The same result can also be obtained from the low-energy theorem of Das, Mathur, and Okubo[15] provided that one uses expression (5) for the pion form factor. We have preferred to use the above method, because it illustrates the question of subtraction in the dispersion relation of $\tilde{H}_5(\nu)$ as well as in the pion form factor.

From formula (30), we notice that for the case when the dispersion relation for $\tilde{H}_5(\nu)$ has a subtraction, one obtains for

$\gamma=\frac{1}{4}$ (as in Schwinger's theory), $b(0)=0$, $r=0$,

$\gamma=0$ (no subtraction[15] in the pion form factor)

$$b(0)=f_\pi/2m_\rho^2, \quad |r|=0.6,$$

and

$$\gamma=\tfrac{1}{10}, \quad b(0)=3f_\pi/10m_\rho^2, \quad |r|=0.36.$$

In contrast, if $\tilde{H}_5(\nu)$ has no subtraction, $b(0)=-f_\pi/m_\rho^2$, so that $|r|=1.2$. Thus, to conclude, an accurate determination of r can settle the question of subtraction in $\tilde{H}_5(\nu)$ as well as in the pion form factor, and, further, it can make a choice between the various values of γ. The present experimental situation regarding r is that, on the basis of 143±15 events, Depommier *et al.*[16] obtained two possible solutions for $|r|$, which are 0.38 and 2. This would indicate $\gamma=\frac{1}{10}$. Further experiments on $\pi\to l\nu\gamma$ decay with better statistics would be of great interest.

[15] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters, **19**, 859 (1967). This paper contains earlier references to $\pi\to l\nu\gamma$ decay.

[16] P. Depommier, J. Heintze, C. Rubbia, and V. Soergel, Phys. Letters **7**, 285 (1963).

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Formal Breakdown of Lorentz Invariance in Two-Dimensional Field Theories*

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Since the field theory which describes the interaction of a massless fermion with a vector meson of bare mass μ_0 in one spatial dimension is known to contain massless boson excitations, there is considerable interest associated with the question of whether these particles can be identified as the Goldstone bosons generated by the breaking of an appropriate symmetry operation of the theory. It is shown that such an interpretation is indeed consistent, independent of the strength of the coupling, provided that one takes Lorentz invariance to be the broken symmetry of the model. Particular attention is given to the limits $\mu_0\to\infty$ (Thirring model) and $\mu_0\to 0$ (Schwinger model). It is found in all cases that despite the apparent breaking of Lorentz invariance, the excitation specrtum has a normal form, and the symmetry breakdown remains entirely unobservable.

I. INTRODUCTION

THE field-theoretical techniques of spontaneously broken symmetries have been used in a number of recent attempts to understand the structure of strong interactions. Invariably the elegance of such formulations has been found to be marred by the problems associated with unwanted massless particles which, according to the Goldstone theorem,[1,2] must necessarily accompany the breaking of a continuous symmetry group associated with a manifestly covariant theory.[3] In order to investigate whether such difficulties might be circumvented by the possible decoupling of the Goldstone boson, the study of various soluble field-theoretical models which are capable of supporting a broken symmetry has recently engaged the attention of a number of authors. In particular, it has been shown[2] that a relativistic theory whose Lagrangian is invariant under transformations of the form

$$\phi(x)\to\phi(x)+\eta, \tag{1.1}$$

where $\phi(x)$ is one of the canonical variables of the theory and η is a constant, always possesses an infinite number of broken-symmetry solutions, each of which describes exactly the same physical system. Such theories have

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[1] J. Goldstone, Nuovo Cimento **19**, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[2] G. S. Guralnik and C. R. Hagen, Nuovo Cimento **43**, 1 (1966).

[3] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, Phys. Rev. Letters **13**, 585 (1964). In the case of gauge theories in which the Goldstone theorem fails to apply in the radiation-gauge formulation, the massless bosons reappear as uncoupled gauge modes in the manifestly covariant formulation of the same theory. This phenomenon is discussed in detail in Ref. 2.

been discussed in some detail in a number of free-field models as well as in the Zachariasen model[2,4] (which itself consists merely of an infinite number of free fields). Although the existence of the gauge group (1.1) in all these examples implies the essential unobservability of the symmetry breaking, models of this type have been quite useful since they possess a number of features which characterize the more interesting (and more speculative) theories of spontaneous symmetry breaking. On the other hand, it has recently been shown by the authors[5] that in certain types of nonrelativistic field theories which have a massless excitation but which are not invariant under (1.1), the broken-symmetry condition

$$\langle 0|\phi(x)|0\rangle=\eta$$

implies that the results of physical measurements will generally depend upon the symmetry-breaking parameter η. One can include in this class both the neutral-scalar theory and the Lee model provided that one requires the mesons of these theories to have vanishing mass. To date, however, the computational difficulties inherent in relativistic theories has precluded the possibility of convincingly generalizing this result to the fully covariant case. Because of the complicated constraints imposed on such a theory by a symmetry-breaking condition, it is far from clear that a nontrivial broken-symmetry solution can be internally consistent, even if one assumes the existence of well-defined solutions which respect the symmetry.

