimportant.

If one assumes that T is negligible outside a cone of opening angle $\theta \ll 1$ and that the energy ϵ is highly relativistic, one may estimate ΔT^0 by applying unitarity as

$$|\Delta T^0| \sim \frac{1}{2\pi^2}\frac{U}{4\pi}\frac{2}{\epsilon}\int d\Omega |f(\theta)|^2 \left[1 + \left|\operatorname{Re}\left(\frac{d\ln T_{fi}}{d\ln\epsilon}\right)_{f=i}\right|\right]$$
$$= \operatorname{Im} T^0 \frac{2U}{\epsilon}\left[1 + \int d\Omega |f(\theta)|^2 \operatorname{Re}\left(\frac{d\ln T_{fi}}{d\ln\epsilon}\right)_{f=i} \Big/ \int d\Omega |f(\theta)|^2\right]. \tag{A4}$$

This result is obtained assuming that T_{fi} yields only elastic scattering, but the order of magnitude is probably not changed by the presence of particle-production channels.

The second term in parentheses in (A4) must still be evaluated. This involves the derivative of T_{fi} with respect to the incident projectile energy ϵ, for a fixed final configuration and scattering angle with initial 3-momentum $p_i = (\epsilon_i^2 - m^2)^{1/2}$. To compute such an off-energy-shell derivative, we must have a model for the scattering amplitude off the energy shell. For elastic scattering, the on-energy shell amplitude appears to be well described as a function of momentum transfer:

$$T\approx T^0e^{at},\quad t=(\epsilon_i-\epsilon_f)^2-(\mathbf{p}_i-\mathbf{p}_f)^2=2(m^2-\epsilon_i\epsilon_f+p_ip_f\cos\theta). \tag{A5}$$

Let us suppose that T has the same behavior when energy is not conserved. We must now find $dt/d\epsilon_i$. This is given by

$$\begin{aligned}\epsilon_i dt/d\epsilon_i&=2(-\epsilon_i\epsilon_f+p_ip_f\cos\theta/v_i^2)\\&\approx t-2m^2+2(\epsilon_i/v_i-p_i)p_f\\&\approx t.\end{aligned} \tag{A6}$$

Substitution of (A6) into (A4) yields the final result,

$$|\Delta T^0|=O[\mathrm{Im}T^0(2U/\epsilon)(1+|-\tfrac{1}{2}|)]. \tag{A7}$$

Therefore we obtain for 20-BeV protons

$$|\Delta T^0/T^0|\approx 3U/\epsilon\approx 1\%, \tag{A8}$$

where U is taken as 40 MeV, the characteristic strength of the shell-model binding potential.

Actually, (A8) is probably an overestimate of the error in T^0. It comes from the assumption that the commutator $[T,V]$ of the binding potential V with the T matrix is the same size as the product TV.

The binding potential is slowly varying over a region corresponding to the wave number of the projectile, and since T depends on the momenta and kinetic energies of the colliding particles, the commutator $[V,T]$ should be small.

The analysis leading to (A8) applies strictly for a single-target particle bound in a fixed potential. When one wants to use T, as we do, to deduce the optical potential V_0 for passage through a many-body system, then the required T has a further constraint. The point is this: V_0 may act arbitrarily often as the projectile passes through the target system, always leaving the target in its ground state. Since V_0 is obtained as a power series in T_α (the scattering matrix for target particle α), it becomes necessary to exclude in the integral equation defining T_α intermediate states in which the target system is not excited. Otherwise these states would be counted more than once in the Born series obtained from V_0. To estimate the resulting correction to T_α, we may go back to the classical picture and ask what fraction of the cross section σ_α on a single-target particle corresponds to the nucleus remaining in the ground state. If the nucleon and the nucleus were quite opaque bodies of similar shape but different size, then this fraction would be given by

$$\begin{aligned}P\sigma_\alpha&=\int d\Omega\frac{d\sigma_\alpha}{d\Omega}|F(\theta)|^2=\int dt|T^0|^2d^{\alpha t}e^{\beta t}\\&=\sigma_\alpha[\alpha/(\alpha+\beta)],\end{aligned}$$

where $F(\theta)$ is the nuclear form factor, i.e., the overlap between the ground-state wave function and the wave function with one nucleon shifted in momentum by $\Delta p=k\theta$. Now α is proportional to r^2, the effective squared radius of a single nucleon interaction, and β is proportional to R^2, the squared nuclear radius. Therefore we get

$$P=\frac{r^2}{r^2+R^2}\approx\frac{r^2}{R^2}\approx\frac{\sigma_\alpha/2\pi}{R^2}=\frac{\sigma_\alpha}{2\pi R^2}. \tag{A9}$$

