Properties of Mandelstam Cuts

D. I. OLIVE AND J. C. POLKINGHORNE
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, England
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The properties of Mandelstam cuts are discussed which can be deduced from expressions suggested by Feynman integrals together with an understanding of the singularity structure of real integrals. The suggestion of Schwarz about their "switching on and off" properties is confirmed, and some expressions for discontinuities are elucidated. The existence of a pole singularity underneath the Reggeon-particle and normal threshold cuts is demonstrated, and the importance of its role in eliminating the Gribov-Pomeranchuk essential singularity is emphasized. This property is also explored in terms of the Feynman-integral model, and the dangers and subtleties of calculations based on orders of the third spectral function are exhibited.

1. INTRODUCTION

IN a recent paper[1] Schwarz has proposed a very attractive relationship between different types of Mandelstam cuts[2] in the Regge plane. In particular he has argued that there should be certain "switching on and off" relationships between the different cuts in order to avoid violating the Froissart bound in cases of high spin while continuing to avoid the need for a Gribov-Pomeranchuk (GP) essential singularity.[3] In this paper we shall show that these relationships do in fact exist, even in cases where the spin is not high enough for the Froissart-bound argument to be applicable. We shall also make some contribution to the understanding of how the presence of GP essential singularity is avoided in perturbation theory.

Our principal tool in the investigation will be an understanding of the singularity structure of integrals with real contours. A full account of the mathematical foundations has been given elsewhere[4] for use in a different physical context, but it will be convenient here to summarize the main ideas and conclusions.

We consider integrals over real contours of the form

$$I(z_\beta)\equiv\int dx_\alpha f(x_\alpha,z_\beta). \tag{1.1}$$

The integrand f has singularities given by the real equations

$$S_i(x_\alpha,z_\beta)=0. \tag{1.2}$$

The contour of integration will have to be slightly displaced to avoid these singularities. It is convenient to specify the way in which this is done by the equivalent procedure of keeping the contour real and adding an infinitesimal imaginary part to each S_i,

$$S_i\rightarrow S_i+i\epsilon_i, \tag{1.3}$$

to specify the direction from which it approaches the contour. This is exactly analogous to the familiar $i\epsilon$ prescription for Feynman integrals.

A singularity of the integral occurs through the well-known pinch mechanism when the implied displacements of the contours become incompatible. If this is due to singularities $S_1,\cdots,S_n$, the equation of the *Landau curve* is given by

$$S_i=0,\quad i=1,\cdots,n,\quad \sum_i \lambda_i\frac{\partial S_i}{\partial x_\alpha}=0, \tag{1.4}$$

where the λ_i are multipliers defined by the equations (to within an over-all multiplicative constant). Eliminating x_α and λ_i from (1.4) gives a single equation in the variables z_β. The conditions for the resulting Landau curve to be actually singular at a given point are:

$$\lambda_i\epsilon_i \text{ has the same sign for } i=1,\cdots,n. \tag{1.5}$$

The curve will become nonsingular at a point where condition (1.5) fails. Then one of the λ_i must vanish at the point and change sign thereafter. The point then corresponds to a contact with a *lower-order singularity*, namely, that generated by equations similar to (1.4) but with the particular singularity S_i omitted. This switching off of a singularity is called, in the jargon, *the hierarchic effect.*

An important notion in discussing integrals of the type (1.1) is that of the *natural continuation* round a singularity. This is the continuation in a sense such that the real contour of integration does not suffer a finite distortion by the singularities. In terms of a suitable variable η, normal to the singularity curve, it is determined by the common sign in (1.5); for details see Ref. 4.

If the singularities are all poles S_i^{-1}, we define the *Cutkosky integral* associated with a singularity as that obtained by replacing S_i^{-1} by $(-2\pi i)\,\mathrm{sgn}\epsilon_i\,\delta(S_i)$, for those S_i participating in the singularity. Then the discontinuity across the singularity is given by

$$\mathrm{disc}I=C_>-(C_<)_{\eta-i\epsilon_{\mathrm{nat}}}, \tag{1.6}$$

where $C_>$ denotes the Cutkosky integral evaluated at the point where we evaluate the discontinuity, and $C_<$

[1] J. H. Schwarz, Phys. Rev. **162**, 1671 (1967). The switching on and off possibility was earlier mentioned in Ref. 9, but its full significance was not realized at that time.
[2] S. Mandelstam, Nuovo Cimento **30**, 1127 (1963); J. C. Polkinghorne, J. Math. Phys. **4**, 1396 (1963).
[3] C. E. Jones and V. Teplitz, Phys. Rev. **159**, 1271 (1967); S. Mandelstam and L. L. Wang, *ibid.* **160**, 1490 (1967); J. B. Bronzan and C. E. Jones, *ibid.* **160**, 1494 (1967).
[4] M. J. W. Bloxham, D. I. Olive, and J. C. Polkinghorne, Cambridge University Report (unpublished).