Aside from these problems of constructing relativistic models in which the breaking of a symmetry leads to physically observable consequences, there is one further generalization of the results obtained in Refs. 2 and 4 which is of considerable interest. In particular, it would be desirable to demonstrate that there exist cases in which the operator $\phi(x)$ need not be one of the canonical variables of the theory under consideration, e.g., that $\phi(x)$ can be bilinear in the fundamental field operators of the system. Within the realm of renormalizable relativistic field theories there is only one set of soluble models known at the present time in which a composite field operator has a particlelike excitation. This class of theories describes the interaction of a massless fermion field with a vector meson in a world of one spatial dimension according to the Lagrangian

$$L=-\tfrac{1}{2}G^{\mu\nu}(\partial_\mu B_\nu-\partial_\nu B_\mu)+\tfrac{1}{4}G^{\mu\nu}G_{\mu\nu}-\tfrac{1}{2}\mu_0{}^2B^\mu B_\mu +\tfrac{1}{2}i\psi\alpha^\mu\partial_\mu\psi+eB_\mu j^\mu, \quad (1.2)$$

which includes as special cases the limits $\mu_0=\infty$ and $\mu_0=0$. Because the solutions of (1.2) are known to contain a zero-mass boson excitation, such a theory might well be expected to support a broken symmetry and we shall consequently investigate in some detail the consequences of casting this massless particle into the role of the Goldstone boson.

II. DERIVATION OF SYMMETRY-BREAKING SOLUTIONS

Since the Green's functions of the theory described by the Lagrangian (1.2) are most easily obtained by means of source techniques, it is useful to introduce the additional coupling terms

$$A^\mu j_\mu+B^\mu J_\mu,$$

where A^μ and J^μ are classical external sources. However, before one can proceed to calculate the solutions of the model, it is further necessary to take into account the long recognized fact that (1.2) is ambiguous[6] because of the ill-defined nature of the formal prescription for the current operator

$$j^\mu(x)=\tfrac{1}{2}\psi\alpha^\mu q\psi,$$

where the α^μ are the two-dimensional set

$$\alpha^0=-\alpha_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$\alpha^1=\alpha_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

and q is the antisymmetric matrix

$$q=\begin{pmatrix}0&-i\\i&0\end{pmatrix},$$

which acts in the charge space of the Hermitian field $\psi(x)$. Recently, it has been shown[7] that one can consistently define the current operator by means of a spacelike limiting procedure, which, in the case $e=0$, may be written as

$$j^\mu(x)=\tfrac{1}{2}\lim_{x\to x'}\psi(x)\alpha^\mu q \times\exp\left[iq\int_{x'}^{x}dx_\mu''\,(\xi A^\mu-\eta\gamma_5A_5{}^\mu)\right]\psi(x'), \quad (2.1)$$

where

$$A_5{}^\mu=\epsilon^{\mu\nu}A_\nu,$$
$$\gamma_5=\alpha^0\alpha^1=\alpha^1,$$

and $\epsilon^{\mu\nu}$ is the antisymmetric tensor defined by

$$\epsilon^{\mu\nu}=-\epsilon^{\nu\mu},$$
$$\epsilon^{01}=+1.$$

The parameters ξ and η, which are required to be real,

[4] W. S. Hellman and P. Roman, Phys. Rev. **143**, 1247 (1964); N. G. Deshpande and S. A. Bludman, *ibid.* **146**, 1186 (1966).

[5] G. S. Guralnik and C. R. Hagen, Phys. Rev. **149**, 1017 (1966); Nuovo Cimento **45**, 959 (1966).

[6] K. Johnson, Nuovo Cimento **20**, 773 (1961).

[7] C. R. Hagen, Nuovo Cimento **51B**, 169 (1967); **51A**, 1033 (1967).

must satisfy the constraint condition

$$\xi+\eta=1$$

if there is to exist an energy-momentum tensor with the correct transformation properties. It is important to note that each distinct value of ξ can be shown[7] to lead to a different solution of (1.2) with the limiting cases $\xi=1$ and $\eta=1$ corresponding, respectively, to the special circumstances in which the vector and axial-vector currents are conserved even in the presence of the sources.

In order to demonstrate the existence of symmetry-breaking solutions of the model, we consider the vacuum-to-vacuum transition amplitude $\langle 0\sigma_1|0\sigma_2\rangle_{A,J,e}$ or more simply $\langle 0|0\rangle_{A,J,e}$. From the action principle, one readily infers the result

$$\langle 0|0\rangle_{A,J,e}=\exp\left\{-ie\int dx\frac{\delta}{\delta A^\mu(x)}\frac{\delta}{\delta J_\mu(x)}\right\}\langle 0|0\rangle_{A,J}, \quad (2.2)$$

where

$$\langle 0|0\rangle_{A,J}\equiv\langle 0|0\rangle_{A,J,0}.$$

With the removal of the coupling term in the exponential there follows a significant simplification, namely the factorization of $\langle 0|0\rangle_{A,J}$ as expressed by

$$\langle 0|0\rangle_{J,A}=\langle 0|0\rangle_A\langle 0|0\rangle_J.$$

Although $\langle 0|0\rangle_J$ can be readily shown from the equations of motion to have the form[7]

$$\langle 0|0\rangle_J=\exp\left\{\tfrac{1}{2}i\int J_\mu(x)G_0{}^{\mu\nu}(x-x')J_\nu(x')dxdx'\right\},$$

where

$$G_0{}^{\mu\nu}(x)=\int\frac{dp}{(2\pi)^2}e^{ipx}\left(g^{\mu\nu}+\frac{p^\mu p^\nu}{\mu_0{}^2}\right)\frac{1}{p^2+\mu_0{}^2-i\epsilon},$$

the amplitude $\langle 0|0\rangle_A$ strictly speaking is not uniquely determined even after one specifies the limiting procedure (2.1) for the current operator.