Thus the σ which we should use in computing the mean free path in the nucleus is

$$\sigma(1-P)=\sigma(1-\sigma/2\pi R^2). \tag{A10}$$

The quantum-mechanical result of Goldberger and Watson has an extra factor of $\frac{1}{2}$, just as did the correction term due to correlations, and for the same reason. Their result is

$$\Delta T^0\approx(if/kR^2)T^0\approx(-\sigma/4\pi R^2)T^0. \tag{A11}$$

For high-energy protons, then, we have

$$\begin{aligned}\Delta T^0/T^0&\approx-4/4\pi R^2\approx-1/150 \text{ for Pb}\\&\approx-1/50 \text{ for Cu}\\&\approx-1/1.2\pi A^{2/3} \text{ for mass number } A.\end{aligned} \tag{A12}$$

The above arguments do not apply in the surface region of the nucleus, where the gradient of the nuclear density becomes appreciable on the scale of nucleon size (as determined from the typical momentum transfer in hadron-nucleon scattering). In fact, one expects the effective potential $f\lambda\rho$ to vary more slowly in the surface than ρ itself. This is easily seen in the formalism of Glauber.[1]

Glauber shows that in the eikonal approximation, in the absence of correlations, the phase shift may be written

$$\begin{aligned}e^{2i\delta(b)}&=\left(1+\frac{i\lambda}{A}\int d^3r\tilde{f}(\mathbf{b}-\mathbf{r}_\perp)\rho(\mathbf{r})\right)^A\\&\approx\exp\left(i\lambda\int d^3r\,\tilde{f}(\mathbf{b}-\mathbf{r}_\perp)\rho(\mathbf{r})\right).\end{aligned} \tag{A13}$$

Here the scattering amplitude on a single nucleon (assumed independent of the nucleon quantum numbers) is

$$f(\mathbf{q})=\int d^2b\, e^{i\mathbf{q}\cdot\mathbf{b}}\tilde{f}(\mathbf{b}). \tag{A14}$$

If $\rho(\mathbf{r})$ varies little in the region where $f(\mathbf{b}-\mathbf{r})$ is appreciable, then we may approximate the integral over $\mathbf{r}_\perp$ by

$$I=\int d^2r_\perp\tilde{f}(\mathbf{b}-\mathbf{r}_\perp)\rho(\mathbf{r})\approx\rho(\mathbf{r}_{||}+\mathbf{b})f(\mathbf{q}=0). \tag{A15}$$

This approximation breaks down at the nuclear surface, where the integral may be written

$$I\equiv\rho^{\mathrm{eff}}f(\mathbf{q}=0), \tag{A16}$$

and ρ^{eff} is a (possibly complex) effective density which goes to zero more slowly than ρ itself.

We conclude that, in the energy range dealt with in this paper (16–20 BeV), corrections to the impulse approximation for T^0 in bulk nuclear matter are only a few percent.

(ii) Correlation effects. In our discussion of correlation effects in the text, we ignored the finite size of the target nucleons. It is interesting to evaluate the resulting error in a simple model.

Instead of a nucleus, let us imagine that the target is a large cube of uniform refractive index and side D. This could be divided into N smaller cubes with side d, satisfying

$$D^3=Nd^3. \tag{A17}$$

If we calculate the scattering amplitude exactly, it is

$$F=(kD^2/2\pi i)(e^{iKD}-1)\cos(\tfrac{1}{2}q_xD)\cos(\tfrac{1}{2}q_yD), \tag{A18}$$

where K is the wave-number shift in the cube.

Now we may calculate the same result, treating the N small cubes as separate "target particles." Ignoring edge effects, we find (A18) reproduced, with

$$\begin{aligned}K\rightarrow K'&=(f\lambda/a^3)(1+i(\mathcal{R}_cf\lambda/a^3)),\\ f&=(k/2\pi i)a^2(e^{iKa}-1).\end{aligned} \tag{A19}$$

Expanding f yields a second-order formula for K':

$$K'=K[1+iK(\tfrac{1}{2}a+\mathcal{R}_c)]+O((Ka)^2). \tag{A20}$$

Now, for a cubic lattice of this type, we have[26]

$$\mathcal{R}_c=-0.72a,$$

yielding

$$K'=K(1-0.22iKa). \tag{A21}$$

Thus the correlation correction calculated in the text, although of the proper sign, is too big to give $K'=K[1+O((Ka)^2)]$.

A similar effect must be expected in scattering from a nucleus, so that the parameter $\mathcal{R}_c$, even if well determined by fits to scattering data, should not be taken as a precise estimate of the correlation distance. As mentioned in the text, the predictions for p-N scattering are insensitive to the absorption in the interior of the nucleus. Thus it turns out that the value of $\mathcal{R}_c$ has little effect on the predicted scattering.