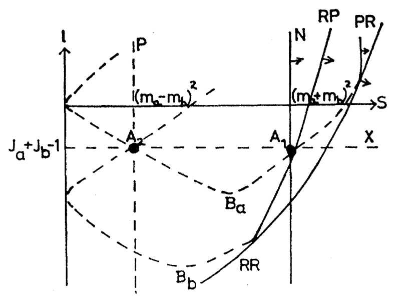


FIG. 1. The singularity structure in the s plane. Parts of curves singular in the natural continuation limit are drawn with solid lines, those nonsingular in this limit with broken lines. The arrows, when rotated through $\frac{1}{2}\pi$ in the complex plane, indicate the sense of the natural continuation.

denotes the continuation of the Cutkosky integral, evaluated on the other side of the singularity, to the point concerned and in the sense indicated.

In many cases, for example Feynman integrals, $C_< = 0$ and so the formula simplifies. Technically this is due to the existence of a *vanishing cycle*, but as we shall see it does not happen automatically when we consider Reggeons.

Finally, we note that singularities of Cutkosky integrals are given by rules similar to the above. The main difference is that if we have $\delta(S_i)$ instead of S_i^{-1} then there is no ϵ_i and the corresponding singularity does not appear in condition (1.5).

2. SINGULARITY MECHANISMS

We shall suppose that the contribution to the scattering amplitude due to two-Reggeon exchange can be written in the form

$$F(l,s)=(\sin\tfrac{1}{2}\pi l)\int d^2k\frac{F_1F_2}{l-\alpha(k^2)-\beta((q-k)^2)+1}\times\frac{1}{\sin\frac{1}{2}\pi\alpha(k^2)}\frac{1}{\sin\frac{1}{2}\pi\beta((q-k)^2)}. \quad (2.1)$$

Here $s=q^2>0$; k is a two-dimensional vector with one time and one space component; α and β are the Regge trajectory functions, which we have taken to be of positive signature and which are assumed to be real analytic functions with positive derivatives in the regions considered. The functions F_i are the functions describing the emission and absorption of the Reggeons, respectively, and are functions of l, s, k^2, and $(q-k)^2$, assumed analytic in the regions concerned.

The form (2.1) is reminiscent of the expression first suggested by Gribov, Pomeranchuk, and Ter-Martirosyan,[5] and confirmed by further investigations based on Feynman integrals.[6–8] However, in these latter papers the problem was investigated in the region $s<0$ and the loop momentum k in the analogous expression was taken as having two space components. We shall show in Sec. 4 that our expression is correctly related to these other *Ansätze*, but in this and the succeeding section we shall be content to investigate the properties of (2.1), considered on its own merits as providing a consistent picture of the singularity structure in $s>0$.

The singularities of the integrand of (2.1) which are of concern arise from the vanishing of denominators. They are

$$\begin{aligned}S_0&\equiv l-\alpha(k^2)-\beta((q-k)^2)+1=0,\\ S_1&\equiv k^2-m_a{}^2=0,\\ S_2&\equiv(q-k)^2-m_b{}^2=0,\end{aligned} \quad (2.2)$$

where in writing equations for S_1 and S_2 we have supposed α takes a correct signature integral value J_a at $k^2=m_a{}^2$, and β similarly take a value J_b at $(q-k)^2=m_b{}^2$. Of course there may be several possible values of J_a and J_b but we consider specific ones.

The Landau curves these singularities generate are given below.

A. Reggeon-Reggeon Cut (RR)

This is generated by S_0 alone and corresponds to the equations

$$\partial S_0/\partial k=0,\quad S_0=0.$$

It occurs at $l=\alpha_{\mathrm{RR}}(s)$, where α_{RR} is the maximum value of $\alpha+\beta-1$ in the region of integration. If the two Reggeons are identical ($\alpha=\beta$), then

$$\alpha_{\mathrm{RR}}=2\alpha(s/4)-1. \quad (2.3)$$

For nonidentical Reggeons the form of α_{RR} depends on the details of the trajectory functions. It always exists and has positive slope if α and β themselves have positive slope.