To demonstrate the origin of this ambiguity, we note that from the action principle it follows that

$$\frac{1}{i}\frac{\delta}{\delta A_\mu(x)}\langle 0|0\rangle_A=\langle 0|j^\mu(x)|0\rangle_A=\tfrac{1}{2}i\lim_{x\to x'}\mathrm{Tr}q\alpha^\mu G(x,x')$$

$$\times\exp\left[-iq\int_{x'}^{x}dx_\mu''(\xi A^\mu-\eta\gamma_5A_5{}^\mu)\right], \quad (2.3)$$

where the Green's function

$$G(x,x')=i\epsilon(x,x')\frac{\langle 0|(\psi(x)\bar\psi(x'))_+|0\rangle_A}{\langle 0|0\rangle_A}$$

satisfies the differential equation

$$\alpha^\mu[(1/i)\partial_\mu-qA_\mu]G(x,x')=\delta(x,x').$$

This equation can be solved[8] by writing

$$G(x,x')=\tilde G_0(x-x')\exp[iq(F(x)-F(x'))],$$

where

$$\alpha^\mu(1/i)\partial_\mu\tilde G_0(x-x')=\delta(x-x'), \quad (2.4)$$

$$\alpha^\mu\partial_\mu F(x)=\alpha^\mu A_\mu(x). \quad (2.5)$$

It is now easy to see that the solution of these equations is not unique and that there is consequently a certain amount of arbitrariness in the matrix element of $j^\mu(x)$. In particular, one notes that the usual causal boundary conditions are capable of specifying $\tilde G_0(x-x')$ only to within an additive constant and that the most general solution of (2.4) is thus of the form

$$\tilde G_0(x-x')=G_0(x-x')+\gamma,$$

where γ is a constant matrix and

$$G_0(x)=[\alpha^1(1/i)\partial_1-(1/i)\partial_0]D(x),$$

$D(x)$ being defined by

$$D(x)=\int\frac{dk}{(2\pi)^2}e^{ikx}\frac{1}{k^2-i\epsilon}.$$

The function $F(x)$ can be inferred from (2.5) to have the structure

$$F(x)=-i\int G_0(x-x')\alpha^\mu A_\mu(x')dx',$$

which enables one to deduce[7]

$$\langle j^\mu(x)\rangle\equiv\frac{\langle 0|j^\mu(x)|0\rangle_A}{\langle 0|0\rangle_A}=\int D_0{}^{\mu\nu}(x-x')A_\nu(x')dx'+\eta^\mu,$$

where

$$D_0{}^{\mu\nu}(x)=-(1/\pi)(\xi\epsilon^{\mu\sigma}\epsilon^{\nu\tau}+\eta g^{\mu\sigma}g^{\nu\tau})\partial_\sigma\partial_\tau D(x)$$

and

$$\eta^\mu=\tfrac{1}{2}i\,\mathrm{Tr}q\alpha^\mu\gamma.$$

This result clearly shows that the ambiguity in (2.4) allows one to consistently impose the broken-symmetry condition

$$\langle j^\mu(x)\rangle|_{A=0}=\eta^\mu \quad (2.6)$$

without requiring the imposition of any constraints upon the theory. On the basis of this result, Eq. (2.3) can be integrated to yield

$$\langle 0|0\rangle_A=\exp\left\{\tfrac{1}{2}i\int A_\mu(x)D_0{}^{\mu\nu}(x-x')A_\nu(x)dxdx\right\}$$

$$\times\exp\left\{i\eta^\mu\int A_\mu(x)dx\right\},$$

[8] J. Schwinger, Phys. Rev. **128**, 2425 (1962).

which differs from the usual form[7] only in the presence of the second exponential.

One can now proceed to calculate the effect of the interaction on the vacuum transition amplitude from Eq. (2.2). The latter can be rewritten in terms of the case $\eta^\mu=0$ as

$$\langle 0|0\rangle_{A,J,e}=\exp\left\{i\eta^\mu\int\left(A_\mu+e\frac{1}{i}\frac{\delta}{\delta J^\mu}\right)dx\right\}\langle 0|0\rangle_{A,J,e}{}^{\eta=0}$$

so that, using the results of Ref. 7, it is trivial to deduce

$$\langle 0|0\rangle_{A,J,e}=\langle 0|0\rangle_{0,0,e}{}^{\eta=0}\exp\left\{i\eta^\mu\int A_\mu(x)dx\right\}\exp\left\{\tfrac{1}{2}i\int[J_\mu(x)+e\eta_\mu]G^{\mu\nu}(x-x')[J_\nu(x')+e\eta_\nu]dxdx'\right\}$$
$$\times\exp\left\{\tfrac{1}{2}i\int A_\mu(x)D^{\mu\nu}(x-x')A_\nu(x')dxdx'\right\}\exp\left\{i\int[J_\mu(x)+e\eta_\mu]M^{\mu\nu}(x-x')A_\nu(x')dxdx'\right\},$$

where

$$G^{\mu\nu}(x)=\left(g^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{\mu^2}\right)\Delta(x)-\frac{e^2}{\pi\mu_0{}^2}\frac{1}{(1-\eta e^2/\pi\mu_0{}^2)(1+\xi e^2/\pi\mu_0{}^2)}\frac{\partial^\mu\partial^\nu}{\mu_0{}^2}D(x),$$