APPENDIX B: INELASTIC REACTIONS

We wish to derive an estimate of $d\sigma/d\Omega$ for the reaction

$$h+\mathfrak{N}\rightarrow h+\mathfrak{N}^*,$$

where $\mathfrak{N}$ is a nucleus, $\mathfrak{N}^*$ is an excited nuclear system, and h is a high-energy hadron.

First, consider the case of very weak interaction between h and $\mathfrak{N}$. In the impulse approximation the scattering amplitude for excitation of the nucleus to state n is

$$F_{n0}(\mathbf{q})=\sum_{\alpha=1}^{A}\langle n|f_\alpha(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}_\alpha}|0\rangle, \tag{B1}$$

where f_α is the scattering amplitude on the αth nucleon, treated as a free particle, and $\mathbf{q}$ is the momentum transfer. For a high-energy proton we take $f_\alpha(\mathbf{q})=f(\mathbf{q})$, independent of the state α of the given nucleon. Then we have

$$\begin{aligned}\frac{d\sigma}{d\Omega_{\text{in}}}&=\frac{d\sigma}{d\Omega_f}\sum_{n\neq0,\alpha,\beta}\int d^3rd^3r'e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}\\ &\qquad\times\langle0|\delta(\mathbf{r}'-\mathbf{r}_\beta)|n\rangle\langle n|\delta(\mathbf{r}-\mathbf{r}_\alpha)|0\rangle\\ &=\frac{d\sigma}{d\Omega_f}\int d^3rd^3r'e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}[\rho^{(2)}(\mathbf{r},\mathbf{r}')-\rho(\mathbf{r})\rho(\mathbf{r}')].\end{aligned} \tag{B2}$$

Here $d\sigma/d\Omega_f=|f(q)|^2$ is the free-particle differential cross section for a stationary nucleon target, $\rho(\mathbf{r})=\sum_{\alpha=1}{}^A\langle0|\delta(\mathbf{r}-\mathbf{r}_\alpha)|0\rangle$ is the nucleon density distribution in the target, and

$$\begin{aligned}\rho^{(2)}(\mathbf{r},\mathbf{r}')&=\sum_{\alpha/\beta=1}^{A}\langle0|\delta(\mathbf{r}'-\mathbf{r}_\beta)\delta(\mathbf{r}-\mathbf{r}_\alpha)|0\rangle\\ &\equiv\rho(\mathbf{r})\rho(\mathbf{r}')+\delta(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r})+C_v(\mathbf{r},\mathbf{r}')\end{aligned} \tag{B3}$$

is the two-nucleon density distribution. $C_v(\mathbf{r},\mathbf{r}')$ is the Van Hove–McVoy two-nucleon correlation function.[27] Its inventors argue that it should vanish for large separation of its arguments.

In contrast, Goldberger and Watson[1] write

$$\begin{aligned}\rho^{(2)}(\mathbf{r},\mathbf{r}')\equiv{}&[(A-1)/A]\rho(\mathbf{r})\rho(\mathbf{r}')\\ &+\delta(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r})+C_G(\mathbf{r},\mathbf{r}'),\end{aligned} \tag{B4}$$

and they suggest that C_G should vanish for large separation of its arguments. Their form is certainly appropriate for a box containing a uniform gas of noninteracting particles. However, for a medium with strong short-range repulsion, the Van Hove–McVoy ansatz appears more plausible, and we shall use it below.

We have, then,

$$\begin{aligned}\frac{d\sigma}{d\Omega_{\text{in}}}&=A\frac{d\sigma}{d\Omega_f}[1+\tilde{C}_v(\mathbf{q})],\\ C_v(\mathbf{q})&=\frac{1}{A}\int d^3rd^3r'C_v(\mathbf{r},\mathbf{r}')e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}.\end{aligned} \tag{B5}$$

One may easily obtain the result $C_v(0)=-1$, which implies

$$\frac{d\sigma}{d\Omega_{\text{in}}}(0°)=0. \tag{B6}$$

[26] G. Placzek, B. Nijboer, and L. Van Hove, Phys. Rev. **82**, 392 (1951).

[27] L. Van Hove and K. McVoy, Phys. Rev. **125**, 1034 (1962).

This follows directly from (B1), since $F_{n0}(0)$ vanishes:

$$F_{n0}(0)=\langle n|Af(0)|0\rangle=Af(0)\delta_{n0}. \quad \text{(B7)}$$

In other words, if h touches only one nucleon and gives no momentum to that nucleon, then $\mathfrak{N}$ remains in the ground state.