B. Reggeon-Particle Cut (RP)

The particle a is Reggeized but particle b is on its mass shell. This singularity is generated by S_0 and S_2. The Landau equations are

$$\lambda_0\frac{\partial S_0}{\partial k}+\lambda_2\frac{\partial S_2}{\partial k}=0, \quad (2.4a)$$

$$S_0=0, \quad (2.4b)$$

$$S_2=0. \quad (2.4c)$$

Equation (2.4a) implies that k is parallel to q, and the remaining equations then give

$$\alpha_{\mathrm{RP}}(s)=\alpha((s^{1/2}-m_b)^2)+J_b-1. \quad (2.5)$$

[5] V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martirosyan, Yadern. Fiz. 2, 361 (1965) [English transl.: Soviet J. Nucl. Phys. 2, 258 (1966)]; Phys. Rev. 139, B184 (1965).

[6] J. C. Polkinghorne, J. Math. Phys. 6, 1960 (1965).

[7] V. N. Gribov (to be published).

[8] J. C. Polkinghorne, Cambridge University Report No. DAMTP 68/5 (unpublished).

Because of the square root this equation is in fact double valued.

C. Particle-Reggeon Cut (PR)

Here the roles of a and b are interchanged. The singularity is generated by S_0 and S_1 and its equation is

$$\alpha_{\mathrm{PR}}(s)=\beta((s^{1/2}-m_a)^2)+J_a-1. \quad (2.6)$$

Although the first authenticated cases[2] of cuts of the type described in Secs. 2 B and 2 C were obtained from Feynman integrals in which the particle was in fact an elementary particle, it has long been understood that the particle could also be one lying on a Regge trajectory.[9]

D. Normal and Pseudothresholds (N,P)

They are generated by S_1 and S_2 and occur at the familiar values

$$s=(m_a\pm m_b)^2. \quad (2.7)$$

E. Fixed-l Singularity (X)

This is generated by S_0, S_1, S_2 together. When these each vanish we have, immediately,

$$l=J_a+J_b-1. \quad (2.8)$$

Since this is generated by three poles in a two-dimensional integration, the discontinuity formula in Sec. 1, yields a δ function. Thus this singularity is a pole. Its existence was noted by Schwarz who did not, however, discuss its significance. It occurs at the first nonsense integer, just like the Gribov-Pomeranchuk pole, but as we shall see, it is distinct from it, being associated with the removal of the GP essential singularity.

Other possible combinations of S's fail to generate meaningful singularities.

The interesting suggestion made by Schwarz[1] is that the RP and PR cuts are switched on and off as singularities by the RR cut. We immediately see that this is possible by the hierarchy mechanism discussed in Sec. 1 since the RP and PR cuts are generated by just one more singularity than the RR cut and hence touch when they meet as discussed below.

Figure 1 illustrates the various curves (a)-(e) in the (s,l) plane. As already noted, the RP and PR curves are two-valued functions of s because of the $s^{1/2}$ in Eqs. (2.5) and (2.6). They pass through the points A_1 and A_2 with (s,l) coordinates $((m_a+m_b)^2, J_a+J_b-1)$, $((m_a-m_b)^2, J_a+J_b-1)$, whatever the functions α and β may be. Further, these curves touch $s=0$, $s^{1/2}$ changing sign at the points of tangency, being positive at A_1. The curves RP and PR have minima B_b and B_a at $s^{1/2}=m_b$ and m_a, respectively.

[9] J. C. Polkinghorne, Phys. Rev. **128**, 2459 (1962). The argument has been rediscovered by Schwarz in Ref. 1.

We have anticipated a point by drawing RR actually touching RP and PR. Schwarz pointed out that this must happen as illustrated if PR and RP are to be nonsingular at $s=0$ (so as not to violate the Froissart bound if the spins J_a and J_b are too high), while one of the two is singular at A_1 in order to vitiate the GP essential singularity argument. We show now that the desired tangency does occur in that the particular one of RP and PR which has steepest slope at A_1, must touch RR between A_1 and the minimum B_a or B_b, whatever functions α, β are involved.