$$D^{\mu\nu}(x)=-\frac{1}{\pi}\left[\epsilon^{\mu\sigma}\epsilon^{\nu\tau}\frac{\xi}{1+\xi e^2/\pi\mu_0{}^2}+g^{\mu\sigma}g^{\nu\tau}\frac{\eta}{1-\eta e^2/\pi\mu_0{}^2}\right]\partial_\sigma\partial_\tau D(x)-\frac{\xi^2}{\pi^2}\frac{e^2}{\mu_0{}^2}\frac{1}{1+\xi e^2/\pi\mu_0{}^2}\epsilon^{\mu\sigma}\epsilon^{\nu\tau}\partial_\sigma\partial_\tau\Delta(x),$$

$$M^{\mu\nu}(x)=-\frac{e}{\pi\mu_0{}^2}\left[\frac{1}{(1-\eta e^2/\pi\mu_0{}^2)(1+\xi e^2/\pi\mu_0{}^2)}\partial^\mu\partial^\nu D(x)+\frac{\xi}{1+\xi e^2/\pi\mu_0{}^2}(\mu^2 g^{\mu\nu}-\partial^\mu\partial^\nu)\Delta(x)\right],$$

and $\Delta(x)$ is the causal Green's function

$$(-\partial^2+\mu^2)\Delta(x)=\delta(x)$$

corresponding to the renormalized meson mass

$$\mu^2=\mu_0{}^2+\xi e^2/\pi.$$

The effect of the interaction on the broken-symmetry condition (2.6) is found immediately by calculating

$$\langle j^\mu\rangle|_{A,J=0}=\eta^\mu+e\int M^{\mu\nu}(x-x')\eta_\nu dx'. \tag{2.7}$$

Since, however, the Fourier transform of $M^{\mu\nu}(x)$ contains terms of the form $k^\mu k^\nu/k^2$, the value of this function at $k=0$ is dependent upon how one takes the limit. It is, consequently, necessary to lend meaning to all expressions involving η^μ by giving η^μ a space-time dependence which will allow one to freely carry out integrations by parts and by taking the limit $\eta^\mu=$ constant only at the end of a given calculation. More specifically, we write

$$\eta^\mu=\eta_1{}^\mu+\eta_2{}^\mu,$$

where

$$\partial_\mu\eta_1{}^\mu=0,$$
$$\epsilon_{\mu\nu}\partial^\mu\eta_2{}^\nu=0.$$

With this prescription, (2.7) becomes

$$\langle j^\mu\rangle|_{A,J=0}=\eta^\mu+\frac{e^2}{\pi\mu_0{}^2}\frac{1}{(1-\eta e^2/\pi\mu_0{}^2)(1+\xi e^2/\pi\mu_0{}^2)}\eta_2{}^\mu-\frac{\xi e^2}{\pi\mu_0{}^2}\frac{1}{1+\xi e^2/\pi\mu_0{}^2}\eta^\mu$$
$$=\eta_1{}^\mu\frac{1}{1+\xi e^2/\pi\mu_0{}^2}+\eta_2{}^\mu\frac{1}{1-\eta e^2/\pi\mu_0{}^2}, \tag{2.8}$$

which displays the different renormalizations undergone by the "vector" and "axial-vector" parts of the symmetry-breaking parameter. Although it may seem peculiar at first sight that the vacuum expectation value of $j^\mu(x)$ should depend upon the way in which one splits the vector η^μ into the two parts $\eta_1{}^\mu$ and $\eta_2{}^\mu$, this can readily be seen to be a consequence of the fact that a constant vector in two dimensions can always be written in the form

$$\eta^\mu=\alpha\epsilon^{\mu\nu}\partial_\nu(\epsilon^{\sigma\tau}x_\sigma\eta_\tau)+\beta\partial^\mu(\eta^\nu x_\nu),$$

where the two parameters α and β are constrained by the single condition

$$\alpha+\beta=1.$$

Since those parts of η^μ which are proportional to α and β are renormalized by $(1+\xi e^2/\pi\mu_0{}^2)^{-1}$ and $(1-\eta e^2/\pi\mu_0{}^2)$, respectively, only in the exceptional cases $\xi=1$ and $\eta=1$ (which correspond to $\eta_2{}^\mu=0$ and $\eta_1{}^\mu=0$, respectively) is the expectation value of $j^\mu(x)$ uniquely specified by the $e=0$ result. Any lingering doubts that the reader has about this decomposition should be resolved by the results of the next section, from which it can be deduced that this is the unique decomposition consistent with the properties of $J^{\mu\nu}$ as the generator of Lorentz transformations.