We now consider the case of strong interaction between h and $\mathfrak{N}$. Let us make the assumption that inelastic excitation occurs in a collision with a single nucleon, and that the interaction of h with the nuclear medium before and after the collision is accounted for by the wave-number shift $K=f\lambda\rho(\mathbf{r})$, with h assumed to follow a straight path through the nucleus. Then we have

$$\begin{aligned}
F_{n0}(\mathbf{q})&=f(\mathbf{q})\int d^3r\, e^{i\mathbf{q}\cdot\mathbf{r}}\sum_\alpha\langle n|\delta(\mathbf{r}-\mathbf{r}_\alpha)|0\rangle e^{2i\delta(b)},\\
2\delta(b)&=\int_{-\infty}^{\infty}dz\, f\lambda\rho(\mathbf{r}),\\
\frac{d\sigma}{d\Omega}&=\sum_{n\neq0}|F_{n0}(\mathbf{q})|^2\\
&=\frac{d\sigma}{d\Omega_{\text{free}}}\int d^3rd^3r'\, e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}e^{2i[\delta(b)-\delta^*(b')]}\\
&\quad\times[\rho(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')+\rho(\mathbf{r})\rho(\mathbf{r}')C_v(\mathbf{r},\mathbf{r}')]\\
&=(d\sigma/d\Omega)_{\text{free}}\mathfrak{A}[1+\tilde{C}_v{}^s(\mathbf{q})].
\end{aligned} \quad \text{(B8)}$$

Here we use

$$\begin{aligned}
\mathfrak{A}&=\int d^3r\, e^{-4\,\text{Im}\,\delta(b)},\\
\mathfrak{A}\tilde{C}_v{}^s(\mathbf{q})&=\int d^3rd^3r'\, e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}\\
&\quad\times e^{2i[\delta(b)-\delta^*(b')]}\rho(\mathbf{r})\rho(\mathbf{r}')C_v(\mathbf{r},\mathbf{r}').
\end{aligned} \quad \text{(B9)}$$

In all the above, we ignore spin and isospin dependence. The appropriate modifications to include these are easy to deduce. See Goldberger and Watson (Ref. 1).

We consider some special cases of correlation functions. First, assume that only a weak long-range correlation is present, i.e., $C_G=0$, or

$$C_v(\mathbf{r},\mathbf{r}')=-(1/A)\rho(\mathbf{r})\rho(\mathbf{r}'). \quad \text{(B10)}$$

Then we find

$$\begin{aligned}
\mathfrak{A}\tilde{C}_v{}^s(\mathbf{q})&=-\frac{1}{A}\left|\int d^3r\, e^{2i\delta(b)}\rho(r)e^{i\mathbf{q}\cdot\mathbf{r}}\right|^2,\\
0&\geq\tilde{C}_v{}^s(\mathbf{q})\geq-1.
\end{aligned} \quad \text{(B11)}$$

In fact, for p-Pb scattering, this $\tilde{C}_v{}^s(0)$ is close to zero and $d\sigma/d\Omega(0^0)$ is finite. How is this compatible with our previous result in the absence of absorption effects? Since h now interacts with many nucleons, it may transfer momentum from several nucleons to one which makes a transition out of the ground state without any net momentum transfer $\mathbf{q}$ to the whole nucleus.

As another possibility, we take $C_v(\mathbf{r},\mathbf{r}')=3\rho(\mathbf{r})\Delta(\mathbf{r}-\mathbf{r}')-(4/A)\rho(\mathbf{r})\rho(\mathbf{r}')$. This is an α-particle cluster model, and $\Delta(\mathbf{s})$ is the density distribution with respect to a given nucleon of the other nucleons in the α particle. If Δ is sharply peaked compared with a nuclear mean free path, we have

$$\frac{d\sigma}{d\Omega_{\text{in}}}\approx\frac{d\sigma}{d\Omega_{\text{free}}}\mathfrak{A}[1+3\tilde{\Delta}(q)], \quad \text{(B12)}$$

where $\tilde{\Delta}(\mathbf{q})$ is the form factor of the α particle, and we neglect the small term due to long-range (ρ-ρ) correlations. This expression should be fairly accurate at moderate $\mathbf{q}$ if α clusters dominate the surface region.

Finally, if the correlations are all of short range compared with the mean free path, we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{in}}\approx\left(\frac{d\sigma}{d\Omega}\right)_{\text{free}}\mathfrak{A}[1+\tilde{C}_v(\mathbf{q})]. \quad \text{(B13)}$$

This is just (B8), with A changed to $\mathfrak{A}$, so that in the short-range limit, even with absorption, the inelastic cross section vanishes at $\mathbf{q}=0$. This fact could be used to test the range of correlations in the nucleus by observing the behavior of $(d\sigma/d\Omega)_{\text{in}}$ at small angles.