This tangency of RP with RR follows if the relative sign of the multipliers λ_0, λ_2 occurring in the Landau equation changes between A_1 and B_b. This is because λ_0 cannot vanish, since S_2 itself generates no singularities, and because when $\lambda_2=0$ on RP we also satisfy the Landau equations for RR and, in fact, have tangency. Evaluating λ_2/λ_0 on the branch of RP where $s^{1/2}$ is positive, we find

$$\lambda_2/\lambda_0=[m_b\beta'(m_b^2)-(s^{1/2}-m_b)\alpha'((s^{1/2}-m_b)^2)]/m_b. \quad (2.9)$$

At B_b, $s^{1/2}=m_b$, so $\lambda^2/\lambda_0=\beta'(m_b^2)$;

at A_1, $s=m_a+m_b$, so

$$\lambda_2/\lambda_0=[m_b\beta'(m_b^2)-m_a\alpha'(m_a^2)]/m_b = \left(\frac{\partial\alpha_{\mathrm{PR}}}{\partial s}-\frac{\partial\alpha_{\mathrm{RP}}}{\partial s}\right)\bigg|_{s^{1/2}=m_a+m_b}\frac{(m_a+m_b)}{m_b}.$$

Thus λ_2 and λ_0 have the same signs at B_b and opposite signs at A_1 providing RP is steeper than PR there. Since similar equations can be written down for PR, this establishes the result.

3. SINGULARITY STRUCTURE

We have been concerned so far with the geometrical properties of the Landau curves associated with (2.1). These are the locations of possible singularities of the integral but in order to determine which parts are actually singular it will be necessary to look more closely at its properties. We shall suppose that the contour is real and that infinitesimal distortions $i\epsilon_0, i\epsilon_1, i\epsilon_2$ are associated with the singularities S_0, S_1, S_2, respectively, in the manner discussed in Sec. 1.

Singularity occurs if the quantities $\lambda_i\epsilon_i$ have a common sign for all the participating S's. The quantities λ_i are calculable but the ϵ_i are so far unknown. Like Schwarz[1] we shall require nonsingularity of RP and PR at $s=0$. For high spins this follows from a desire to avoid violating the Froissart bound, but alternative arguments which can always be applied are given in Sec. 4 to show that this is always true. By (2.9), $\lambda_2/\lambda_0 =[\beta'(m_b^2)+\alpha'(m_b^2)]>0$ at $s=0$ on RP. Similarly, λ_1/λ_0 is positive at $s=0$ on PR. Hence we must have

$$-\epsilon_0=\epsilon_1=\epsilon_2. \quad (3.1)$$

It then follows from (2.9) that the steeper of RP and PR is singular at $s=(m_a+m_b)^2$. The curve RR is always singular, having only one ϵ. As usual, the normal threshold is singular and the pseudothreshold nonsingular.

For X, $l=J_a+J_b-1$, we find $\lambda_0:\lambda_1:\lambda_2=1:\alpha'(m_a^2):\beta'(m_b^2)$ at all points except A_1 and A_2. Since all λ's thus have a common sign, (3.1) tells us that X is not singular.

We now see that the simple $i\epsilon$ prescriptions (3.1) which satisfy our requirements in fact follow from the Feynman $i\epsilon$ prescription. This associates $-i\epsilon$ with m^2, or equivalently $+i\epsilon$ with k^2 in the propagator $(k^2-m^2+i\epsilon)^{-1}$. Doing likewise with $\alpha(k^2)$, and remembering α' is positive, we find

$$\alpha(k^2+i\epsilon)=\alpha(k^2)+i\epsilon.$$

Thus the Feynman prescription is equivalent to associating $+i\epsilon$ with each trajectory function. By inspection of the expressions for S_0, S_1, and S_2 this indeed gives (3.1). Note that this is not what we get if we insert Regge poles into (+) and (−) bubbles of a unitary integral [as in the Amati-Fubini-Stanghellini (AFS) argument[10]].

We can also evaluate the natural distortion for our singularities, telling us how to detour the singularities while maintaining flat contours. These are indicated by arrows in Fig. 1 (assuming the Feynman prescription).