A special case of the vector-meson theory which is of particular interest is obtained by taking the limit

$$e\to\infty\,,\quad \mu_0\to\infty\,,\quad e^2/\mu_0{}^2\to\lambda\,.$$

This corresponds to the case of the Thirring model which is formally described by the Lagrangian $L=\frac{1}{2}i\psi\alpha^\mu\partial_\mu\psi+\frac{1}{2}\lambda j^\mu j_\mu$. In this case the broken-symmetry condition (2.8) trivially becomes

$$\langle j^\mu\rangle|_{A,J=0}=\eta_1{}^\mu\frac{1}{1+\lambda\xi/\pi}+\eta_2{}^\mu\frac{1}{1-\lambda\eta/\pi}\,. \tag{2.9}$$

We emphasize here that there is no connection between the broken-symmetry solution (2.9) and that previously obtained by Leutwyler[9] for the Thirring model. In particular, it is to be noted that Leutwyler does not allow the possibility of a broken-symmetry solution in the free-field limit, that is for $\lambda=0$ he finds $\eta_1{}^\mu=\eta_2{}^\mu=0$. This approach consequently leads to the incorrect claim that the construction of nonzero $\langle j^\mu(x)\rangle$ can be achieved only for the particular values of λ corresponding to the singular points ($\lambda=-\pi/\xi$ and $\lambda=\pi/\eta$) of (2.9). On the other hand, the solutions obtained here and in Ref. 7 specifically require that λ be restricted such that $(1+\lambda\xi/\pi)^{-1}$ and $(1-\lambda\eta/\pi)^{-1}$ be positive and bounded out but do not otherwise limit the strength of the coupling.

It is instructive to note that from the expressions[7] for the divergence and curl of $j^\mu(x)$ in the source-free limit, i.e.,

$$\partial_\mu j^\mu=0\,,$$
$$\epsilon^{\mu\nu}\partial_\mu j_\nu=(e\xi/2\pi)\epsilon_{\mu\nu}G^{\mu\nu}\,,$$

one finds with the help of the field equations

$$G^{\mu\nu}=\partial^\mu B^\nu-\partial^\nu B^\mu\,,$$
$$\partial_\nu G^{\mu\nu}=ej^\mu-\mu_0{}^2B^\mu$$

the result

$$\partial^2(-\partial^2+\mu^2)j^\mu=0\,. \tag{2.10}$$

In the limit in which one obtains the Thirring model, only the massless mode persists, i.e.,

$$\partial^2 j^\mu=0\,, \tag{2.11}$$

while in the limit $\mu_0\to 0$ (Schwinger model) one must require[7] $\xi=1$, and (2.10) reduces to

$$(-\partial^2+e^2/\pi)j^\mu=0\,. \tag{2.12}$$

Although (2.10) and (2.11) imply the possibility of consistently taking $\langle j^\mu(x)\rangle\neq0$ in the source-free limit, (2.12) clearly demonstrates that in the Schwinger model such a condition requires

$$(e^2/\pi)\langle j^\mu(x)\rangle=0\,,$$

i.e., the broken symmetry is inconsistent because of the absence of a zero-mass particle. Although this result can also be extracted from (2.8) by setting $\eta_2{}^\mu=0$ and taking the limit $\mu_0\to 0$, it is somewhat more instructive to establish this result by a direct calculation.

To do this we begin with the Lagrangian

$$L=-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+\tfrac{1}{2}i\psi\alpha^\mu\partial_\mu\psi+ej^\mu A_\mu+A^\mu J_\mu+j^\mu a_\mu\,,$$

where J^μ and a^μ are external sources. As before, one obtains

$$\langle0|0\rangle_{a,J,e}=\exp\left\{-ie\int\frac{\delta}{\delta a^\mu(x)}\frac{\delta}{\delta J_\mu(x)}dx\right\}\langle0|0\rangle_{a,J} \tag{2.13}$$

and

$$\langle0|0\rangle_{a,J}=\langle0|0\rangle_a\langle0|0\rangle_J\,.$$

Since it is necessary to impose current conservation on $j^\mu(x)$, one takes $\xi=1$ so that

$$\langle0|0\rangle_a=\exp\left\{\tfrac{1}{2}i\int a_\mu(x)D^{\mu\nu}(x-x')a_\nu(x')dxdx'\right\}\times\exp\left\{i\eta^\mu\int a_\mu(x)dx\right\}\,,$$

where

$$D^{\mu\nu}(x)=-(1/\pi)\epsilon^{\mu\sigma}\epsilon^{\nu\tau}\partial_\sigma\partial_\tau D(x)\,.$$

One also finds by straightforward calculation the result

$$\langle0|0\rangle=\exp\left\{\tfrac{1}{2}i\int J_\mu(x)D_R{}^{\mu\nu}(x-x')J_\nu(x')dxdx'\right\}\,,$$

where $D_R{}^{\mu\nu}(x)$ is the radiation-gauge Green's function

$$D_R{}^{\mu\nu}=\delta_0{}^\mu\delta_0{}^\nu\nabla^{-2}\,.$$

Substitution into (2.13) yields

$$\langle0|0\rangle_{a,J,e}=\langle0|0\rangle_{0,0,e}\exp\left\{i\eta^\mu\int a_\mu(x)dx\right\}\exp\left\{\tfrac{1}{2}i\int[J_\mu(x)+e\eta_\mu]D_R{}^{\mu\nu}(x-x')[J_\nu(x')+e\eta_\nu]dxdx'\right\}$$
$$\times\exp\left\{\tfrac{1}{2}i\int dx\left[a_\mu(x)+e\int D_{\mu\alpha}{}^R(x-x'')(J^\alpha(x'')+e\eta^\alpha)dx''\right]\Delta^{\mu\nu}(x-x')\right.$$
$$\left.\times\left[a_\nu(x')+e\int dx'''D_{\nu\beta}{}^R(x'-x''')[J^\beta(x''')+e\eta^\beta]dx''\right]dx'\right\}\,,$$

[9] H. Leutwyler, Helv. Phys. Acta 38, 431 (1965).

where

$$\Delta^{\mu\nu}(x)=-(1/\pi)\epsilon^{\mu\sigma}\epsilon^{\nu\tau}\partial_\sigma\partial_\tau\Delta(x)$$

and

$$(-\partial^2+e^2/\pi)\Delta(x)=\delta(x).$$

Since there are no independent dynamical variables associated with the electromagnetic field in two dimensions, all boson matrix elements of the theory can be generated by the single source $a^\mu(x)$. We thus set $J^\mu=0$ and obtain

$$\langle 0|0\rangle_{a,0,e}=\langle 0|0\rangle_{0,0,e}\exp\left\{i\eta^\mu\int a_\mu(x)dx\right\}$$

$$\times\exp\left\{\tfrac{1}{2}i\int\left[a_\mu(x)+e^2\int D(x-x'')\eta_\mu dx''\right]\Delta^{\mu\nu}(x-x')\right.$$

$$\left.\times\left[a_\nu(x')+e^2\int D(x'-x''')\eta_\nu dx'''\right]dxdx'\right\},$$

where we have given η^μ a space-time dependence which will allow integrations by parts (subject to the condition $\partial_\mu\eta^\mu=0$). One now finds

$$\langle j^\mu(x)\rangle|_{a=0}=\eta^\mu+e^2\int\Delta^{\mu\nu}(x-x')D(x'-x'')dx'dx''\eta_\nu$$

$$=\eta^\mu[1-(e^2/\pi)\Delta(k=0)]$$

$$=0,$$

which clearly illustrates the cancellation of the free-field broken symmetry through the effect of the interaction. This rather remarkable result thus shows that the soluble models in two dimensions provide some support for the hypothesis[2] that broken-symmetry solutions can occur in a given quantum-field theory if and only if the system has a massless excitation.

III. CONSISTENCY OF SYMMETRY BREAKING WITH THE PROPERTIES OF THE LORENTZ GROUP GENERATORS

It is not difficult to understand why most attention to spontaneous symmetry breaking has been confined to the case of spin-zero Goldstone particles. In particular, the formidable mathematical (and physical) difficulties associated with spin can readily be displayed by means of the commutator of a field Φ of nonzero spin with $J^{\mu\nu}$, the generators of the Lorentz group

$$(1/i)[J^{\mu\nu},\Phi(x)]=(x^\mu\partial^\nu-x^\nu\partial^\mu)\Phi(x)+\sigma^{\mu\nu}\Phi(x),$$

where $\sigma^{\mu\nu}$ is a numerical matrix. It follows from this expression that the requirement $\langle 0|\Phi|0\rangle\neq 0$ necessarily implies the formal breakdown of Lorentz invariance. In general, this is a very dangerous requirement since (assuming the symmetry-breaking parameter $\langle\Phi\rangle$ appears in a nontrivial manner) the result is likely to be an unacceptable departure from Lorentz invariance in the excitation spectrum of the theory. As is seen from the solution of the simple model theories discussed in this paper, the symmetry breaking in these cases is sufficiently trivial so as to preclude an observable breakdown of Lorentz invariance. Nevertheless, it is necessary that this breaking be consistent with the Lorentz group properties and we must consequently confirm that the equation

$$(1/i)[J^{\mu\nu},j^\lambda(x)]=(x^\mu\partial^\nu-x^\nu\partial^\mu)j^\lambda(x)+g^{\mu\lambda}j^\nu(x)-g^{\lambda\nu}j^\mu(x),$$

where

$$J^{01}=\int d\mathbf{x}\,[x^0T^{01}(x)-\mathbf{x}T^{00}(x)]$$

is consistent with $\langle j^\lambda\rangle\neq 0$ and hence (using $\langle 0|P^1=0$) that

$$i\int d\mathbf{x}'\mathbf{x}'\langle 0|[T^{00}(x'),j^\lambda(x)]|0\rangle=g^{0\lambda}\langle j^1(x)\rangle-g^{\lambda 1}\langle j^0(x)\rangle \quad (3.1)$$

is satisfied. We shall do this only in the case of the massive vector-meson model and note that the results for the other models can be obtained by appropriate limiting procedures. Incidentally, a similar equation must be valid for $\langle 0|B^\mu|0\rangle$ since the field equation $\partial_\nu G^{\mu\nu}=ej^\mu-\mu_0{}^2B^\mu$ requires that $\langle B^\mu\rangle=(e/\mu_0{}^2)\langle j^\mu\rangle$. However, verification of (3.1) is tantamount to verification of the same equation with $j^\mu\to B^\mu$.

We evaluate (3.1) by first finding

$$(1/i)(\delta/\delta A_\lambda(x))\langle 0|T^{00}(x')|0\rangle|_{A=J=0}.$$

In this process we neglect all terms proportional to $\langle j^\lambda(x)\rangle$ since they cannot contribute to (3.1). From Ref. 7 we find that

$$T^{00}(x)=\tfrac{1}{2}(G^{01})^2+\tfrac{1}{2}\mu_0{}^2[a_1B_1{}^2+a_2B_0{}^2]-\tfrac{1}{2}i\psi\alpha^1\partial_1\psi-eB_1j^1-(e^2\xi/\pi)B_1{}^2,$$

where the shorthand notation $a_1\equiv 1+\xi e^2/\pi\mu_0{}^2$ and $a_2\equiv 1-\eta e^2/\pi\mu_0{}^2$ has been introduced.