Because of its possible relationship to the Gribov-Pomeranchuk singularity we look at the points of X, in boundary values other than the natural one, that is underneath the various cuts. To do this we must study the relevant discontinuity formulas which take us under the cuts. For the normal threshold cut, it is known that there is a vanishing cycle in $s>(m_a+m_b)^2$, so that its discontinuity is given by a single Cutkosky integral,

$$\begin{aligned}\mathrm{disc}_N F&=(\sin\tfrac{1}{2}\pi l)\int\frac{F_1F_2}{l-J_a-J_b+1}d^2k\,(-2\pi i)\delta(\sin\tfrac{1}{2}\pi\alpha(k^2))(-2\pi i)\delta(\sin\tfrac{1}{2}\pi\beta(q-k)^2)\\&=(\sin\tfrac{1}{2}\pi l)\frac{F_1F_2}{l-J_a-J_b+1}\,\frac{-8}{\alpha'(m_a^2)\beta'(m_b^2)\{[s-(m_a+m_b)^2][s-(m_a-m_b)^2]\}^{1/2}},\end{aligned}\tag{3.2}$$

where we understand $k^2=m_a^2$, $(q-k)^2=m_b^2$ in F_1 and F_2. This clearly has a pole at $l=J_a+J_b-1$, since we recall that our choice of signatures for the trajectories α and β ensures that both J_a and J_b are even and so $\sin\frac{1}{2}\pi l$ does not vanish at this point. The existence of the pole in the discontinuity then implies that F has a similar pole when evaluated in the boundary value under the normal threshold cut, but over the RP cut. (We understand "over" to correspond to the natural boundary value and "under" to be opposite to "over.")

In a similar way an identical pole is discovered underneath the RP cut (while over the N cut) which, as shown in Sec. 4, also has a vanishing cycle.

Thus the pole X has properties exactly opposite to those expected of the GP pole,[3] which is singular in the natural boundary value but which then must not be found underneath the normal or RP cuts if the GP essential singularity is to be avoided. The role of X is clearly to cancel the GP pole which would otherwise appear on these lower sheets. We study in Sec. 5 how this cancellation manifests itself in terms of Feynman integrals.

Since no other singular curves pass through the intersection of RR with X and since X is not singular in the natural boundary value, it follows that X cannot be singular under the RR cut and therefore the discontinuity across this cut cannot be singular at X either. However, a Cutkosky integral for RR, just obtained by substituting $2\pi i\delta(S_0)$ for S_0^{-1}, is, according to the rules given in Sec. 1, singular at X. The reason is that, because of the Lorentz metric, there is not a real vanishing cycle for RR. This will be shown in Sec. 4. Thus the discontinuity is, according to (1.6), the difference of two Cutkosky integrals. Each possesses a pole at X but their residues cancel, as can be shown by evaluating the discontinuities at X by a further application of the Cutkosky rule (there are vanishing cycles again in this case).

Thus the discontinuity across the RR cut possesses some subtle features.

4. TWO-REGGEON CONTRIBUTION

In this section we shall discuss the form (2.1) assumed for the two-Reggeon contribution and how it is related to other suggested forms. The best founded of these latter discussions are based on properties of Feynman integrals[6–8] and they give an expression similar to (2.1) but with s taken negative and with the two-dimensional loop integration being over anti-Euclidean vectors k (both components spacelike).

If our expression (2.1) is continued into $s<0$, it proves possible to make a Wick rotation which then reduces (2.1) to the same form as these other expressions. This is because in $s<0$ we can write $q=(0,q)$, so that the component of q is associated with the space component k_1 rather than the time component k_0. Then each singularity equation (2.2) has the form

$$f(k_0^2+i\epsilon)=0\tag{4.1}$$

[10] D. Amati, S. Fubini, and A. Stanghellini, Nuovo Cimento **26**, 2896 (1962).

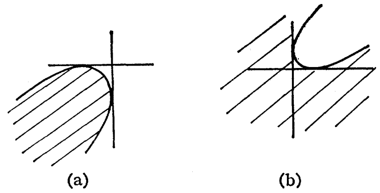


FIG. 2. The curves $\lambda=0$ (a) when $s<0$, (b) when $s>0$.

as far as the k_0 and $i\epsilon$ dependence is concerned, since we have assumed α' and β' are positive.

It is now necessary to assume that for each S_i the roots of (4.1) are real when $\epsilon=0$ and k_1, q, and l are real. A sufficient condition for this is that the trajectory functions are Herglotz functions, that is functions whose imaginary parts are of the same sign as the imaginary part of their argument. This is true in potential theory if trajectories do not cross. The status of the assumption in relativistic quantum mechanics is not known. If there were nonreal k_0^2 roots of (4.1) extra terms would be obtained on making the Wick rotation. There might then have associated with them extra complex Regge singularities but the existence of the latter would not in fact spoil the analysis of the interrelation of the real singularities already given.