We also need an explicit expression for the two-point Fermi Green's function $G(x,x')$ in the presence of interactions. Using the methods of Ref. 7, one finds

$$G(x,x')\equiv G_{A,J,e}{}^\eta(x,x')=G_{0,0,e}{}^\eta(x,x')\exp\left\{iq\int A^\mu(x'')[N_\mu(x'',x)-N_\mu(x'',x')]dx''\right\}$$

$$\times\exp\left\{iq\int[J^\mu(x'')+e\eta^\mu][M_\mu(x'',x)-M_\mu(x'',x')]dx''\right\}, \quad (3.2)$$

where

$$G_{0,0,e^\eta}(x,x')=[\gamma+G_0(x,x')]\exp\left\{-i\pi\left(\frac{e^2}{\pi\mu_0^2}\right)\frac{1}{a_1a_2}[D(x-x')-D(0)]\right\}\exp\left\{\frac{-ie^2}{\mu_0^2a_1}[\Delta(x-x')-\Delta(0)]\right\}$$

and we have introduced the functions[7]

$$N^\mu(x,x')=[(1/a_2)\partial^\mu+(1/a_1)\epsilon^{\mu\nu}\gamma_5\partial_\nu]D(x-x')$$

and

$$M^\mu(x,x')=(e/\mu_0^2)[(1/a_2)\partial^\mu+(1/a_1)\epsilon^{\mu\nu}\partial_\nu\gamma_5]D(x-x')-(1/a_1)\epsilon^{\mu\nu}\gamma_5\partial_\nu\Delta(x-x').$$

Although it is a straightforward exercise to compute the higher-order Fermi Green's functions, we omit the results of this calculation since they are not relevant to our discussion of Eq. (3.2).

One can now evaluate

$$(1/i)(\delta/\delta A_\lambda(x))\langle 0|T^{00}(x')|0\rangle|_{A=J=0}$$

term by term using the expression given in Sec. II for $\langle 0|0\rangle_{J,A,e^\eta}$, Eq. (3.2), and the usual source techniques. Terms of the form $\langle 0|(B^\mu B^\nu j^\lambda)_+|0\rangle$ are readily found to yield

$$\frac{-i\delta}{\delta J_\mu(x')}\frac{-i\delta}{\delta J_\nu(x')}\frac{-i\delta}{\delta A_\lambda(x)}\langle 0|0\rangle|_{A=J=0}=-iM^{\mu\lambda}(x,'x)\int G^{\nu\sigma}(x',x'')e\eta_\sigma dx''-iM^{\nu\lambda}(x',x)\int G^{\mu\sigma}(x',x'')e\eta_\sigma dx''+\text{I.T.},$$

where I.T. represents irrelevant terms proportional to $\langle 0|j^\lambda(x)|0\rangle$.

Although the expression $\int G^{\mu\sigma}(x',x'')e\eta_\sigma dx''$ may be evaluated directly (using our specified integration technique), it is more convenient to use

$$\int G^{\mu\sigma}(x',x'')e\eta_\sigma dx''=\langle 0|B^\nu|0\rangle=(e/\mu_0^2)\langle 0|j^\nu|0\rangle$$

to deduce

$$\frac{-i\delta}{\delta J_\mu(x')}\frac{-i\delta}{\delta J_\nu(x')}\frac{-i\delta}{\delta A_\lambda(x)}\langle 0|0\rangle\bigg|_{A=J=0}=-iM^{\mu\lambda}(x',x)\frac{e}{\mu_0^2}\langle 0|j^\nu|0\rangle-iM^{\nu\lambda}(x'mx)\frac{e}{\mu_0^2}\langle 0|j^\mu|0\rangle+\text{I.T.}\quad(3.3)$$

From this representation it is easily seen that

$$\langle 0|((G^{01})^2j^\lambda)_+|0\rangle=0$$

since in forming this from (3.3) one derivative always acts on $\langle 0|j^\nu|0\rangle$. Just as directly we find that

$$\frac{-i\delta}{\delta A_\lambda(x)}\frac{-i\delta}{\delta J_1(x')}\frac{-i\delta}{\delta A_1(x')}\langle 0|0\rangle|_{A=J=0}=-iM^{1\lambda}(x',x)\langle 0|j^1|0\rangle-\frac{ie}{\mu_0^2}D^{1\lambda}(x',x)\langle 0|j^1|0\rangle.$$

Finally we compute

$$-\tfrac{1}{2}i\langle 0|(\bar\psi(x')\alpha^1\partial_1\psi(x')j^\lambda(x))_+|0\rangle=-\frac{1}{2}\frac{-i\delta}{\delta A_\lambda(x)}(\partial_{x''}{}^1\,\text{Tr}G(x',x'')\alpha^1\langle 0|0\rangle)_{x'\to x''}.$$

Using (3.2), we find that

$$-\tfrac{1}{2}i\frac{-i\delta}{\delta A_\lambda(z)}\partial_{x''}{}^1\,\text{Tr}G(x',x'')\alpha^1|_{x'\to x''}=-\tfrac{1}{2}i\left\{\text{Tr}\alpha^1G_{0,0,e^\eta}(x',x'')q\partial_1N^\lambda(x,x'')+\frac{i}{2\pi}\text{Tr}\left[\int dx'''e\eta^\mu\partial_1M_\mu(x''',x'')\right]\times\partial_1N^\lambda(x,x'')|_{x\to x''}\right\},$$

the evaluation of which results in the expression

$$-\frac{1}{2}\frac{-i\delta}{\delta A_\lambda(x)}[\partial_{x''}{}^1\,\text{Tr}G(x',x'')\alpha^1]|_{x'=x''}\langle 0|0\rangle=i\partial^\lambda\partial^1D(x'-x)\left[\frac{(\eta_2)^1}{a_2}\frac{e^2}{\pi\mu_0^2}+\frac{\eta^1}{a_2}\right]+i\epsilon^{\lambda\nu}\epsilon^{0\sigma}\partial_\nu\partial_\sigma\left[-D(x'-x)+\frac{\xi e^2}{\pi\mu_0^2}\Delta(x'-x)\right]\times\left[\frac{\eta^0}{a_1}-\frac{(\eta_1)^0}{a_1^2}\frac{e^2}{\pi\mu_0^2}\right]+\text{I.T.}$$

Thus, it follows that

$$\frac{1}{i}\frac{\delta}{\delta A_\lambda(x)}\langle 0|T^{00}(x')|0\rangle|_{A=J=0}$$

$$=\tfrac{1}{2}\mu_0^2\left[-\frac{2iea_1}{\mu_0^2}M^{1\lambda}(x',x)\langle 0|j^1|0\rangle-\frac{2iea_2}{\mu_0^2}M^{0\lambda}(x',x)\langle 0|j_0|0\rangle\right]-i\epsilon^{\lambda\nu}\epsilon^{0\sigma}\partial_\nu\partial_\sigma\left[D(x'-x)+\frac{-\xi e^2}{\pi\mu_0^2}\Delta(x'-x)\right]$$

$$\times\left[\frac{(\eta_1)^0}{(a_1^2)}\frac{e^2}{\pi\mu_0^2}-\frac{\eta^0}{a_1}\right]+i\langle 0|j^1|0\rangle\left[e\left(1+\frac{2e^2\xi}{\pi\mu_0^2}\right)M^{1\lambda}(x',x)+\frac{e^2}{\mu_0^2}D^{1\lambda}(x',x)\right].$$

Grouping terms and using the expressions for $M^{\mu\nu}$ and $D^{\mu\nu}$, this becomes

$$(1/i)(\delta/\delta A_\lambda(x))\langle 0|T^{00}(x')|0\rangle=i\langle 0|j^1|0\rangle\left(-\frac{e^4\xi}{\mu_0^4\pi^2}-\frac{e^2}{\pi\mu_0^2}\right)\frac{\partial^1\partial^\lambda D(x'-x)}{a_1a_2}+i\partial^\lambda\partial^1 D(x'-x)\left[\frac{\eta^1}{a_2}+\frac{(\eta_2)^1e^2}{a_1^2\pi\mu_0^2}\right]$$

$$+\left[\frac{\eta^0}{a_1}-\frac{(\eta_1)^0e^2}{a_1^2\pi\mu_0^2}\right]i\partial^\lambda\partial^0 D(x'-x)+\frac{ie^2}{\pi\mu_0^2a_1}\partial^0\partial^\lambda D(x'-x)\langle 0|j^0|0\rangle+\text{I.T.}$$

Note the important result that all terms proportional to the massive propagator $\Delta(x'-x)$ vanish, as required by the Goldstone theorem. Using Eq. (2.8) the above expression finally becomes

$$(\delta/\delta A_\lambda(x))\langle 0|T^{00}(x')|0\rangle=-\partial^1\partial^\lambda D(x'-x)\langle 0|j^1|0\rangle$$
$$-\partial^0\partial^\lambda D(x'-x)\langle 0|j|0\rangle+\text{I.T.}$$

From this it is immediately seen that Eq. (3.1) is valid and that we have consequently succeeded in constructing a consistent broken-symmetry theory. As mentioned in the previous section, experimentation with the above expressions reveals that we have defined the integrals involving η^μ in the correct manner.

CONCLUSIONS

In this paper we have established the consistency of a broken symmetry involving the nonvanishing vacuum-expectation value of a bilinear operator. This is the only relativistic example presently known to exist which is nonperturbative and consequently not intimately involved with the intricacies of a cutoff. It should be emphasized that the broken-symmetry requirement in these models has a minimal effect on the Green's functions of the theories, and in no way changes the spectrum. In particular, the Green's functions involving only boson operators are identical to the corresponding Green's functions evaluated at $\eta^\mu=0$, while the Green's functions involving Fermi operators are changed only by the inclusion of the constant γ in $\tilde{G}_0$ and the replacement of $J^\mu(x)$ by $J^\mu(x)+e\eta^\mu$. It must be pointed out, however, that despite the relatively slight changes induced by the symmetry breaking, the structure involved is much more complicated than in the so-called naturally occurring cases discussed in Ref. 2. This is borne out by the fact that $\langle j^\mu\rangle$ is dependent on the strength of the coupling parameter e^2/μ_0^2.

Finally we note that by making an appropriate identification the formal Green's function expressions of the Thirring model with a broken symmetry can easily be shown to correspond to their analog in four dimensions—the Bjorken model[10,11] with vanishing fermion bare mass. The actual interpretation and comparision to electrodynamics is not identical, however, because of peculiarities intrinsic to a two-dimensional space-time.

[10] G. S. Guralnik, Phys. Rev. **136** B1404 (1964); **136**, B1417 (1964).

[11] J. D. Bjorken, Ann. Phys. (N. Y.) **24**, 174 (1963).