If the roots of (4.1) are real when $\epsilon=0$, then either k_0 is pure imaginary or it has the form

$$k_0=\hat{k}_0-i\epsilon\,\mathrm{sgn}\hat{k}_0,\quad \hat{k}_0 \text{ real.} \tag{4.2}$$

This disposition of singularities makes it possible to rotate the k_0 contour through $\frac{1}{2}\pi$ so that

$$\int d^2k\to i\int d^2k',$$

with k' anti-Euclidean.

It is possible to give an alternative discussion based on an integration over invariants. It is convenient to start in this case in $s<0$. The two-dimensional anti-Euclidean integration is then written in the form

$$\int\frac{ds_1\,ds_2}{[-\lambda(s,s_1,s_2)]^{1/2}}. \tag{4.3}$$

Here s_1 and s_2 are the squares of the momenta associated with α and β, respectively,

$$\lambda=s^2+s_1^2+s_2^2-2ss_1-2ss_2-2s_1s_2, \tag{4.4}$$

and the region of integration is $\lambda<0$. This is the interior of a parabola lying in the third quadrant of the s_1s_2 plane, as shown in Fig. 2(a). When $s=0$ this degenerates into a coincident line pair and for $s>0$ the parabola lies in the first quadrant, as shown in Fig. 1(b). Its interior is no longer the region of integration, which has become complex and is bounded by complex points of the curve $\lambda=0$.[11] Cauchy's theorem in its generalized form allows this region to be freely distorted, provided that its boundary is always some part of the curve $\lambda=0$ and that no singularities are crossed. This enables us, under the assumptions discussed above, to make the region of integration real again, in which case it becomes the outside of $\lambda=0$. This is the analog of the Wick rotation in reverse since the integral described is indeed the result of transforming from the Lorentz integration to invariants in $s>0$.

[11] Compare several similar cases discussed in detail by I. T. Drummond, Nuovo Cimento **29**, 720 (1963).

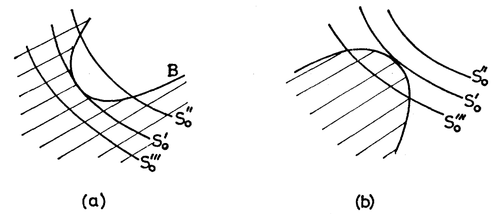


FIG. 3. Diagrams relevant to RR in (a) $s>0$, (b) $s<0$.

If the singularity structure is discussed in terms of invariants the analysis is a little more general than that discussed in Sec. 1 since there is now a boundary B given by $\lambda=0$ which also participates in generating Landau curves. There is, of course, no $i\epsilon$ associated with B, so it does not figure in (1.5). The singularities are generated as follows:

RR by B, S_0;
RP by B, S_0, S_2;
PR by B, S_0, S_1;
P, N by B, S_1, S_2;
X by S_0, S_1, S_2.

It is clear that if we adopt the form suggested by Feynman integrals in $s<0$, then the poles S_1 and S_2 are nowhere near the undistorted contour in the neighborhood of $s=0$ and this provides a more general argument than that based on the Froissart bound for the nonsingularity properties used at the beginning of Sec. 3.

The invariant formulation is also convenient for seeing the vanishing cycle properties used in Sec. 3. For axample, consider the RR singularity, which corresponds to S_0 touching B. In Fig. 3(a) we have $s>0$, and S_0' corresponds to S_0 evaluated at the value of l given by the singularity, S_0'' and S_0''' to values of l on either side of singularity. Since both S_0'' and S_0''' intersect the shaded region of integration, there is no vanishing cycle. However, for $s<0$ the situation would be as in Fig. 3(b) and there would be a vanishing cycle in the region $l<\alpha_{\mathrm{RR}}(s)$. Other singularities can be discussed by drawing similar diagrams.

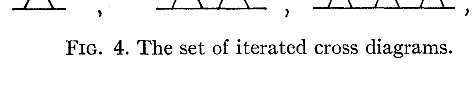
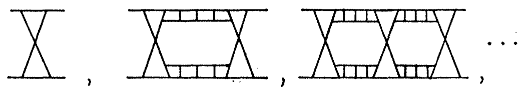


FIG. 4. The set of iterated cross diagrams.



(a) (b) (c)

FIG. 5. The set of diagrams to be considered.

5. PERTURBATION-THEORY MODEL

In this section we see what understanding can be obtained of the mechanism for removing essential singularities in the model of relativistic quantum mechanics provided by Feynman integrals.[12] The original Gribov-Pomeranchuk[13] argument for the existence of an essential singularity at $l=-1$ stemmed from two-particle unitarity in this channel and the presence of a ρ_{tu} double spectral function.

The set of Feynman diagrams in Fig. 4, the iterated cross, supply these ingredients and do indeed lead to an essential singularity when summed. Mandelstam showed that if the s-channel particles are Reggeized, rather than elementary, cuts exist, with just such properties as to vitiate the argument for essential singularity.[2,3]

The corresponding set of diagrams to be considered is shown in Fig. 5 where the crosses are now joined by Regge-pole generating ladders. This is the simplest set, but the crosses could be replaced by any diagram with a third spectral function which was two-Reggeon irreducible and the ladders by any set of Regge-pole generating diagrams.

Diagrams of the form of Fig. 5(a) give a pole in the positive-signature amplitude of the form

$$G_1/(l+1)\,, \tag{5.1}$$

where G_1 is an amplitude whose precise definition is not required here. Diagrams like Fig. 5(b) yield an expression[8]

$$G_2(s,l)/(l+1)\,, \tag{5.2}$$

where G_2 contains a term of the form (2.1). Because of the properties of the X singularity discussed in Sec. 3, (5.2) will have a single pole at $l=-1$ on top of the normal threshold cut, but a double pole beneath it.

In fact, it turns out, using the normal threshold discontinuity equation (3.2) and the special properties of G_2,[2] that

$$\mathrm{disc}_{\mathrm{N}}G_2=\rho G_1^2/(l+1)\,, \tag{5.3}$$

where ρ is a calculable phase space factor. Thus to the order we have considered, namely, up to "second order in ρ_{tu}" the amplitude evaluated underneath the normal threshold cut (but over the RP cut) has the form, near $l=-1$,

$$a_{\mathrm{II}}\sim(G_1+G_2)/(l+1)+\rho G_1^2/(l+1)^2\,, \tag{5.4}$$

while above this cut

$$a_{\mathrm{I}}\sim(G_1+G_2)/(l+1)\,. \tag{5.5}$$

At first sight this appears disastrous, for instead of cancelling the pole in a_{II} we have added to it a double pole. The explanation is found by considering the unitarity equation which gives a_{II} in terms of a_{I} to all orders in ρ_{tu}

$$a_{\mathrm{II}}=a_{\mathrm{I}}/(1-\rho a_{\mathrm{I}})=a_{\mathrm{I}}+\rho a_{\mathrm{I}}^2+\rho^2a_{\mathrm{I}}^3+\cdots\,. \tag{5.6}$$

To all orders in ρ_{tu}, $a_{\mathrm{I}}=G/(l+1)$, where $G=G_1+G_2+\cdots$, and so by (5.6),

$$a_{\mathrm{II}}=\frac{G}{l+1-\rho G}=\frac{G}{l+1}+\frac{\rho G^2}{(l+1)^2}+\cdots, \tag{5.7}$$

that is, a sequence of multiple poles. If we collect terms involving up to second order in ρ_{tu}, we do indeed obtain (5.4) even though to all orders, near $l=-1$,

$$a_{\mathrm{II}}\sim-1/\rho\,.$$

Thus in the model theory of Fig. 5 there are two summations, the first of which, the sum over ladders, introduces new singularities and the second of which, the sum over crosses, removes these singularities, as far as a_{II} is concerned.

It is clear that great care must be taken when using arguments based on "powers of the third spectral function." Unitarity in the s channel is of the greatest importance and we shall only satisfy this requirement if we consider infinite sequences of diagrams, like the whole set of Fig. 5, which then require us to work to all orders in ρ_{tu}.

The analysis of the higher diagrams, like 5(c), is not easy but we expect[14] that the contribution to a_{I} near $l=-1$ in nth order of ρ_{tu} is $G_n/(l+1)$, where G_n is an integral of the form

$$(\sin\tfrac{1}{2}\pi l)\int d^2k_1\cdots d^2k_{n-1}\hat{X}(q^2,\,l;\,m_a^2,\,m_b^2,\,k_1^2,\,(q-k_1)^2)X(q^2,\,l;\,k_1^2,\,(q-k_1)^2,\,k_2^2,\,(q-k_2)^2)\times\cdots$$
$$\times\frac{\hat{X}(q^2;\,l;\,k_{n-1}^2,\,(q-k_{n-1})^2,\,m_a^2,\,m_b^2)}{\prod_i\{[l-\alpha(k_i^2)-\beta((q-k_i)^2)+1]\sin\frac{1}{2}\pi\alpha(k_i^2)\sin\frac{1}{2}\pi\beta((q-k_i)^2)\}}. \tag{5.8}$$

[12] For a discussion of this model and its implications for complex angular momentum theory see R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge University Press, New York, 1966), Chap. 3.

[13] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Letters 2, 239 (1962).

[14] Compare Ref. 7. A simple case analogous to (c) but involving an elementary particle instead of the intermediate cross has been analyzed. See P. Osborne and J. C. Polkinghorne, Nuovo Cimento 47, 526 (1966).

This can be analyzed according to the previous techniques and exhibits the same singularities and $i\epsilon$ prescriptions in the natural boundary value as (2.1), provided we assume the Feynman $i\epsilon$ prescription as discussed above. Such properties are normally preserved under infinite summation so that Fig. 1 now applies to the whole model theory defined by Fig. 5 and, we suggest, to the complete theory.

Under the normal threshold cut, (5.8) has poles of order up to $(l+1)^{-n+1}$. The normal threshold discontinuity is, using a generalization of (3.2),[4]

$$\mathrm{disc}_N G_n = \rho \sum_{r=1}^{n-1} G_r^{\mathrm{I}} G_{n-r}^{\mathrm{II}}/(l+1)\,.$$

Thus our behavior near $l=-1$ agrees with unitarity which, in nth order, reads

$$a_{\mathrm{I}}^n - a_{\mathrm{II}}^n = \rho \sum_{r=1}^{n-1} a_{\mathrm{I}}^r a_{\mathrm{II}}^{n-r}\,.$$

Notice the importance of the two-dimensional nature of the integration in this for it means that the residues at the poles factor into products. This is not the case for analogous calculations for the diagrams of Fig. 4. The residues in that case still contain integrations and this is what gives rise to the GP essential singularity in that case.[12]

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Parity-Conserving Hyperon Nonleptonic Decays in $SU(3)$

M. O. Taha*

Institute for Advanced Study, Princeton, New Jersey

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General restrictions on the $SU(3)$ invariants are found such that an experimentally satisfactory one-parameter solution of the parity-conserving nonleptonic hyperon decays is obtained under octet dominance and CP invariance. Both normal and abnormal charge conjugation are considered. The pole model is discussed as an example. In this model the restrictions determine the $B\bar{B}M$ and spurion F/D ratios, for which reasonable values are obtained.

I. INTRODUCTION

It is known that within the framework of strict $SU(3)$ symmetry, the situation with respect to the (p-wave) parity-conserving nonleptonic hyperon decays remains in its pre-$SU(3)$ state of the $\Delta I=\frac{1}{2}$ rule when one applies octet dominance and CP invariance, together with the charge-conjugation properties of the current-current theory of the weak interaction. With these assumptions, one $SU(3)$ restriction is imposed on the parity-violating amplitudes.[1] This is the well-known Lee-Sugawara[2] relation. It is found to be experimentally good for both the parity-violating and the parity-conserving decays, and can be deduced for the parity-conserving decays as well under one or the other of some further symmetry assumptions.[2–4]

In a recent paper,[5] a somewhat general result is obtained on the conditions under which an $SU(3)$ model leads to a one-parameter solution to the parity-violating hyperon nonleptonic decays in agreement with experiment. In particular, it is shown that beyond the above assumptions, it is sufficient to neglect the decuplet contributions in both the $B\pi$ and $B\bar{B}$ channels. It is our purpose in this paper to apply similar considerations to the parity-conserving amplitudes.

We start by finding what can be deduced about the parity-conserving decays under the assumptions of Ref. 5 mentioned above. Two relations among the four independent amplitudes result:

$$2B(\Xi^- \to \Lambda\pi^-) = B(\Lambda \to p\pi^-) + \sqrt{3}B(\Sigma^+ \to p\pi^0)\,, \quad (1.1)$$

$$B(\Sigma^+ \to p\pi^0) = -\sqrt{3}B(\Lambda \to p\pi^-)\,. \quad (1.2)$$

The relation (1.1) is the Lee triangle known to be in agreement with experiment. Relation (1.2) is also well satisfied experimentally.[6]

Using (1.1) and (1.2) together with

$$B(\Sigma^- \to n\pi^-) = 0 \quad (1.3)$$

to characterize the experimental situation, we then state necessary and sufficient conditions on the $SU(3)$ invariants, such that an experimentally satisfactory one-parameter solution is obtained. A special case of

* On leave of absence from Downing College, Cambridge, England, and the University of Khartoum, Sudan.

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